INDIAN JOURNAL OF SCIENCE AND TECHNOLOGY

RESEARCH ARTICLE

# Laceability Partition Dimension of Some Special Graphs

**Manjula M[1]*, Leena N Shenoy[2], Sowmya Krishna[3]**

**1** Research Scholar, Department of Mathematics, B.N.M.I.T., Bengaluru and Visvesvaraya Technological University, Belagavi, Karnataka, India
**2** Associate Professor, Department of Mathematics, B.N.M.I.T., Bengaluru and Visvesvaraya Technological University, Belagavi, Karnataka, India
**3** Assistant Professor, Department of Mathematics, Dayananda Sagar College of Engineering, Bengaluru, Affiliated to Visvesvaraya Technological University, Belagavi, Karnataka, India

OPEN ACCESS

**Received:** 12-06-2024

**Accepted:** 21-07-2024

**Published:** 06-08-2024

**Citation:** Manjula M , Shenoy LN, Krishna S (2024) Laceability Partition Dimension of Some Special Graphs. Indian Journal of Science and Technology 17(31): 3183-3189. https://doi.org/10.17485/IJST/v17i31.1955

* **Corresponding author**.

m.m.manjula@gmail.com

**Funding:** None

**Competing Interests:** None



Published By Indian Society for Education and Environment (iSee)

**ISSN**
Print: 0974-6846
Electronic: 0974-5645

## Abstract

**Objectives:** Laceability partition dimension of a connected graph $G$ is the minimum number of partitions of vertex set such that the subgraph induced by each partition is laceable in the case of a bipartite graph $G$ and random Hamiltonian laceable in case of non-bipartite graph $G$. **Methods:** Mathematical Induction method and tracing Hamiltonian path. **Findings:** This study presents the laceability partition dimension (lpd) of some special graphs namely the Crown graph, Windmill graph, Dutch windmill graph, Cocktail party graph, shadow graph, and image graph. **Novelty:** This study discusses the laceability partition dimension of the graphs through which we discover a Hamiltonian path within a smaller structure whenever Hamiltonian laceability doesn't exist in large networks which has significance in communication networking that strives for optimal message routing efficiency.

**Keywords:** Hamiltonian path; Bipartite graph; Laceability; Perfect matching; Shadow distance graph

## 1 Introduction

Graph theory, as a branch of mathematics, has emerged as a powerful tool for modeling and analyzing complex systems across various disciplines, ranging from computer science to biology, sociology, and beyond.

The concept of laceability has garnered significant attention due to its implications in diverse fields such as computer graphics, robotics, materials science, and spatial analysis. Understanding the structural properties and the behavior of laceable graphs provides valuable insights into the underlying connectivity of complex networks and facilitates the design and optimization of systems in practical contexts.

The referred graphs here are both connected and finite. Let $V(G)$ be the set of vertices and $E(G)$ be the set of edges of the graph $G$. The diameter of $G$, $diam(G)$ is the longest distance among any two vertices of $G$. A subgraph H of a graph $G$ is a graph whose vertex set $V(H) \subseteq V(G)$ and the edge set $E(H) \subseteq E(G)$. A graph is an induced subgraph of $G$, if $u, v \in H$ and $uv \in E(G)$, then the edge $uv \in H$ as well.

Partitioning denotes the process of partitioning the graph $G = (V, E)$ with $V$ vertices and $E$ edges into fewer components with specific properties.

The edge partition dimension of a connected graph and the edge partition dimension of multipartite graphs are discussed in[1]. Here, we are eager to study whether there exists a Hamiltonian path between every pair of vertices within each partition. Matching in a graph $G$ refers to an independent set of edges in $G$, where no two edges in the set are adjacent. If a graph $G$ of order 2k has a matching $M$ of cardinality $k$, then this matching $M$ is called a perfect matching, as $M$ matches every vertex of $G$ to some vertex of $G$.

A graph $G$ is a bipartite graph if $V\ (G)$ can be partitioned into two subsets $U$ and $V$ called partite sets, such that for every edge of $G$ joins a vertex of $U$ and a vertex of $V$. A sequence of arcs or edges that passes through each vertex exactly once without returning to the starting point or vertex is called a Hamiltonian path. A bipartite graph $G$ with its bipartition $U,\ V$ is called balanced if $|U|\ =\ |V|$ and almost-balanced if $|U|\ =\ |V|\pm 1$.

Bipartite graph G with its bipartitions U and V is called Hamiltonian laceable if

a) $G$ is balanced and there exists a Hamiltonian path between every $u \in U$ and every $v \in V$ , or

b) $G$ is almost balanced and there exists a Hamiltonian path between every two distinct vertices $u,\ u^{'} \in U$ , where $U$ is the larger bipartite set.

A connected, simple, non-bipartite graph $G(V,\ E)$ is random Hamiltonian laceable, if there exists a $u - v$ H-path for every $u,\ v\ \in V$.[2,3].

Hamiltonian laceability and laceable properties of some of the graphs such as star graph, hypercubes, variants of hypercubes, etc have been discussed.[4–6]. The graphs which are not laceable motivated us to find the laceability partition dimension of the graphs. In this paper, we discuss the lpd of some of the special graphs.

## 2 Methodology

In this section Crown graph, Windmill graph, Dutch windmill graph, Cocktail party graph, shadow graph and image graph are considered for the study through Mathematical Induction method and tracing Hamiltonian path.

### 2.1 Crown graph

**Definition 2.1.** The crown graph $H_{n,n}$ is the graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching[7].

Consider a crown graph $H_{n,n}\ =\ G(U \cup V,\ E)$ which is a bipartite graph obtained by removing perfect matching edges $(u_i,\ v_j)$ for $i\ =\ j$, where $u_i\ \in U$ and $v_j\ \in V$ .

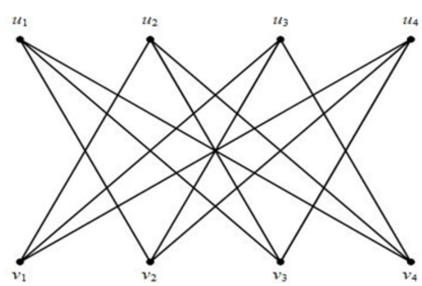


**Fig 1. Crown graph $H_{4,4}$**

### 2.2 Windmill graph

**Definition 2.2.** The Windmill graph $W_d(k,\ n)$ is an undirected graph obtained by taking $k$-copies of the complete graph $K_n$ with a vertex in common[7].

### 2.3 Dutch Windmill graph

**Definition 2.3.** The Dutch Windmill graph ${D_n}^k$ is the graph obtained by taking $k$-copies of the cycle graph $C_n$ with a vertex in common[7].

### 2.4 Shadow graph

**Definition 2.4.** The Shadow graph $S(G)$ of a graph $G$ is obtained from $G$ by adding for each vertex $v$ of $G$, a new vertex $v'$ called the shadow vertex of $v$, and joining $v'$ to the neighbors of $v$ in $G$.

### 2.5 Shadow distance graph

**Definition 2.5.** The shadow distance graph of $G$, denoted by $D_{sd}(G, D_s)$ is constructed from $G$ with the following conditions:

- consider two copies of $G$ say $G$ itself and $G'$
- if $u \in V(G)$, then we denote the corresponding vertex as $u' \in V(G')$
- the vertex set of $D_{sd}(G, D_s)$ is $V(G) \cup V(G')$
- The edge set of $D_{sd}(G, D_s)$ is $E(G) \cup E(G') \cup E_{ds}$ where $E_{ds}$ is the set of all edges between two distinct vertices $u \in V(G)$ and $v' \in V(G')$ that satisfy the condition $d(u, v) \in D_s$ in G.[8].

In this paper, we have considered the shadow distance graph of the Crown graph $H_{n,n}$.

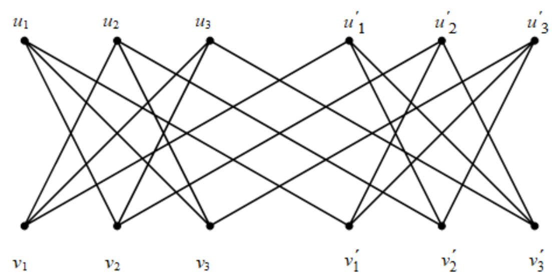


**Fig 2.** $D_{sd}(H_{3,3}, 3)$

### 2.6 Cocktail Party Graph

**Definition 2.6.** The Cocktail Party graph $\overline{L_n}$ is the graph consisting of two rows of paired vertices in which all vertices but the paired ones are connected with a graph edge[7].

### 2.7 Image graph

**Definition 2.7.** The image graph of a connected graph $G$, denoted by $Img(G)$ is the graph obtained by joining the vertices of the original graph $G$ to the corresponding vertices of a copy of $G$[8]

In this paper, we have considered the image graph of Crown graph.

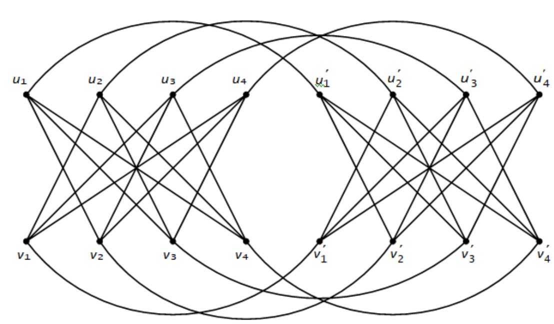


**Fig 3.** $Img(H_{4,4})$

## 3 Results and Discussion

**Remark 3.1.** For $n \geq 3$, $diam(H_{n,n})$ is 3.

**Remark 3.2.** The graph $H_{2,2}$ is disconnected.

**Lemma 3.1.** For the graph $H_{3,3}$, $lpd(H_{3,3})$ is 2.

**Proof**. The graph $H_{3,3} = G(U \cup V, E)$ is a bipartite graph such that $U \cap V = \emptyset$. For every $u \in U$ and $\in V$, there does not exist Hamiltonian path, and hence $H_{3,3}$ is a non-Hamiltonian laceable graph.

The graph $G$ can be partitioned into two induced subgraphs such that two vertices from one partite and one vertex from the other partite are considered. Here the partitioned subgraphs are almost balanced graphs and laceable. Hence, $lpd(H_{3,3}) = 2$.

**Theorem 3.2.** For the Crown graph $H_{n,n}$ where $n \geq 4$, $lpd(H_{n,n})$ is 1.

**Proof**. The graph $H_{n,n}$= G (U ∪ V, E) is balanced bipartite graph such that $|U| = |V|$. For all $u_i \in U$ and $v_j \in V$ with $1 \leq i, j \leq n$, there exists a $u_i - v_j$ Hamiltonian path $P$ since such a path can be traversed by moving alternatively between $U$ and $V$ as $d(u_i) = d(v_j) = n - 1$.

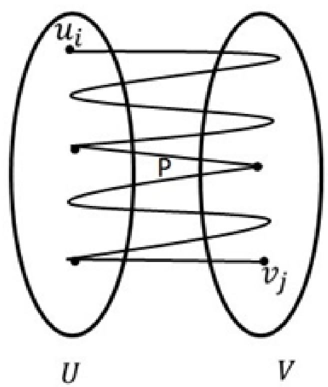


**Fig 4.** $u_i - v_j$ **Hamiltonian path in** $H_{n,n}$

**Remark 3.3.** For $n \geq 3$, $diam(\mathrm{W}_d(k, n))$ is 2.

**Theorem 3.3.** For Windmill graph $\mathrm{W}_d(k, n)$, the $lpd(\mathrm{W}_d(k, n))$ is k, for $k \geq 2$ and $n \geq 3$.

**Proof** . Consider the graph $G = W_d(k, n)$, having $k(n-1) + 1$ vertices. The graph $G$ is non-bipartite and is non-random Hamiltonian laceable.

We intend to find $lpd(G)$ by mathematical induction. For $n = 3$, the result is obvious. For $n = 4$, the graph $G$ is partitioned into three subgraphs specifically one copy of $K_4$ and two copies of $K_3$. Since the graphs $K_3$ and $K_4$ are random Hamiltonian laceable, lpd of $K_3$ and $K_4$ is one. Therefore, $lpd(W_d(k, 4)) = 3$ for $k = 3$ which is shown in Figure 5.

Assuming the result is true for $n \geq 5$, we need to prove the result is true for $n = m$. The graph $G$ is partitioned into two subgraphs, one being the complete graph $K_m$ and the rest being (k − 1) copies of the complete graph $K_{m-1}$. Since both $K_m$ and $K_{m-1}$ are random Hamiltonian laceable graphs, $lpd(W_d(k, m)) = 1 + (k-1) = k$.

Hence, the proof.

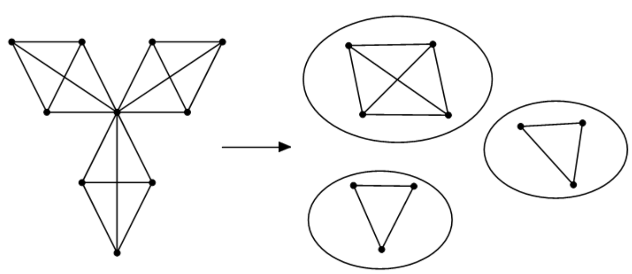

**Fig 5. Windmill graph** $W_d(3, 4)$

**Theorem 3.4**. For Dutch Windmill graph ${D_n}^k$, for $n \geq 3$

$$lpd({D^k}_n) = \begin{cases} n, & for\, n = 3 \\ 0, & for\, n > 3 \end{cases}$$

**Proof**. Consider the graph $G = {D_n}^k$ having$k(n-1) + 1$ vertices.

For $n = 3$, structure is almost same as Windmill graph and hence it almost follows same proof as in Theorem 3.3

For $n > 3$, the only way graph can be partitioned is all partitions having $P_2$'s except one or two partitions having isolated vertex depending on $nk$ being even or odd respectively. Thus making the graph's $lpd = 0$

**Remark 3.4.** For $n \geq 3$, $diam(S(P_n))$ is $n - 1$.

**Theorem 3.5.** For the Shadow graph $S(P_n)$,

$$lpd = \begin{cases} n, & in\, n\, is\, even \\ 0, & otherwise \end{cases}$$

**Proof**. Consider the graph $G = S(P_n)$having $2n$ vertices. The graph $G$ is non-bipartite and is non-random hamiltonian laceable graph.

**Case 1:** when $n$ is even.

For $n \geq 2$, the shadow graph of a path $P_n$ can be partitioned into $n$ number of $K_2$' s making the graph's $lpd = n$.

**Case 2:** when $n$ is odd.

For $n \geq 2$, the shadow graph of a path $P_n$ can be partitioned into $n - 1$ number of $K_2$'s and two isolated vertices, thus making the graph's $lpd = 0$.

**Observation 3.6.** For the Petersen graph, lpd is 5 whose proof follows from the above theorem.

**Theorem 3.7.** For the Shadow distance graph, $D_{sd}(H_{n,n}, 3)$, lpd is 1.

**Proof.** Let $G(U \cup V, E)$ and $G'(U' \cup V', E')$ be two copies of a bipartite graph $H_{n,n}$ with $4n$ vertices. Then $U \cup U'$ and $V \cup V'$ will be the 2 partites of the bipartite graph $D_{sd}(H_{n,n}, 3)$ where $U = \{u_1, u_2 \ldots, u_n\}$,$U' = \{u_1', u_2' \ldots, u_n'\}$, $V = (v_1, v_2 ..., v_n\}$and $V' = (v_1', v_2' ..., v_n'\}$.

We now execute the Hamiltonian path from $u \in U \cup U'$ to $v \in V \cup V'$ in $D_{sd}(H_{n,n}, 3)$ in all possible cases as follows.

**Case 1:** $u = u_i \in U\, and\, v = v_j \in V, 1 \leq i, j, \leq n$

**Subcase (i)**: For $i \neq j$

Starting from $u$, cover all the vertices except $v$ in G by traversing alternatively between sets $U$ and $V$, forming $P_1 : u - u_k$ Hamiltonian path. There exists an edge $(u_k, v_k')$, where $v_k' \in G'$ such that a path $P_2 : v_k' - u_j'$, where all the vertices of $G'$ is covered. Concatenation of the paths $P_1$ and $P_2$ along with the edge $\left(u_j', v_j\right)$ forms the required $u - v$ Hamiltonian path.

**Subcase (ii):** For $i = j$

Starting from $u_i$, reach the vertex $v_i'$ by an incident edge, then cover the vertex which is adjacent to $v_i'$ in $U'$ and reach a vertex in $V$ which is adjacent to the vertex in $U'$. Repeat this process till we reach $u_j'$. There exists a path that covers the remaining vertices of $G$ and $G'$ forming the required $u - v$ Hamiltonian path.

**Case 2:** $u = u_i \in U$ and $v = vj' \in V'$, $1 \leq i, j \leq n$

Starting from $u$, cover all the vertices of $G$ by traversing alternatively between sets $U$ and $V$ and reach the final vertex $v_k$, where $k \neq j$. Then move to $G'$ and cover all vertices and reach the destination vertex $v = vj'$which gives the required $u - v$ Hamiltonian path.

**Case 3:** $u = u_i' \in U'$ and $v = v_j \in V$, $1 \leq i, j \leq n$

Since the graph is vertex-transitive, the proof follows in similar way as in Case 2.

**Case 4:** $u = u_i' \in U'$ and $v = v_j' \in V'$, $1 \leq i, j \leq n$

Since the graph is vertex-transitive, the proof follows in a similar way as in Case 1.

**Remark 3.5.** Diameter of the graph $\bar{L}_n$ is 2.

**Theorem 3.8.** For the Cocktail Party graph $\bar{L}_n$ for $n \geq 2$, $lpd(\bar{L}_n) = 1$.

**Proof**. Consider the graph $G(V, E) = \bar{L}_n$ which is the non-bipartite graph with $2n$ vertices. The result is obvious for $n = 2$. Consider graph $\bar{L}_n$ for$n \geq 3$. Let$M = (m_1, m_2, m_3 \ldots m_i)$ and $N = (n_1, n_2, n_3 \ldots n_j)$, where $i = j$, be the two vertex sets of graphs, $\bar{L}_n$. Let $H(V_1, E_1)$ be a subgraph of $G(V, E)$ such that $V_1$ = V and $E_1 \subset$ E obtained by removing edges which is incident on the vertices of same set. Let $e_i$ be the set of edges which is removed.

Then the subgraph $H$ so obtained is isomorphic to Crown graph. From the theorem (3.2) the lpd of Crown graph is one. This proves that the subgraph $H(V_1, E_1)$ has lpd one. Hence, the graph $G(V, E) = H(V_1, E_1) + e_i$ has lpd one as the connectivity is increased. Hence, the proof.

**Theorem 3.9.** For the image graph, $Img(H_{n,n})$ with $4n$ vertices, lpd is 1.

**Proof**. Let $G(U \cup V, E)$ and $G'(U' \cup V', E')$ be two copies of the bipartite graph $H_{n,n}$. Then $U \cup V'$ and $V \cup U'$ will be the 2 partites of the bipartite graph $Img(H_{n,n})$ where $U = \{u_1, u_2 \ldots, u_n\}$, $U' = \{u_1', u_2' \ldots, u_n'\}$, $V = \{v_1, v_2 \ldots, v_n\}$ and $V' = \{v_1', v_2' \ldots, v_n'\}$.

We now execute the Hamiltonian path from $u \in U \cup V'$ to $v \in V \cup U'$ in $Img(H_{n,n})$ in all possible cases as follows.

**Case 1:** $u = u_i \in U$ and $v = v_j \in V$, $1 \leq i, j \leq n$

Starting from $u$, cover all the vertices except $v$ in $G$ by traversing alternatively between sets $U$ and $V$ and reach $u_k$, where $u_k$ is adjacent to $v$. Then move to the vertex $u_k'$ in $G'$ and cover all vertices and reach $v' \in U'$. From $v'$ finally, reach the destination vertex $v = v_j$ which gives the required *u* − *v* Hamiltonian path.

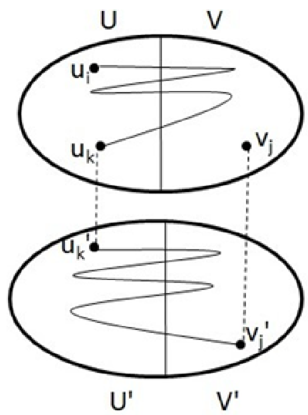


**Fig 6.** $u_i - v_j$ **Hamiltonian path in** $Img(H_{n,n})$

**Case 2:** $u = u_i \in U$ and $v = u_j' \in U'$, $1 \leq i, j \leq n$

**Subcase (i):** $i = j$

Starting from $u$, cover all the vertices of $G$ by traversing alternatively between sets $U$ and $V$ and reach the final vertex $v_k$. Then move to $v_k' \in G'$ and covering all vertices by traversing in reverse order to reach the destination vertex $v = u_j'$ which gives the required $u - v$ Hamiltonian path.

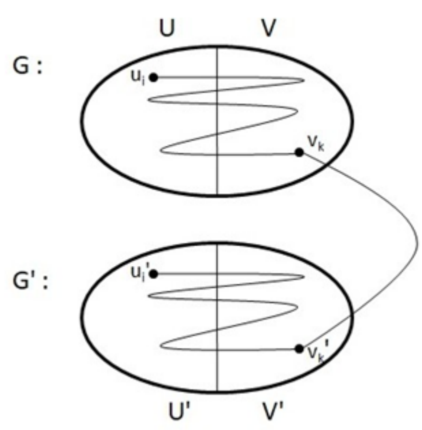


**Fig 7.** $u_i - u_i'$ **Hamiltonian path in** $Img(H_{n,n})$

**Subcase (ii)**: $i \neq j$

Starting from $u$, cover all the vertices of $G$ by traversing alternatively between sets $U$ and $V$ and reach vertex $v_j$. Then move to $v_j^{'} \in G^{'}$ and cover all vertices in $G'$ and reach the destination vertex $v = u_j^{'}$ which gives the required $u - v$ Hamiltonian path.

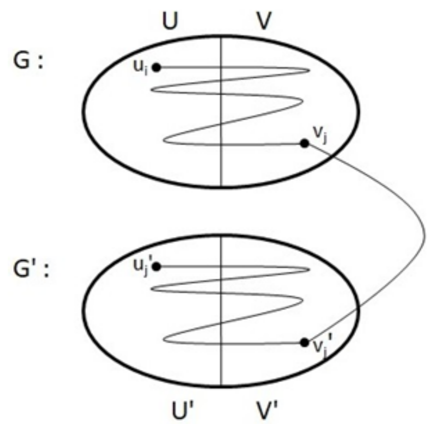


**Fig 8.** $u_i - u_j^{'}$ **Hamiltonian path in** $Img(H_{n,n})$

**Case 3:** $u = v_i' \in V^{'}$ and $v = u_j^{'} \in U^{'}$, $1 \leq i, j \leq n$

Since the graph is vertex-transitive, the proof follows in a similar way as in Case 1.

**Case 4:** $u = v_i^{'} \in V^{'}$ and $v = v_j \in V$, $1 \leq i, j \leq n$

Since the graph is vertex-transitive, the proof follows in a similar way as in Case 2.

## 4 Conclusion

Many research has been done on finding the Hamiltonian laceability and random Hamiltonian laceability of bipartite and non-bipartite graphs respectively. This study has found the laceability partition dimension of graphs that were not laceable. The result could have been improved by algorithmic approach for finding the Hamiltonian paths. This study concludes by posing the following Open Problem: Laceability partition dimension for generalized shadow distance graph $D_{sd}(G, D_s)$.

## References

1) Kuziak D, Maritz E, Vetrik T, Yero IG. The edge partition dimension of graphs. *Discrete Math Lett*. 2023;12:34–39. Available from: https://doi.org/10.47443/dml.2023.010.
2) Shudanxue Q, Deng P, Li. Fault-Tolerant Strongly Hamiltonian Laceability and Hyper-Hamiltonian Laceability of Cayley Graphs Generated by Transposition Trees. *The Computer Journal*. 2023;66(2):384–398. Available from: https://doi.org/10.1093/comjnl/bxab167.
3) Sowmya K, Shenoy LN, AIP Conference Proceedings. Study on Hamiltonian Laceability Properties of Cayley Graphs, . vol. 2516. AIP Publishing. 2022. Available from: https://doi.org/10.1063/5.0108601.
4) Devi MRS, Girisha A. Hamiltonian Laceability in Book Graphs. *Journal of Physics Conference Series*. 2022;(1):2161–2161. Available from: https://doi.org/10.1088/1742-6596/2161/1/012032.
5) Annapoorna MS, Murali R, Manjunath G. Tadpole graphs and their laceability. *International Journal for Innovative Engineer- ing and Management Research*. 2021;p. 1762–176. Available from: https://doi.org/10.48047/ijiemr/v10/i04/44.
6) Nagarathnamma KG, Leena N, Shenoy, Sowmya K. Modified Detour Index of Hamiltonian Connected (Lace- able) Graphs. *Indian Journal of Science and Technology*. 2024;17(19):1923–1934. Available from: https://doi.org/10.17485/IJST/v17i19.1033.
7) Goudar VM, and S. Edge Geodetic Parameters of Some Special Graphs. *International Journal of Applied Engineering Research*. 2019;14. Available from: https://doi.org/10.26637/MJM0603/0027.
8) Mekala A, Kumar R, Murali. Edge domination in some image graphs. *Gedrag and Organisative Review*. 2020;33. Available from: https://lemma-tijdschriften.com/gallery/goj-1580.pdf.