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RESEARCH ARTICLE

Certain Applications of New Caristi Contraction Coupled Solution in S_b-Metric spaces

D Swapna[1]*, **V Nagaraju**[2]

1 Research scholar, Department of Mathematics, University College of science, Osmania University, Telangana, 500007, Hyderabad, India
2 Professor, Department of Mathematics, University College of Science, Osmania University, Telangana, 500007, Hyderabad, India

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* **Corresponding author**.

swapnadharpally@gmail.com

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Abstract

Objective: To initiate some new Caristi-type contractions on complete S_b-metric spaces. **Methods:** Caristi-type contractions of four mappings together with ω-compatibility in S_b- metric space used to derive some common coupled fixed-point theorems. **Findings:** We obtained new common coupled fixed-point results in complete S_b- metric spaces by using Caristi-type contractions which generalize some existing results in the literature and validated them with an appropriate example. **Novelty:** By defining a new Caristi type contraction, we have established a common coupled fixed-point result and have given some applications to integral equations as well as homotopy theory of common coupled fixed point results in S_b- metric spaces.

Keywords: Caristi type mappings; Coupled fixed point; Complete S_b-metric spaces; lower semi continuous mappings; $\omega-$compatible mappings

1 Introduction

The notion of Banach contraction principle was introduced by S. Banach in 1922. It is a fundamental tool in nonlinear analysis and has been extended and generalized in various types of metric spaces by several researchers. One such most useful and important generalization of the contraction principle was given by Caristi. Afterwards this result was improved and extended to various generalized spaces by different authors (see, e.g.,[1–4]).

Recently Sedghi et al. by using the concept of S and b-metric spaces, S_b-metric space was introduced, and established common fixed point results in S_b-metric spaces. Subsequently, many authors established so many results on S_b-metric spaces (see, e.g.,[5–7]).

In 1987, Guo and Lakshmikantham first developed the idea of coupled fixed points. Later, employing a weak contractivity type assumption, Bhaskar and Lakshmikantham developed a novel fixed point theorem for mixed monotone mapping in a metric space driven with partial ordering. For the first time, weak compatibility was introduced in 1998 by Jungck and Rhoades, and they demonstrated that the inverse is not true, even though compatible mappings are weakly compatible. Recently, several types of metric spaces have been developed, and some coupled fixed point results have been studied (see, e.g.,[8–10]).

Nowadays, the fixed point theory has been attracting several mathematics researchers due to its applications in the various fields of mathematics such as differential equations, integral equations, and homotopy theory etc.

Motivated by the work done recently on fixed point theory with different approaches by numerous mathematicians, the present study is aimed at providing coupled common fixed point theorems in the context of S_b-metric spaces by defining and using a new type of contraction mapping of the Caristi type which improves several existing results in the literature. We also gave instances of proper and pertinent applications to homotopy theory and integral equations.

2 Methodology

First, we recall some basic definitions and results.

2.1 Definition[6]:

Given a non-empty set A, let $\vartheta \geq 1$ be a real number. Assume the following conditions are met by a mapping $\varsigma_b : A \to [0, \infty)$:

$$(\varsigma_b 1) \quad 0 < \varsigma_b(\alpha, œ, \beta) \, for \, all \, \alpha, œ, \beta \in A \, with \, \alpha \neq œ \neq \beta,$$

$(\varsigma_b 2) \quad \varsigma_b(\alpha, œ, \beta) = 0 \Leftrightarrow \alpha = œ = \beta,$

$(\varsigma_b 3) \quad \varsigma_b(\alpha, œ, \beta) \leq \vartheta(\varsigma_b(\alpha, \alpha, \theta) + \varsigma_b(œ, œ, \theta) + \varsigma_b(\beta, \beta, \theta))$ for all $\alpha, œ, \beta, \theta \in A$.

The combination (A, ς_b) is referred to as a S_b-metric space, and the function ς_b is called a S_b-metric on A.

2.1.1 Example [6]:

Let $S_a(\alpha, œ, \beta) = S(\alpha, œ, \beta)^{\vartheta}$, where $\tau > 1$ is a real number, and let (A, S) be an S-metric space. Then, S_a with $\vartheta = 2^{2(\tau-1)}$ is a S_b-metric.

2.2. Definition[6]:

Assume the S$_b$-metric space (A, ς_b). In A, consider a sequence $\{\alpha_z\}$.

(1) If there is a $z_0 \in N$ such that $\varsigma_b(\alpha_z, \alpha_z, \alpha_w) < \epsilon$ for any $w, z \geq z_0$, then $\{\alpha_z\}$ is ς_b-Cauchy sequence.

(2) If, for every $\epsilon > 0$, there exists a positive integer z_0 such that $\varsigma_b(\alpha_z, \alpha_z, \alpha) < \epsilon$ or $\varsigma_b(\alpha, , \alpha, \alpha_z) < \epsilon$ for all $z \geq z_0$, then$\{\alpha_z\}$ is ς_b-convergent to a point $\alpha \in A$.

We denote this as $\lim\limits_{z\to\infty} \alpha_z = \alpha$.

(3) If every ς_b-Cauchy sequence is ς_b-convergent in A, then (A, ς_b) is called a complete S_b-metric space.

2.3. Lemma[6]:

In a S$_b$-metric space, we have

$\varsigma_b(\alpha, \alpha, œ) \leq \vartheta\varsigma_b(œ, œ, \alpha)$ and $\varsigma_b(œ, œ, \alpha) \leq \vartheta\varsigma_b(\alpha, \alpha, œ)$.

2.4. Lemma[6]:

In a S$_b$-metric space, we have

$\varsigma_b(\alpha, \alpha, œ) \leq 2\vartheta\varsigma_b(\alpha, \alpha, \beta) + \vartheta^2\varsigma_b(\beta, \beta, œ)$.

2.5 Lemma[6] :

Assuming that $\{\alpha_z\}$ is a ς_b-convergent to α and that (A, ς_b) is a S_b-metric space with $\vartheta \geq 1$. We obtain

(i) $\frac{1}{2\vartheta}\varsigma_b(œ, œ, \alpha) \leq \lim\limits_{z\to\infty} inf \, \varsigma_b(œ, œ, \alpha_z) \leq 2\vartheta\varsigma_b(œ, œ, \alpha)$ and

(ii) $\frac{1}{\vartheta^2}\varsigma_b(\alpha, \alpha, œ) \leq \lim\limits_{z\to\infty} inf \, \varsigma_b(\alpha_z, \alpha_z, œ) \leq \vartheta^2\varsigma_b(\alpha, \alpha, œ)$

for all $œ \in A$. In particular, if $\alpha = œ$, then we have $\lim\limits_{z\to\infty} \varsigma_b(\alpha_z, \alpha_z, œ) = 0$.

2.6. Definition [7]:

Consider the S_b-metric space (A, ς_b). A pair $(X,\ Z)$ is called

(a) coupled fixed point of $\Omega : A^2 \to A$ if $\Omega(X, Z) = X, \Omega(Z, X) = Z$ for $X, Z \in A$;

(b) coupled coincident point of $\Omega : A^2 \to A$ and $\Lambda : A \to A$ if $F(X, Z) = \Lambda X, \quad \Omega(Z, X) = \Lambda Z$;

(c) coupled common points of $\Omega : A^2 \to A$ and $\Lambda : A \to A$ if $\Omega(X, Z) = \Lambda X = X, \quad \Omega(Z, X) = \Lambda Z = Z$;

(d) the pair (Ω, Λ) is weakly compatible if $\Lambda(\Omega(X, Z)) = \Omega(\Lambda X, \Lambda Z)$ and $\Lambda(\Omega(Z, X)) = \Omega(\Lambda Z, \Lambda X)$ whenever for all $X, Z \in I$ such that $\Omega(X, Z) = \Lambda X, \quad \Omega(Z, X) = \Lambda Z$.

3 Results and Discussion

We derive a coupled fixed-point theorem with the help of a new Caristi-type contraction in this part.

3.1 Theorem:

Consider a complete S_b-metric space (A, ς_b) with coefficient $\vartheta > 1$. Let there be four mappings, $\Gamma, \Omega : A^2 \to A, \Lambda, \Phi : A \to A$ with

$$\begin{aligned}&2\vartheta^4 \varsigma_b(\Gamma(\alpha, œ), \Gamma(\alpha, œ), \Omega(\beta, \varrho)) \leq \psi(\varkappa(\Lambda\alpha))\varkappa(\Lambda\alpha) - \varkappa(\Gamma(\alpha, œ)) + \psi(\wp(\Lambda œ))\wp(\Lambda œ) - \wp(\Gamma(œ, \alpha)) \\ &+\psi(\gamma(\Phi\beta))\gamma(\Phi\beta) - \gamma(\Omega(\beta, \varrho)) + \psi(\chi(\Phi\varrho))\chi(\Phi\varrho) - \chi(\Omega(\varrho, \beta))\end{aligned} \tag{1}$$

for all $\alpha, œ, \beta, \varrho \in A$ and $\varkappa, \wp, \gamma, \chi : A \to [0,\ \infty)$.

are lower semi-continuous functions and, $\psi : (-\infty, +\infty) \to (0, 1)$ be continuous function and satisfying

(i_0) $\Gamma(A^2) \subseteq \Phi(A)$ and $\Omega(A^2) \subseteq \Lambda(A)$,

(i_1) (Γ, Λ) and (Ω, Φ) are ω-compatible,

(i_2) either $\Lambda(A)$ or $\Phi(A)$ is a complete subspace of A.

Then, in A, there exists a unique common coupled fixed point of Γ, Ω, Λ, and Φ.

Proof: Let $\beta_0, \zeta_0 \in A$ be arbitrary, and from (i_0) we construct the sequences $\{\beta_z\}, \{\zeta_z\}, \{\alpha_z\}$ and $\{œ_z\}$, in A as

$$\Gamma(\beta_{2z}, \zeta_{2z}) = \Phi\beta_{2z+1} = \alpha_{2z}, \quad \Gamma(\zeta_{2z}, \beta_{2z}) = \Phi\zeta_{2z+1} = œ_{2z}$$

$$\Omega(\beta_{2z+1}, \zeta_{2z+1}) = \Lambda\beta_{2z+2} = \alpha_{2z+1}, \quad \Omega(\zeta_{2z+1}, \beta_{2z+1}) = \Lambda\zeta_{2z+2} = œ_{2z+1}, \quad where \quad z = 0, 1, 2, \ldots\ .$$

In A, we now demonstrate that, Γ, Ω, Λ, and Φ have a CCFP(common coupled fixed point). For every z, let us assume that $\varsigma_b(\alpha_{2z}, \alpha_{2z}, \alpha_{2z+1}) > 0$ and $\varsigma_b(œ_{2z}, œ_{2z}, œ_{2z+1}) > 0$. In the absence of this, there is some positive integer z such that $\alpha_{2z} = \alpha_{2z+1}$, $œ_{2z} = œ_{2z+1}$, and so on. The proof is finished, and $(\alpha_{2z}, œ_{2z})$ is a coupled fixed point of $\Gamma, \Omega, \Lambda, \Phi$.

Using Equation (1), we obtain for each $z \in N$

$$\begin{aligned}&0 < \varsigma_b(\alpha_{2z+2}, \alpha_{2z+2}, \alpha_{2z+1}) \leq 2\vartheta^4 \varsigma_b(\Gamma(\beta_{2z+2}, \zeta_{2z+2}), \Gamma(\beta_{2z+2}, \zeta_{2z+2}), \Omega(\beta_{2z+1}, \zeta_{2z+1})) \\ &\leq \psi(\varkappa(\Lambda\beta_{2z+2}))\varkappa(\Lambda\beta_{2z+2}) - \varkappa(\Gamma(\beta_{2z+2}, \zeta_{2z+2})) \\ &+\psi(\wp(\Lambda\zeta_{2z+2}))\wp(\Lambda\zeta_{2z+2}) - \wp(\Gamma(\zeta_{2z+2}, \beta_{2z+2})) \\ &+\psi(\gamma(\Phi\beta_{2z+1}))\gamma(\Phi\beta_{2z+1}) - \gamma(\Omega(\beta_{2z+1}, \zeta_{2z+1})) \\ &+\psi(\chi(\Phi\zeta_{2z+1}))\chi(\Phi\zeta_{2z+1}) - \chi(\Omega(\zeta_{2z+1}, \beta_{2z+1})) \\ &\leq \psi(\varkappa(\alpha_{2z+1}))\varkappa(\alpha_{2z+1}) - \varkappa(\alpha_{2z+2}) \\ &+\psi(\wp(œ2z+1))\wp(œ_{2z+1}) - \wp(œ_{2z+2}) \\ &+\psi(\gamma(\alpha_{2z}))\gamma(\alpha_{2z}) - \gamma(\alpha_{2z+1}) \\ &+\psi(\chi(œ_{2z}))\chi(œ_{2z}) - \chi(œ_{2z+1}) \\ &< \varkappa(\alpha_{2z+1}) - \varkappa(\alpha_{2z+2}) + \wp(œ_{2z+1}) - \wp(œ_{2z+2})\end{aligned}$$

$$+\gamma(\alpha_{2z}) - \gamma(\alpha_{2z+1}) + \chi(œ_{2z}) - \chi(œ_{2z+1}) \tag{2}$$

By similar arguments, we obtain

$$\begin{aligned}&\varsigma_b(œ_{2z+2}, œ_{2z+2}, œ_{z+1}) < \varkappa(\alpha_{2z+1}) - \varkappa(\alpha_{2z+2}) + \wp(œ_{2z+1}) - \wp(œ_{2z+2}) \\ &+\gamma(\alpha_{2z}) - \gamma(\alpha_{2z+1}) + \chi(œ_{2z}) - \chi(œ_{2z+1}).\end{aligned} \tag{3}$$

Combining Equation (2) and Equation (3), we can get

$$max\left\{\left\{\begin{array}{l}\varsigma_b(\alpha_{2z+2},\alpha_{2z+2},\alpha_{2z+1}),\\ \varsigma_b(œ_{2z+2},œ_{2z+2},œ2z+1)\end{array}\right.\right\}< X(\alpha_{2z+2})-X(\alpha_{2z+2})+ \\ -\wp(œ_{2z+2})+\gamma(\alpha_{2z})- \\ \gamma(\alpha_{2z+1})+X(œ_{2z})-X(œ_{2z}+1). \tag{4}$$

and

$$(\varkappa(\alpha_{2z+2})+\wp(œ_{2z+2}))+(\gamma(\alpha_{2z+1})+\chi(œ_{2z+1}))<(\varkappa(\alpha_{2z+1})+\wp(œ_{2z+1}))+(\gamma(\alpha_{2z})+\chi(œ_{2z})) \tag{5}$$

It can be seen from this, the sequences $(\{\varkappa(\alpha_{2z})\},\{\wp(œ_{2z})\})$ and $(\{\gamma(\alpha_{2z})\},\{\chi(œ_{2z})\})$ of non-negative real numbers that are not increasing. As a result, they have to converge to $w_1,w_2,w_3,w_4\geq 0$ in turn.

Let $w_1>0$, $w_2>0$, $w_3>0$, or $w_4>0$. We obtain contradiction by letting $z\to\infty$ in Equation (5). Thus,

$$\lim_{z\to\infty}\varkappa(\alpha_{2z})=0=\lim_{z\to\infty}\wp(œ_{2z})\ and\ \lim_{z\to\infty}\gamma(\alpha_{2z})=0=\lim_{z\to\infty}\chi(œ_{2z}) \tag{6}$$

For every two positive integers z and w with $w>z$, using Equations (3) and (4),

$$\begin{aligned}&\varsigma_b(\alpha_{2z},\alpha_{2z},\alpha_{2w})\leq\vartheta(\varsigma_b(\alpha_{2z},\alpha_{2z},\alpha_{2z+1})+\varsigma_b(\alpha_{2z},\alpha_{2z},\alpha_{2z+1})+\varsigma_b(\alpha_{2w},\alpha_{2w},\alpha_{2z+1}))\\&\leq 2\vartheta\varsigma_b(\alpha_{2z},\alpha_{2z},\alpha_{2z+1})+\vartheta^2\varsigma_b(\alpha_{2z+1},\alpha_{2z+1},\alpha_{2w})\\&\leq 2\vartheta\varsigma_b(\alpha_{2z},\alpha_{2z},\alpha_{2z+1})+2\vartheta^3\varsigma_b(\alpha_{2z+1},\alpha_{2z+1},\alpha_{2z+2})+\vartheta^4\varsigma_b(\alpha_{2z+2},\alpha_{2z+2},\alpha_{2w})\\&\leq 2\vartheta\varsigma_b(\alpha_{2z},\alpha_{2z},\alpha_{2z+1})+2\vartheta^3\varsigma_b(\alpha_{2z+1},\alpha_{2z+1},\alpha_{2z+2})\\&+\cdots+\vartheta^{2q-1}\varsigma_b(\alpha_{2w-1},\alpha_{2w-1},\alpha_{2w})\\&<2\vartheta\begin{pmatrix}\varkappa(\alpha_{2z-1})-\varkappa(\alpha_{2z})+\wp(œ_{2z-1})-\wp(œ_{2z})\\+\gamma(\alpha_{2z})-\gamma(\alpha_{2z+1})+\chi(œ_{2z})-\chi(œ_{2z+1})\end{pmatrix}\\&+2\vartheta^3\begin{pmatrix}\varkappa(\alpha_{2z})-\varkappa(\alpha_{2z+1})+\wp(œ_{2z})-\wp(œ_{2z+1})\\+\gamma(\alpha_{2z+1})-\gamma(\alpha_{2z+2})+\chi(œ_{2z+1})-\chi(œ_{2z+2})\end{pmatrix}+\cdots\cdots+\vartheta^{2q-1}\begin{pmatrix}\varkappa(\alpha_{2w-2})-\varkappa(\alpha_{2w-1})+\wp(œ_{2w-2})-\wp(œ_{2w-1})\\+\gamma(\alpha_{2w-1})-\gamma(\alpha_{2w})+\chi(œ_{2w-1})-\chi(œ_{2w})\end{pmatrix}\end{aligned}$$

as $z,w\to\infty$, and from Equation (6) we have $\lim_{z,w\to\infty}\varsigma_b(\alpha_{2z},\alpha_{2z},\alpha_{2w})=0$.

Similarly, we can prove $\lim_{z,w\to\infty}\varsigma_b(œ_{2z},œ_{2z},œ_{2w})=0$. Hence, $\{\alpha_{2z}\}$ and $\{œ_{2z}\}$ are ς_b-Cauchy sequences in complete ς_b-metric spaces (A,ς_b). Suppose $\Lambda(A)$ is a complete subspace of (A,ς_b), so that $\{\alpha_{2z}\}$, $\{œ_{2z}\}\subseteq\Lambda(A)$. Then the sequences $\{\alpha_{2z}\}$, $\{œ_{2z}\}$ are convergent to $\alpha,œ$ respectively in $\Lambda(A)$ with

$$\lim_{z\to\infty}\alpha_{2z+1}=\alpha\quad\lim_{z\to\infty}œ_{2z+1}=œ \tag{7}$$

Since $\Lambda:A\to A$ and $\alpha,œ\in\Lambda(A)$, there exist $l,\beta\in A$ such that $\Lambda l=\alpha$ and $\Lambda\beta=œ$. Since $\varkappa,\wp,\gamma,\chi$ are lower semi-continuous functions, hence

$$\lim_{z\to\infty}\varkappa(\alpha_{2z+1})=\varkappa(\alpha),\quad\lim_{z\to\infty}\wp(œ_{2z+1})=\wp(œ),\quad\lim_{z\to\infty}\gamma(\alpha_{2z})=\gamma(\alpha),\quad\lim_{z\to\infty}\chi(œ_{2z})=\chi(œ).$$

From Equation (6), we get $\varkappa(\alpha)=\wp(œ)=\gamma(\alpha)=\chi(œ)=0$. We claim that $\Gamma(l,\beta)=\alpha$ and $\Gamma(\beta,l)=œ$. Suppose $\Gamma(l,\beta)\neq\alpha$ and $\Gamma(\beta,l)\neq œ$ by Lemma (2.5), we have that

$$\frac{1}{2\vartheta}\varsigma_b(\Gamma(l,\beta),\Gamma(l,\beta),\alpha)\quad\leq\lim_{z\to\infty}inf\varsigma_b(\Gamma(l,\beta),\Gamma(l,\beta),\alpha_p).$$

Now from Equation (1), we have

$$\begin{aligned}&\vartheta^3\varsigma_b(\Gamma(l,\beta),\Gamma(l,\beta),\alpha)\leq\lim_{z\to\infty}inf2\vartheta^4\varsigma_b(\Gamma(l,\beta),\Gamma(l,\beta),\alpha_{2z})\\&\leq\lim_{z\to\infty}inf\begin{pmatrix}\psi(\varkappa(\Lambda l))\varkappa(\Lambda l)-\varkappa(\Gamma(l,\beta))+\psi(\wp(\Lambda\beta))\wp(\Lambda\beta)-\wp(\Gamma(\beta,l))\\+\psi(\gamma(\Phi\beta_{2z}))\gamma(\Phi\beta_{2z})-\gamma(\Omega(\beta_{2z},\zeta_{2z}))\\+\psi(\chi(\Phi\zeta_{2z}))\chi(\Phi\zeta_{2z})-\chi(\Omega(\zeta_{2z},\beta_{2z}))\end{pmatrix}\\&\leq\lim_{z\to\infty}inf\begin{pmatrix}\psi(\varkappa(\Lambda l))\varkappa(\Lambda l)-\varkappa(\Gamma(l,\beta))+\psi(\wp(\Lambda\beta))\wp(\Lambda\beta)-\wp(\Gamma(\beta,l))\\+\psi(\gamma(\alpha_{2z-1}))\gamma(\alpha_{2z-1})-\gamma(\alpha_{2z})\\+\psi(\chi(œ_{2z-1}))\chi(œ_{2z-1})-\chi(œ_{2z})\end{pmatrix}\\&<\varkappa(\Lambda l)-\varkappa(\Gamma(l,\beta))+\wp(\Lambda\beta)-\wp(\Gamma(\beta,l))\\&<\varkappa(\Lambda l)-\wp(\Lambda\beta)=0.\end{aligned}$$

Consequently, $\varsigma_b(\Gamma(l,\beta),\Gamma(l,\beta),\alpha)=0$ implies $\Gamma(l,\beta)=\alpha=\Lambda l$. We obtain $\Gamma(\beta,l)=\text{œ}=\Lambda\beta$ in the same manner. We have $\Gamma(\alpha,\text{œ})=\Lambda\alpha$ and $\Gamma(\text{œ},\alpha)=\Lambda\text{œ}$ since $\{\Gamma,\Lambda\}$ is a weakly compatible pair. We now establish that $\Lambda\text{œ}=\text{œ}$ and $\Lambda\alpha=u\alpha$. We know from Lemma (2.5) that

$$\frac{1}{2\vartheta}\varsigma_b(\Lambda\alpha,\Lambda\alpha,\alpha)\le \lim_{z\to\infty} inf\, \varsigma_b(\Lambda\alpha,\Lambda\alpha,\alpha_{2z}).$$

Now from Equation (1), we have

$$\begin{aligned}
&\vartheta^3\varsigma_b(\Lambda\alpha,\Lambda\alpha,\alpha)\le \lim_{z\to\infty} inf 2\vartheta^4\varsigma_b(\Gamma(\alpha,\text{œ}),\Gamma(\alpha,\text{œ}),\Omega(\beta_{2z},\zeta_{2z}))\\
&\le \lim_{z\to\infty} inf \begin{pmatrix}\psi(\varkappa(\Lambda\alpha))\varkappa(\Lambda\alpha)-\varkappa(\Gamma(\alpha,\text{œ}))+\psi(\wp(\Lambda\text{œ}))\wp(\Lambda\text{œ})-\wp(\Gamma(\text{œ},\alpha))\\ +\psi(\gamma(\Phi\beta_{2z}))\gamma(\Phi\beta_{2z})-\gamma(\Omega(\beta_{2z},\zeta_{2z}))\\ +\psi(\chi(\Phi\zeta_{2z}))\chi(\Phi\zeta_{2z})-\chi(\Omega(\zeta_{2z},\beta_{2z}))\end{pmatrix}\\
&\le \lim_{z\to\infty} inf \begin{pmatrix}\psi(\varkappa(\Lambda\alpha))\varkappa(\Lambda\alpha)-\varkappa(\Gamma(\alpha,\text{œ}))+\psi(\wp(\Lambda\text{œ}))\wp(\Lambda\text{œ})-\wp(\Gamma(\text{œ},\alpha))\\ +\psi(\gamma(\alpha_{2z-1}))\gamma(\alpha_{2z-1})-\gamma(\alpha_{2z})\\ +\psi(\chi(\text{œ}_{2z-1}))\chi(\text{œ}_{2z-1})-\chi(\text{œ}_{2z})\end{pmatrix}\\
&<\varkappa(\Lambda\alpha)-\varkappa(\Gamma(\alpha,\text{œ}))+\wp(\Lambda\text{œ})-\wp(\Gamma(\text{œ},\alpha))=0.
\end{aligned}$$

In this way, we may demonstrate that $\Lambda\text{œ}=\text{œ}$ and that $\Lambda\alpha=\alpha$. Consequently, $\Gamma(\alpha,\text{œ})=\Lambda\alpha=\alpha$ and $\Gamma(\text{œ},\alpha)=\Lambda\text{œ}=\text{œ}$ follow. As a result, $(\alpha,\text{œ})$ is Λ and Γ's CCFP(common coupled fixed point). There exist $\mathrm{X},Z\in A$ such that $\Gamma(\alpha,\text{œ})=\alpha=\Phi\mathrm{X}$ and $\Gamma(\text{œ},\alpha)=\text{œ}=\Phi Z$ because $\Gamma(A^2)\subseteq\Phi(A)$.

Now consider

$$\begin{aligned}
&\varsigma_b(\alpha,\alpha,\Omega(\mathrm{X},Z))=\varsigma_b(\Gamma(\alpha,\text{œ}),\Gamma(\alpha,\text{œ}),\Omega(\mathrm{X},Z))\\
&\le 2\vartheta^4\varsigma_b(\Gamma(\alpha,\text{œ}),\Gamma(\alpha,\text{œ}),\Omega(\mathrm{X},Z))\\
&\le \psi(\varkappa(\Lambda\alpha))\varkappa(\Lambda\alpha)-\varkappa(\Gamma(\alpha,\text{œ}))+\psi(\wp(\Lambda\text{œ}))\wp(\Lambda\text{œ})-\wp(\Gamma(\text{œ},\alpha))\\
&+\psi(\gamma(\Phi\mathrm{X}))\gamma(\Phi\mathrm{X})-\gamma(\Omega(\mathrm{X},Z))+\psi(\chi(\Phi Z))\chi(\Phi Z)-\chi(\Omega(Z,\mathrm{X}))\\
&<\varkappa(\Lambda\alpha)-\varkappa(\Gamma(\alpha,\text{œ}))+\wp(\Lambda\text{œ})-\wp(\Gamma(\text{œ},\alpha))\\
&+\gamma(\Phi\mathrm{X})-\gamma(\Omega(\mathrm{X},Z))+\chi(\Phi Z)-\chi(\Omega(Z,\mathrm{X}))\\
&<\gamma(\alpha)+\chi(\text{œ})=0.
\end{aligned}$$

$\varsigma_b(\alpha,\alpha,\Omega(\mathrm{X},Z))=0$ implies $\Omega(\mathrm{X},Z)=\alpha$ as a result. In a similar way, we obtain $\Omega(Z,\mathrm{X})=\text{œ}$. We have $\Omega(\alpha,\text{œ})=\Phi\alpha$ and $\Omega(\text{œ},\alpha)=\Phi\text{œ}$ since $\{\Omega,\Phi\}$ is a weakly compatible pair. The proofs of $\Phi\alpha=\alpha$ and $\Phi\text{œ}=\text{œ}$ are now given. We know from Lemma (2.5) that

$$\frac{1}{\vartheta^2}\varsigma_b(\alpha,\alpha,\Phi\alpha)\le \lim_{z\to\infty} inf\, \varsigma_b(\alpha_{2z},\alpha_{2z},\Phi\alpha)$$

Now from Equation (1), we have

$$\begin{aligned}
&2\vartheta^2\varsigma_b(\alpha,\alpha,\Phi\alpha)\le \lim_{z\to\infty} inf\, 2\vartheta^4\varsigma_b(\Gamma(\beta_{2z},\zeta_{2z}),\Gamma(\beta_{2z},\zeta_{2z}),\Omega(\alpha,\text{œ}))\\
&\le \lim_{z\to\infty} inf \begin{pmatrix}\psi(\varkappa(\Lambda\beta_{2z}))\varkappa(\Lambda\beta_{2z})-\varkappa(\Gamma(\beta_{2z},\zeta_{2z}))\\ +\psi(\wp(\Lambda\zeta_{2z}))\wp(\Lambda\zeta_{2z})-\wp(\Gamma(\zeta_{2z},\beta_{2z}))\\ +\psi(\gamma(\Phi\alpha))\gamma(\Phi\alpha)-\gamma(\Omega(\alpha,\text{œ}))+\psi(\chi(\Phi\text{œ}))\chi(\Phi\text{œ})-\chi(\Omega(\text{œ},\alpha))\end{pmatrix}\\
&\le \lim_{z\to\infty} inf \begin{pmatrix}\psi(\varkappa(\alpha_{2z-1}))\varkappa(\alpha_{2z-1})-\varkappa(\alpha_{2z})+\psi(\wp(\text{œ}_{2z-1}))\wp(\text{œ}_{2z-1})-\wp(\text{œ}_{2z})\\ +\psi(\gamma(\Phi\alpha))\gamma(\Phi\alpha)-\gamma(\Omega(\alpha,\text{œ}))+\psi(\chi(\Phi\text{œ}))\chi(\Phi\text{œ})-\chi(\Omega(\text{œ},\alpha))\end{pmatrix}\\
&<\gamma(\Phi\alpha)-\gamma(\Omega(\alpha,\text{œ}))+\chi(\Phi\text{œ})-\chi(\Omega(\text{œ},\alpha))=0.
\end{aligned}$$

$\Phi\alpha=\alpha$ is thus implied by $\varsigma_b(\alpha,\alpha,\Phi\alpha)=0$. Thus, $\Omega(\alpha,\text{œ})=\Phi\alpha=\alpha$, $\Omega(\text{œ},\alpha)=\Phi\text{œ}=\text{œ}$, and $\Phi\text{œ}=\text{œ}$ are obtained in a similar manner. Hence, $(\alpha,\text{œ})$ is the CCFP(coupled common fixed point) for Φ,Γ,Ω, and Λ. We will demonstrate the common coupled fixed point's uniqueness in A in the section that follow. Assume that Γ,Ω,Λ, and Φ have another coupled fixed point $(\alpha',\text{œ}')$ for this reason. Next, we have from Equation (1),

$$\begin{aligned}
&\varsigma_b\left(\alpha,\alpha,\alpha'\right)\le 2\vartheta^4\varsigma_b\left(\alpha,\alpha,\alpha'\right)=2\vartheta^4\varsigma_b\left(\Gamma(\alpha,\text{œ}),\Gamma(\alpha,\text{œ}),\Omega(\alpha',\text{œ}')\right)\\
&\le\psi(\varkappa(\Lambda\alpha))\varkappa(\Lambda\alpha)-\varkappa(\Gamma(\alpha,\text{œ}))+\psi(\wp(\Lambda\text{œ}))\wp(\Lambda\text{œ})-\wp(\Gamma(\text{œ},\alpha))\\
&+\psi\left(\gamma(\Phi\alpha')\right)\gamma(\Phi\alpha')-\gamma\left(\Omega(\alpha',\text{œ}')\right)+\psi\left(\chi(\Phi\text{œ}')\right)\chi(\Phi\text{œ}')-\chi\left(\Omega(\text{œ}',\alpha')\right)\\
&<\gamma(\Phi\alpha)-\gamma(\Omega(\alpha,\text{œ}))+\chi(\Phi\text{œ})-\chi(\Omega(\text{œ},\alpha))\\
&+\gamma(\Phi\alpha')-\gamma\left(\Omega(\alpha',\text{œ}')\right)+\chi(\Phi\text{œ}')-\chi\left(\Omega(\text{œ}',\alpha')\right)\\
&=0.
\end{aligned}$$

Consequently, $\varsigma_b\left(\alpha,\alpha,\alpha'\right)=0$. Similarly, we obtain $\varsigma_b\left(\text{œ},\text{œ},\text{œ}'\right)=0$ which indicates $\alpha=\alpha'$.

Specifically, $\text{œ}=\text{œ}'$. Hence, there is only one UCCFP (unique common coupled fixed point)by Γ,Ω,Λ, and Φ.

$$\varsigma_b(\alpha,\alpha,\text{œ})\le 2\vartheta^4\varsigma_b(\alpha,\alpha,\text{œ})=2\vartheta^4\varsigma_b(\Gamma(\alpha,\text{œ}),\Gamma(\alpha,\text{œ}),\Omega(\text{œ},\alpha))$$

$\leq \psi(\varkappa(\Lambda\alpha))\varkappa(\Lambda\alpha) - \varkappa(\Gamma(\alpha, œ)) + \psi(\wp(\Lambda œ))\wp(\Lambda œ) - \wp(\Gamma(œ, \alpha))$
$+\psi(\gamma(\Phi œ))\gamma(\Phi œ) - \gamma(\Omega(œ, \alpha)) + \psi(\chi(\Phi\alpha))\chi(\Phi\alpha) - \chi(\Omega(\alpha, œ))$
$< x(\wedge\alpha) - x(\Gamma(\alpha, œ)) + \wp(\wedge œ) - \wp(\Gamma(œ, \alpha))$
$+\gamma(\Phi\alpha') - \gamma(\Omega(œ, \alpha)) + \chi(\Phi œ') - \chi\left(\Omega(œ', \alpha')\right) = 0.$
Consequently, $\varsigma_b(\alpha, \alpha, œ) = 0$ and so the unique common fixed point of Γ, Ω, Λ, and Φ is of the type (α, α).

3.2 Corollary:

Consider a complete S_b-metric space(A, ς_b) with coefficient $\vartheta > 1$. Given two mappings, $\Lambda : A \to A$ and $\Gamma : A^2 \to A$, such that
$2\vartheta^4\varsigma_b(\Gamma(\alpha, œ), \Gamma(\alpha, œ), \Gamma(l, \beta)) \leq \psi(\varkappa(\Lambda\alpha))\varkappa(\Lambda\alpha) - \varkappa(\Gamma(\alpha, œ)) + \psi(\wp(\Lambda œ))\wp(\Lambda œ) - \wp(\Gamma(œ, \alpha))$
$+\psi(\gamma(\Lambda l))\gamma(\Lambda l) - \gamma(\Gamma(l, \beta)) + \psi(\chi(\Lambda\beta))\chi(\Lambda\beta) - \chi(\Gamma(\beta, l))$
for all $\alpha, œ, l, \beta \in A$ and $\varkappa, \wp, \gamma, \chi : A \to [0, \infty)$re lower semi-continuous functions and, $\psi : (-\infty, +\infty) \to (0, 1)$ is a continuous function,
(i_0) $\Gamma(A^2) \subseteq \Lambda(A)$,
(i_1) pair (Γ, Λ) is ω-compatible,
(i_2) $\Lambda(A)$ is a complete subspace of A.
Then, in A, there is a unique common coupled fixed point for Γ and Λ.

3.3 Corollary:

Consider a complete S_b-metric space (A, ς_b) with coefficient $\vartheta > 1$. Given two mappings, $\Lambda : A \to A$ and $\Gamma : A^2 \to A$, such that
$2\vartheta^4\varsigma_b(\Gamma(\alpha, œ), \Gamma(\alpha, œ), \Gamma(l, \beta)) \leq \varkappa(\psi(\Lambda\alpha, \Lambda l))\psi(\Lambda\alpha, \Lambda l) - \psi(\Gamma(\alpha, œ), \Gamma(l, \beta))$
$+\varkappa(\phi(\Lambda œ, \Lambda\beta))\phi(\Lambda œ, \Lambda\beta) - \psi(\Gamma(œ, \alpha), \Gamma(\beta, l))$
for all $\alpha, œ, l, \beta \in A$ and $\psi, \phi : A \times A \to [0, \infty)$are lower semi-continuous functions and,
$\varkappa : (-\infty, +\infty) \to (0, 1)$ is a continuous function,
(i_0) $\Gamma(A^2) \subseteq \Lambda(A)$,
(i_1) pair (Γ, Λ) is ω-compatible,
(i_2) either $\Lambda(A)$ is a complete subspace of A.
Then, in A, there is a UCCFP(unique common coupled fixed point) for Γ and Λ.

3.4 Corollary :

Consider a complete S_b-metric space $(A, \varsigma_b)$with coefficient $\vartheta > 1$. Assume that there exists a mapping $\Gamma : A^2 \to A$ such that
$\varsigma_b(\Gamma(\alpha, œ), \Gamma(\alpha, œ), \Gamma(l, \beta)) \leq \varkappa(\psi(\alpha, l))\psi(\alpha, l) + \varkappa(\phi(œ, \beta))\phi(œ, \beta)$
for all $\alpha, œ, l, \beta \in A$ and $\psi, \phi : A \times A \to [0, \infty)$ are lower semi-continuous functions and,
$\varkappa : (-\infty, +\infty) \to (0, 1)$ is a continuous function. Then there is a unique coupled fixed point of Γ in A.

3.5 Example:

Let$\varsigma_b : A^3 \to [0, \infty)$ be a mapping defined as
$\varsigma_b(\alpha, œ, \beta) = ((œ + \beta - 2\alpha| + (œ - \beta|)^2$.
So clearly (A, ς_b) is complete S_b-metric space with $\vartheta = 2$. Define $\Gamma, \Omega : A^2 \to A$ and
$\Lambda, \Phi : A \to A$ by $\Gamma(l, \beta) = \frac{l^2+\beta^2}{2^5}$, $\Omega(l, \beta) = \frac{l^2+\beta^2}{2^7}$ and $\Lambda l = \frac{l^2}{2}$, $\Phi l = \frac{l^2}{4}$.
Let $\psi : (-\infty, +\infty) \to (0, 1)$ as $\psi(z) < z$. Define four lower semi-continuous functions
$\varkappa, \wp, \gamma, \chi : A \to (0, \infty)$ as $\varkappa(l) = l$, $\wp(l) = 2l$, $\gamma(l) = 4l$ and $\chi(l) = 8l$. Then clearly the $(i_0), (i_1), (i_2)$ conditions of Theorem 3.1 are fulfilled. Now for any $\imath, \jmath, ð, Ʊ \in A$

$$L.H.S = 2\vartheta^4(\varsigma_b(\Gamma(l, \jmath), \Gamma(l, \jmath), \Omega(ð, Ʊ))$$

$2\vartheta^4((\Gamma(\imath, \jmath) + \Omega(ð, Ʊ) - 2\Gamma(\imath, \jmath)| + (\Gamma(\imath, \jmath) - \Omega(ð, Ʊ)|)^2$
$= 2\vartheta^4\left(2\left|\frac{\imath^2+\jmath^2}{2^5} - \frac{ð^2+Ʊ^2}{2^7}\right|\right)^2 = \frac{1}{2^7}((4\imath^2 + 4\jmath^2 - ð^2 - Ʊ^2|)^2$

$$R.H.S \;=\; \psi(x(\wedge_l))x(\wedge_l) - x(\Gamma(l,j)) + \psi(\wp(\wedge_1))\,\wp(\wedge j) - \wp(\Gamma(j,l))$$

$+\psi(\gamma(\Phi\eth))\gamma(\Phi\eth) - \gamma(\Omega(\eth,\mho)) + \psi(\chi(\Phi\mho))\chi(\Phi\mho) - \chi(\Omega(\mho,\eth))$
$= \psi\left(\varkappa\left(\frac{\imath^2}{2}\right)\right)\varkappa\left(\frac{\imath^2}{2}\right) - \varkappa\left(\frac{\imath^2+\jmath^2}{2^5}\right) + \psi\left(\wp(\frac{\jmath^2}{2})\right)\wp(\frac{\jmath^2}{2}) - \wp\left(\frac{\jmath^2+\imath^2}{2^5}\right)$
$+\psi\left(\gamma(\frac{\eth^2}{4})\right)\gamma(\frac{\eth^2}{4}) - \gamma\left(\frac{\eth^2+\mho^2}{2^7}\right) + \psi\left(\chi(\frac{\mho^2}{4})\right)\chi(\frac{\mho^2}{4}) - \chi\left(\frac{\mho^2+\eth^2}{2^7}\right)$
$< \frac{\imath^4}{4} - \frac{\imath^2+\jmath^2}{2^5} + \jmath^4 - \frac{\jmath^2+\imath^2}{2^4} + \eth^4 - \frac{\eth^2+\mho^2}{2^5} + 4\mho^4 - \frac{\mho^2+\eth^2}{2^4}$
$= \frac{1}{2^5}(8\imath^4 + 32\jmath^4 + 32\eth^4 + 128\mho^4 - 3\imath^2 - 3\jmath^2 - 3\eth^2 - 3\mho^2)$

For all $\imath,\jmath,\eth,\mho \in$, $L.H.S < R.H.S$. As a result, Γ,Ω,Λ, and Φ have a unique common coupled fixed point $(0,0)$, since the requirement of Theorem (3.1) holds.

3.6. Applications

As an application to Corollary 3.4, we examine the existence of a unique solution to an initial value problem in this section.

Consider the initial value problem

$$\frac{d\alpha}{dt} = \Gamma(t,\alpha(t),\alpha(t)), \quad t \in I = [0,1], \quad \alpha(0) = \alpha_0 \tag{8}$$

where $\Gamma : I \times R^2 \to R$ and $\chi_0 \in R$.

3.6.1. Theorem

Consider the initial value problem in Equation (8) with $\Gamma \in C(I \times \mathbb{R}^2)$

$$\int_0^t \Gamma(s,\, \alpha(s),\, \zeta(s))ds = \frac{1}{5} max \begin{cases} \int_0^t \Gamma(s,\, \alpha(s),\, \alpha(s))ds, \\ \int_0^t \Gamma(s,\, \zeta(s),\, \zeta(s))ds \end{cases}$$

Then the initial value problem in Equation (8) has a unique solution in $C\left(I \times \mathbb{R}^2\right)$.

Proof: In the case of the initial value problem (Equation (8)), the integral equation is

$\alpha(t) = \alpha_0 + \int\limits_0^t \Gamma(s,\alpha(s),\alpha(s))ds.$

Let $A = C\left(I \times R^2\right)$ and let $\varsigma_b\left(\alpha,\text{œ},\beta\right) = |\alpha - \beta| + |\text{œ} - \beta|$ for $\alpha,\text{œ},\beta \in A$. Then (A,ς_b) is a complete S_b-metric space with $\vartheta = 2$.

Define $\psi,\phi : A \times A \to [0,\infty)$ by $\psi(\alpha,\text{œ}) = |\alpha - \text{œ}|$ and $\phi(\alpha,\text{œ}) = 2\,|\alpha - \text{œ}|$ and $\varkappa : (-\infty,+\infty) \to (0,1)$ by $\varkappa(t) = \frac{2}{5}$. Define $\Lambda : A^2 \to A$ by

$\Lambda(\alpha,\zeta)(t) = \alpha_0 + \int\limits_0^t \Gamma(s,\alpha(s),\zeta(s))ds,$

Now

$\varsigma_b(\Lambda(\rho,\varrho)(t),\Lambda(\rho,\varrho)(t),\Lambda(\mu,\nu)(t)) = 2\left(\Lambda\left(\rho,\varrho\right)(t) - \Lambda\left(\mu,\nu\right)(t)\right|$
$= 2\left|\int\limits_0^t \Gamma(s,\rho(s),\varrho(s))ds - \int\limits_0^t \Gamma(s,\mu(s),\nu(s))ds\right|$
$= \; 2\left|\frac{1}{5}max\begin{Bmatrix} \int_0^t \Gamma\left(s,\rho\left(s\right),\rho\left(s\right)\right)ds, \\ \int_0^t \Gamma\left(s,\varrho\left(s\right),\varrho\left(s\right)\right)ds \end{Bmatrix} - \frac{1}{5}max\begin{Bmatrix} \int_0^t \Gamma\left(s,\mu\left(s\right),\mu\left(s\right)\right)ds, \\ \int_0^t \Gamma\left(s,\nu\left(s\right),\nu\left(s\right)\right)ds \end{Bmatrix}\right|$
$\leq \frac{2}{5}max\begin{Bmatrix} \left|\int_0^t \Gamma\left(s,\rho\left(s\right),\rho\left(s\right)\right)ds - \int_0^t \Gamma\left(s,\mu\left(s\right),\mu\left(s\right)\right)ds\right| \\ \left|\int_0^t \Gamma\left(s,\varrho\left(s\right),\varrho\left(s\right)\right)ds - \int_0^t \Gamma\left(s,\nu\left(s\right),\nu\left(s\right)\right)ds\right| \end{Bmatrix}$
$\leq \frac{2}{5}max\begin{Bmatrix} \left|\rho\left(t\right) - \mu\left(t\right)\right|, \\ \left|\varrho\left(t\right) - \nu\left(t\right)\right| \end{Bmatrix}$
$\leq \varkappa\left(\psi\left(\rho,\mu\right)\right)\psi\left(\rho,\mu\right) + \varkappa\left(\phi\left(\varrho,\nu\right)\right)\phi\left(\varrho,\nu\right)$

Hence, there is only one solution to the Equation (8) in A, based on Corollary 3.4.

We now provide the primary outcome on the application to homotopy theory.

3.6.2 Theorem:

I and $\bar{\mathrm{I}}$ are two open and closed subsets of A such that $\mathrm{I} \subseteq \bar{\mathrm{I}}$. Let (A, ς_b) be a complete S_b-metric space. Assume that $\mathrm{A}_b : \bar{\mathrm{I}}^2 \times [0,1] \to A$ is an operator that satisfies the subsequent requirements:

$(\tau_0)\alpha \neq \mathrm{A}_b(\alpha, œ, \chi)$ and $œ \neq \mathrm{A}_b(œ, \alpha, \chi)$ for each $\alpha, œ \in \partial\mathrm{I}$ and $\chi \in [0,1]$ (here $\partial\mathrm{I}$ denotes the boundary of I in A),

$(\tau_1)\, \varsigma_b(\mathrm{A}_b(\alpha, œ, \chi), \mathrm{A}_b(\alpha, œ, \chi), \mathrm{A}_b(l, \beta, \chi)) \leq \varkappa(\psi(\alpha, l))\psi(\alpha, l) - \psi(\mathrm{A}_b(\alpha, œ, \chi), \mathrm{A}_b(l, \beta, \chi))$
$+\varkappa(\phi(œ, \beta))\phi(œ, l) - \phi(\mathrm{A}_b(œ, \alpha, \chi), \mathrm{A}_b(\beta, l, \chi))$

$\forall\, \alpha, œ, l, \beta \in \bar{\mathrm{I}}, \chi \in [0,1]$ and $\psi, \phi : A \times A \to [0, \infty)$ are lower semi continuous functions and, $\varkappa : (-\infty, +\infty) \to (0,1)$ is a continuous function.

(τ_2) there exists $M \geq 0$ such that
$\varsigma_b(\mathrm{A}_b(l, \beta, \chi), \mathrm{A}_b(l, \beta, \chi), \mathrm{A}_b(l, \beta, \mu)) \leq M\,|\chi - \mu|$
for every $l, \beta \in \bar{\mathrm{I}}$ and $\chi, \mu \in [0,1]$.

Then $\mathrm{A}_b(.,0)$ has a coupled fixed point $\Leftrightarrow \mathrm{A}_b(.,1)$ has a coupled fixed point.

Proof: Consider the set
$\Upsilon = \{\chi \in [0,1] : \alpha = \mathrm{A}_b(\alpha, œ, \chi) \text{ and } œ = \mathrm{A}_b(œ, \alpha, \chi) \ \ for\ some\ \ \alpha, œ \in \mathrm{I}\}$.

We have $(0,0) \in \Upsilon^2$ since $\mathrm{A}_b(.,0)$ has a coupled fixed point in I. Thus, Υ is a set that is not empty. We will demonstrate that Υ is both open and closed in $[0,1]$. Consequently, $\Upsilon = [0,1]$ by virtue of connectedness. Consequently, there is a coupled fixed point for $\mathrm{A}_b(.,1)$ in I. We first demonstrate the closure of Υ in $[0,1]$. Using $\chi_z \to \chi \in [0,1]$ as $z \to \infty$, let $(\chi_z)_{z=1}^{\infty} \subseteq \Upsilon$. We have to demonstrate that $\chi \in \Upsilon$. $\exists \alpha_z, œ_z \in \mathrm{I}$ with $\alpha_z = \mathrm{A}_b(\alpha_z, œ_z, \chi_z)$ and $œ_z = \mathrm{A}_b(œ_z, \alpha_z, \chi_z)$, since $\chi_z \in \Upsilon$ for $z = 1, 2, 3, \cdots$.

Think About

$$\begin{aligned}
&\varsigma_b(\alpha_z, \alpha_z, \alpha_{z+1}) = \varsigma_b(\mathrm{A}_b(\alpha_z, œ_z, \chi_z), \mathrm{A}_b(\alpha_z, œ_z, \chi_z), \mathrm{A}_b(\alpha_{z+1}, œ_{z+1}, \chi_{z+1}))\\
&\quad \leq 2\vartheta\varsigma_b(\mathrm{A}_b(\alpha_z, œ_z, \chi_z), \mathrm{A}_b(\alpha_z, œ_z, \chi_z), \mathrm{A}_b(\alpha_{z+1}, œ_{z+1}, \chi_z))\\
&\qquad +\vartheta^2\varsigma_b(\mathrm{A}_b(\alpha_{z+1}, œ_{z+1}, \chi_z), \mathrm{A}_b(\alpha_{z+1}, œ_{z+1}, \chi_z), \mathrm{A}_b(\alpha_{z+1}, œ_{z+1}, \chi_{z+1}))\\
&\quad \leq 2\vartheta\varsigma_b(\mathrm{A}_b(\alpha_z, œ_z, \chi_z), \mathrm{A}_b(\alpha_z, œ_z, \chi_z), \mathrm{A}_b(\alpha_{z+1}, œ_{z+1}, \chi_z)) + +\vartheta^2 M\,(\chi_z - \chi_{z+1}|.
\end{aligned}$$

Letting $z \to \infty$, we get

$$\begin{aligned}
&\lim_{z\to\infty} \tfrac{1}{2\vartheta}\varsigma_b(\alpha_z, \alpha_z, \alpha_{z+1}) \leq \lim_{z\to\infty} \varsigma_b(\mathrm{A}_b(\alpha_z, œ_z, \chi_z), \mathrm{A}_b(\alpha_z, œ_z, \chi_z), \mathrm{A}_b(\alpha_{z+1}, œ_{z+1}, \chi_z)) + 0\\
&\quad \leq \lim_{z\to\infty} \begin{pmatrix} \varkappa(\psi(\alpha_z, \alpha_{z+1}))\psi(\alpha_z, \alpha_{z+1}) - \psi(A_b(\alpha_z, œ_z, \chi_z), A_b(\alpha_{z+1}, œ_{z+1}, \chi_z)) \\ +\varkappa(\phi(œ_z, œ_{z+1}))\phi(œ_z, œ_{z+1}) - \phi(A_b(œ_z, \alpha_z, \chi_z), A_b(œ_{z+1}, \alpha_{z+1}, \chi_z)) \end{pmatrix}\\
&\quad < \lim_{z\to\infty} \begin{pmatrix} \psi(\alpha_z, \alpha_{z+1}) - \psi(\mathrm{A}_b(\alpha_z, œ_z, \chi_z), \mathrm{A}_b(\alpha_{z+1}, œ_{z+1}, \chi_z)) \\ +\phi(œ_z, œ_{z+1}) - \phi(\mathrm{A}_b(œ_z, \alpha_z, \chi_z), \mathrm{A}_b(œ_{z+1}, \alpha_{z+1}, \chi_z)) \end{pmatrix}\\
&\qquad < \lim_{z\to\infty} \begin{pmatrix} \psi(\alpha_z, \alpha_{z+1}) - \psi(\mathrm{A}_b(\alpha_z, œ_z, \chi_z), \mathrm{A}_b(\alpha_{z+1}, œ_{z+1}, \chi_{z+1})) \\ +\phi(œ_z, œ_{z+1}) - \phi(\mathrm{A}_b(œ_z, \alpha_z, \chi_z), \mathrm{A}_b(œ_{z+1}, \alpha_{z+1}, \chi_{z+1})) \end{pmatrix}\\
&\quad \lim_{z\to\infty} (\psi(\alpha_z, \alpha_{z+1}) - \psi(\alpha_z, \alpha_{z+1})) + \lim_{z\to\infty} (\phi(œ_z, œ_{z+1}) - \phi(œ_z, œ_{z+1}))\\
&\quad = 0.
\end{aligned}$$

So

$$\lim_{z\to\infty} \varsigma_b(\alpha_z, \alpha_z, \alpha_{z+1}) = 0 \ and\, similarly\, we\, prove\ \lim_{z\to\infty} \varsigma_b(œ_z, œ_z, œ_{z+1}) = 0. \tag{9}$$

We now demonstrate that in (A, ς_b), $\{\alpha_z\}$ and $\{œ_z\}$ are ς_b-Cauchy sequences. Assume, however, that $\{\alpha_z\}$ and $\{œ_z\}$ are not ς_b- Cauchy. $z_k > w_k$ is the result of monotonic rising sequences of natural numbers $\{w_k\}$ and $\{z_k\}$, when $\epsilon > 0$.

$$\varsigma_b(\alpha_{w_k}, \alpha_{w_k}, \alpha_{z_k}) \geq \epsilon, \ \ \varsigma_b(œ_{w_k}, œ_{w_k}, œ_{z_k}) \geq \epsilon \tag{10}$$

and

$$\varsigma_b(\alpha_{w_k}, \alpha_{w_k}, \alpha_{z_k-1}) < \epsilon, \ \ \varsigma_b(œ_{w_k}, œ_{w_k}, œ_{z_k-1}) < \epsilon. \tag{11}$$

Utilizing the Lemma (2.4) and (Equation (10)), (Equation (11)), we derive

$$\begin{aligned}
&\epsilon \leq \varsigma_b(\alpha_{w_k}, \alpha_{w_k}, \alpha_{z_k})\\
&\leq 2\vartheta\varsigma_b(\alpha_{w_k}, \alpha_{w_k}, \alpha_{w_k+1}) + \vartheta^2\varsigma_b(\alpha_{w_k+1}, \alpha_{w_k+1}, \alpha_{z_k}).
\end{aligned}$$

Using the Lemma (2.4) and letting $k \to \infty$, we have

$$\begin{aligned}
&\epsilon \leq \lim_{k\to\infty} \vartheta^2\varsigma_b(\alpha_{w_k+1}, \alpha_{w_k+1}, \alpha_{z_k})\\
&\leq \lim_{k\to\infty} \vartheta^2\varsigma_b\left(\mathrm{A}_b(\alpha_{w_{k+1}}, œ_{w_{k+1}}, \chi_{w_{k+1}}), \mathrm{A}_b(\alpha_{w_{k+1}}, œ_{w_{k+1}}, \chi_{w_{k+1}}), \mathrm{A}_b(\alpha_{z_k}, œ_{z_k}, \chi_{z_k})\right)
\end{aligned}$$

$\leq \lim_{k\to\infty} 2\vartheta^3 \varsigma_b\left(\mathrm{A}_b(\alpha_{w_{k+1}}, æ_{w_{k+1}}, \chi_{w_{k+1}}), \mathrm{A}_b(\alpha_{w_{k+1}}, æ_{w_{k+1}}, \chi_{w_{k+1}}), \mathrm{A}_b(\alpha_{z_k}, æ_{z_k}, \chi_{w_{k+1}})\right)$
$+ \lim_{k\to\infty} \vartheta^4 \varsigma_b\left(\mathrm{A}_b(\alpha_{z_k}, æ_{z_k}, \chi_{w_{k+1}}), \mathrm{A}_b(\alpha_{z_k}, æ_{z_k}, \chi_{w_{k+1}}), \mathrm{A}_b(\alpha_{z_k}, æ_{z_k}, \chi_{z_k})\right)$
which implies that
$\frac{\epsilon}{2\vartheta^3} \leq \lim_{k\to\infty} \varsigma_b\left(\mathrm{A}_b(\alpha_{w_{k+1}}, æ_{w_{k+1}}, \chi_{w_{k+1}}), \mathrm{A}_b(\alpha_{w_{k+1}}, æ_{w_{k+1}}, \chi_{w_{k+1}}), \mathrm{A}_b(\alpha_{z_k}, æ_{z_k}, \chi_{w_{k+1}})\right)$
$\leq \lim_{k\to\infty} \begin{pmatrix} \varkappa\left(\psi(\alpha_{w_{k+1}}, \alpha_{z_k})\right)\psi(\alpha_{w_{k+1}}, \alpha_{z_k}) - \psi\left(\mathrm{A}_b(\alpha_{w_{k+1}}, æ_{w_{k+1}}, \chi_{w_{k+1}}), \mathrm{A}_b(\alpha_{z_k}, æ_{z_k}, \chi_{w_{k+1}})\right) \\ +\varkappa\left(\phi(æ_{w_{k+1}}, æ_{z_k})\right)\phi(æ_{w_{k+1}}, æ_{z_k}) - \phi\left(\mathrm{A}_b(æ_{w_{k+1}}, \alpha_{w_{k+1}}, \chi_{w_{k+1}}), \mathrm{A}_b(æ_{z_k}, \alpha_{z_k}, \chi_{w_{k+1}})\right) \end{pmatrix}$
$< \lim_{k\to\infty} \begin{pmatrix} \psi(\alpha_{w_{k+1}}, \alpha_{z_k}) - \psi\left(\mathrm{A}_b(\alpha_{w_{k+1}}, æ_{w_{k+1}}, \chi_{w_{k+1}}), \mathrm{A}_b(\alpha_{z_k}, æ_{z_k}, \chi_{w_{k+1}})\right) \\ +\phi(æ_{w_{k+1}}, æ_{z_k}) - \phi\left(\mathrm{A}_b(æ_{w_{k+1}}, \alpha_{w_{k+1}}, \chi_{w_{k+1}}), \mathrm{A}_b(æ_{z_k}, \alpha_{z_k}, \chi_{w_{k+1}})\right) \end{pmatrix}$
$< \lim_{k\to\infty}\left(\psi(\alpha_{w_{k+1}}, \alpha_{z_k}) - \psi(\alpha_{w_{k+1}}, \alpha_{z_k})\right) + \lim_{k\to\infty}\left(\phi(æ_{w_{k+1}}, æ_{z_k}) - \phi(æ_{w_{k+1}}, æ_{z_k})\right)$
$= 0.$

Therefore, $\epsilon < 0$, indicating a paradox. Consequently, A ς_b-Cauchy sequence in (A, ς_b) is $\{\alpha_z\}$. Similarly, by demonstrating the completeness of (A, ς_b), we can show that $\{æ_z\}$ is a ς_b-Cauchy sequence in (A, ς_b) and that $\alpha, æ \in \mathrm{I}$ exist.

$$\lim_{z\to\infty} \alpha_z = \alpha = \lim_{z\to\infty} \alpha_{z+1} \; and \; \lim_{z\to\infty} æ_z = æ = \lim_{z\to\infty} æ_{z+1}. \tag{12}$$

From Lemma (2.5), we have
$\frac{1}{2\vartheta}\varsigma_b(\mathrm{A}_b(\alpha, æ, \chi), \mathrm{A}_b(\alpha, æ, \chi), \alpha) \leq \lim_{z\to\infty} inf\, \varsigma_b(\mathrm{A}_b(\alpha, æ, \chi), \mathrm{A}_b(\alpha, æ, \chi), \mathrm{A}_b(\alpha_z, æ_z, \chi))$
$< lim\ inf \begin{pmatrix} x(\psi(\alpha, \alpha_z))\psi(\alpha, \alpha_z) - \psi(A_b(\alpha, æ, x)\ A_b(\alpha_z, æ_z, x) \\ +x(\Phi(æ, _z))\Phi(æ, æ_z) - \Phi(A_b(æ, \alpha, x),\ A_b(æ_z, \alpha_z, x)) \end{pmatrix}$
$< \lim_{k\to\infty} inf \begin{pmatrix} \psi(\alpha, \alpha_z) - \psi(\alpha, \alpha_z) \\ \psi(\alpha, \alpha_z) - \psi(\alpha, \alpha_z) \end{pmatrix}$
$= 0.$

$\alpha = \mathrm{A}_b(\alpha, æ, \chi)$ follows as a consequence, and in a similar vein, we can demonstrate $æ = \mathrm{A}_b(æ, \alpha, \chi)$. Consequently, $\chi \in \Upsilon$. Υ is therefore closed in [0, 1]. Assume $\chi_0 \in \Upsilon$. After that, $\alpha_0 = \mathrm{A}_b(\alpha_0, æ_0, \chi_0)$ and $æ_0 = \mathrm{A}_b(æ_0, \alpha_0, \chi_0)$ exist in I. Given that Υ is open, $B_{\varsigma_b}(\alpha_0, \delta) \subseteq \Upsilon$ and $B_{\varsigma_b}(æ_0, \delta) \subseteq \Upsilon$. Such occurs for $\delta > 0$. To ensure that $|\chi - \chi_0| \leq \frac{1}{M^z} < \epsilon$, choose $\chi \in (\chi_0 - \epsilon, \chi_0 + \epsilon)$. Then, given
$\alpha \in \overline{B_b(\alpha_0, \delta)} = \{\alpha \in A/\psi(\alpha, \alpha_0) = \varsigma_b(\alpha, \alpha, \alpha_0) \leq \delta\}$ and
$æ \in \overline{B_b(æ_0, \delta)} = \{æ \in A/\psi(æ, æ_0) = \varsigma_b(æ, æ, æ_0) \leq \delta\}$
$\varsigma_b(\mathrm{A}_b(\alpha, æ, \chi), \mathrm{A}_b(\alpha, æ, \chi), \alpha_0)$
$= \varsigma_b(\mathrm{A}_b(\alpha, æ, \chi), \mathrm{A}_b(\alpha, æ, \chi), \mathrm{A}_b(\alpha_0, æ_0, \chi_0))$
$\leq \frac{2\vartheta}{M^{n-1}} + \vartheta^2 \varsigma_b(\mathrm{A}_b(\alpha, æ, \chi_0), \mathrm{A}_b(\alpha, æ, \chi_0), \mathrm{A}_b(\alpha_0, æ_0, \chi_0)).$
Letting $z \to \infty$, we obtain
$\frac{1}{\vartheta^2}\varsigma_b(\mathrm{A}_b(\alpha, æ, \chi), \mathrm{A}_b(\alpha, æ, \chi), \alpha_0) \leq \varsigma_b(\mathrm{A}_b(\alpha, æ, \chi_0), \mathrm{A}_b(\alpha, æ, \chi_0), \mathrm{A}_b(\alpha_0, æ_0, \chi_0))$
$\leq \begin{pmatrix} \varkappa(\psi(\alpha, \alpha_0))\psi(\alpha, \alpha_0) - \psi(A_b(\alpha, æ, \chi_0), A_b(\alpha_0, æ_0, \chi_0)) \\ +\varkappa(\phi(æ, æ_0))\phi(æ, æ_0) - \phi(A_b(æ, \alpha, \chi_0), A_b(æ_0, \alpha_0, \chi_0)) \end{pmatrix}$
$< \psi(\alpha, \alpha_0) + \phi(æ, æ_0)$
$= \varsigma_b(\alpha, \alpha, \alpha_0) + \varsigma_b(æ, æ, æ_0)$
$\leq 2\delta$
and similarly
$\frac{1}{\vartheta^2}\varsigma_b(\mathrm{A}_b(æ, \alpha, \chi), \mathrm{A}_b(æ, \alpha, \chi), æ_0) \leq 2\delta + \varsigma_b(\alpha_0, \alpha_0, \alpha_0) + \varsigma_b(æ_0, æ_0, æ_0)$
Therefore, we conclude that
$max \begin{Bmatrix} \varsigma_b(\mathrm{A}_b(\alpha, æ, \chi), \mathrm{A}_b(\alpha, æ, \chi), \alpha_0) \\ \varsigma_b(\mathrm{A}_b(æ, \alpha, \chi), \mathrm{A}_b(æ, \alpha, \chi), æ_0) \end{Bmatrix} \leq 2\vartheta^2\delta$

For any fixed $\chi \in (\chi_0 - \epsilon, \chi_0 + \epsilon)$, then for each $\mathrm{A}_b(., \chi) : \overline{B_b(\alpha_0, \delta)} \to \overline{B_b(\alpha_0, \delta)}$ and $\mathrm{A}_b(., \chi) : \overline{B_b(æ_0, \delta)} \to \overline{B_b(æ_0, \delta)}$. Then, Theorem (5) meets all of its requirements. From this, we infer that there is a coupled fixed point for $\mathrm{A}_b(., \chi)$ in $\bar{\mathrm{I}}$. This connected fixed point, however, needs to be in I. For all $\chi \in (\chi_0 - \epsilon, \chi_0 + \epsilon)$, $\chi \in \Upsilon$. Thus, Υ is open in [0, 1] since $(\chi_0 - \epsilon, \chi_0 + \epsilon) \subseteq \Upsilon$. We employ the identical method for the opposite inference.

4 Conclusion

This study has proved common coupled fixed point results in S_b-metric space by using Caristi type contractions which generalize several existing results in the literature and also presented applications to integral equations and to homotopy theory.

References

1) Shukla S, Rai S. Caristi type fixed point theorems in 1-M-complete fuzzy metric-like spaces. *J Anal*. 2023;31:2247–2263. Available from: https://doi.org/10.1007/s41478-023-00562-x.
2) Kishore GNV, Ravi P, Agarwal BRS, Srinivasa RVN. Caristi type cyclic contraction and common fixed point theorems in bipolar metric spaces with applications. . *Fixed Point Theory and Applications*. 2018;21:1–13. Available from: https://doi.org/10.1186/s13663-018-0646-z.
3) Kishore GN, Srinuvasa BR, Radenovic$'$ S, Huang H. INLINE:56c45f896ba24b08b6f9ba3cd98c4648:ENDEQNBLOCK S, Huang H. Caristi type cyclic contraction and coupled fixed point results in bipolar metric spaces. *Sahand Communications in Mathematical Analysis (SCMA)*. 2020;17(1):1–22. Available from: https://doi.org/10.22130/scma.2018.79219.369.
4) Abdul MD, Hossain H, Akter H, Mondal S. A study of Caristi's fixed point theorem on normed space and its applications. *Advances in Pure Mathematics*. 2021;11(3):169–179. Available from: https://doi.org/10.4236/apm.2021.113012.
5) Tas N, Ozgur N. New Generalized Fixed Point Results on Sb-Metric Spaces. *Konuralp Journal of Mathematics*. 2021;9(1):24–32. Available from: https://dergipark.org.tr/en/download/article-file/1000285.
6) Mangapathi N, Rao S, Rao B, Rao KRK, Pasha MI. On certain fixed point theorems in Sb-metric spaces with applications. *International Journal of Analysis and Applications*. 2023;21(18):1–15. Available from: https://doi.org/10.28924/2291-8639-21-2023-18.
7) Singh AK, Singh MR. Some coupled fixed point theorems in cone b-metric spaces. *Journal of Mathematical and Computational Science*. 2020;10(4):891–905. Available from: https://doi.org/10.28919/jmcs/4462.
8) Adilakshmi G, Kishore GNV. A new approach of finding fixed point results in partial metric spaces and applications to graph theory. *AIP Conference Proceedings*. 2023;2796(090001). Available from: https://doi.org/10.1063/5.0149113.
9) Zlatanov B. A variational principle and coupled fixed points. *J Fixed Point Theory Appl*. 2019;21(69). Available from: https://doi.org/10.1007/s11784-019-0706-y.
10) Cevik C, Özeken Ç. Coupled fixed point results for new classes of functions on ordered vector metric space. *Acta Math Hungar*. 2024;172:1–18. Available from: https://doi.org/10.1007/s10474-024-01393-3.