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RESEARCH ARTICLE

Solving Fuzzy Time Delay Malthusian Growth model using Runge - Kutta Method Based on Centroidal Mean

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Abstract

Background/Objective: A mathematical formula for population growth is known as the Malthusian growth model. According to this theory, the pace of growth corresponds to the current population. In terms of functionality, this is similar to exponential growth, in which population size increases at a fixed rate over regular intervals. The main aim of this article is to solve the Malthusian growth model in a fuzzy environment using the fourth-order Runge-Kutta method based on Centroidal Mean (RK4CeM). **Methods:** The fourth order Runge – Kutta method based on Centroidal Mean is used to obtain an approximate solution for the Malthusian growth model. In this study, we deal with the problem of fuzzy delay differential equation with constant delay using the fourth order Runge – Kutta method based on Centroidal mean where the linear interpolation is used to approximate the delay term. **Findings:** A numerical example of fuzzy time delay Malthusian growth model has been considered to demonstrate the efficiency of the proposed method. The solution graph of the proposed method is well comparable with the exact solution of the problem. **Novelty:** The numerical solutions and solution graph of the proposed method are well comparable with the exact solution and Euler method. This ensures the accuracy of the computed results. Hence, the proposed method is effective and suitable for solving fuzzy delay differential equations.

Keywords: Fuzzy Delay Differential Equations; RungeKutta Method; Centroidal Mean; Triangular Fuzzy Number; Malthusian growth model

1 Introduction

Fuzzy Time Delay Malthusian Growth Model incorporates fuzziness and delays into the classical Malthusian growth model. It reflects more realistic scenarios where population growth is influenced by imprecise parameters and time delays. Fuzzy Time Delay The Malthusian Growth Model is a kind of Fuzzy Delay Differential Equation (FDDE) that incorporates fuzzy logic and delays. These equations are more complex than standard differential equations due to the presence of fuzziness and delays.

Rajakumari and Sharmila [1,2] proposed the Runge Kutta method based on a linear combination of variety of Means to find the solution of a hybrid fuzzy differential equation. Mary Jansirani et.al [3] executed a composite Runge-Kutta method based on a variety of means by using intuitionistic fuzzy differential equations. In a delay differential equation, the derivative of the unknown function at a given time is stated in terms of the value of the function at a previous time. Senu et.al [4] developed and proposed a two-derivative Runge-Kutta method with Newton interpolation to solve delay differential equations. Ponammal et.al [5] studied the topic using a numerical method called the natural continuous extension fourth-order Runge-Kutta method. This method is likely employed to efficiently and accurately solve the Pantograph equation, which can be challenging due to its delayed nature.

Yookesh.T.L., et.al [6] presented variational iteration method to solve time delay differential equation under uncertainty condition. Hamza et.al [7] proposed a technique for finding an approximate solution of FDDE equations using the implicit Runge-Kutta method of order seven method with generalized differentiability. Muthukumar et.al [8] implemented the fifth order Runge-Kutta Method to solve FDDE. Prasanth Bharathi et.al [9] used Runge- Kutta method of order four to solve fuzzy mixed delay differential equations. Prasanth Bharathi et.al [10] implemented the Runge-Kutta method of order four to solve fuzzy neutral delay differential equations. Nguyen [11] explained the FDDE under granular differentiability with applications such as fuzzy time delay Malthusian growth model. A.F. Jameel et.al [12] presented paper to find an approximate analytical solution for first order fuzzy delay differential equations.

In this work, the focus is on solving the Fuzzy Time Delay Malthusian Growth Model using the Runge Kutta method based on the Centroidal Mean. This study introduces and employs a novel numerical approach, the Runge Kutta method based on the Centroidal Mean, to address the complexities of solving a FDDE. To verify the effectiveness and viability of this method, a comparative study with the Euler method is conducted.

The Runge Kutta method based on the Centroidal Mean is a new approach specifically designed for FDDEs. In this work an algorithm is proposed to handle fuzziness and delays in the solution process. The proposed algorithm involves modifications to the traditional Runge Kutta method based on Centroidal Mean to specifically address the fuzziness and delay aspects of the problem. The Euler method is a simpler numerical method often used for solving differential equations. By comparing the Runge Kutta method based on the Centroidal Mean with the Euler method, the study aims to demonstrate the superior accuracy and efficiency of the new approach.

This paper is structured as follows: in section 2, the definitions and basic concepts are explained. In section 3 the procedure is proposed to solve a fuzzy delay differential equation using the fourth order Runge-Kutta method based on centroidal Mean. In section 4, a numerical example is illustrated using the proposed method. The conclusion is given in section 5.

2 Definitions and Basic Concepts

This section gives definitions and necessary basic concepts which will be used throughout this paper.

2.1 Fuzzy Number

Fuzzy set f is a fuzzy subset of R, i e , $f : R \to [0,1]$satisfying the following conditions:

- f is normal, i.e., $x_0 \in R$ with $f(x_0) = 1$;
- f is a convex fuzzy set, i.e., $f(\lambda x + (1-\lambda)y) \geq min[f(x), f(y)], \forall\, x, y \in R,\ \forall \lambda \in [0,1]$
- f is upper semi continuous on R

2.2 Triangular Fuzzy Number

The triangular fuzzy number is a fuzzy interval represented as A = (a_1, a_2, a_3) where $a_1$and a_2 are two end points and a_2 is a peak point. The α cut interval for a fuzzy number A as A_α, the obtained interval A_α is defined as $A_\alpha = \left[a_1^{(\alpha)}, a_3^{(\alpha)}\right]$ then

$$\mu_A(x) = \begin{cases} 0, & x < a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3 \\ 0, x > a_3 & \end{cases}$$

The crisp interval is obtained as follows, $\forall \alpha \in [0,1]$ by α cut operations. Then

$$A_\alpha = [(a_2 - a_1)\alpha + a_1, -(a_3 - a_2)\alpha + a_3]$$

2.3 Fuzzy Delay Differential Equation

Delay differential equation is of the form,

$$\mathrm{y}'(\mathrm{x}) = \mathrm{f}(\mathrm{x}, \mathrm{y}(\mathrm{x}), \mathrm{y}(\mathrm{x}-\tau_1), \mathrm{y}(\mathrm{x}-\tau_2), \ldots, \mathrm{y}(\mathrm{x}-\tau_{\mathrm{i}})), x \geq x_0$$

$$y(x) = \phi(x), \quad x \geq x_0 \tag{1}$$

where τ_i is a constant for each i = 1, 2, 3, . . ., n. In a Delay Differential Equation, a fuzzy parameter could be a coefficient, an initial condition, or both. It is known as a fuzzy delay differential equation (FDDE). The function $y(x)$ is a fuzzy derivative of $y(x)$ at $x \in [-\tau, T]$ and $\phi(x)$is a fuzzy number corresponding to the given initial function when $x < 0$. It is represented in the following form

$$\underline{y}'(x) = f\left(x, \underline{y}(x), \underline{y}(x-\tau_1), \ldots \underline{y}(x-\tau_i)\right), x \geq x_0$$
$$\underline{y}(x) = \underline{\phi}(x)$$
$$\overline{y}'(x) = f(x, \overline{y}(x), \overline{y}(x-\tau_1), \ldots \overline{y}(x-\tau_i)), x \geq x_0,$$
$$\overline{y}(x) = \overline{\phi}(x) \tag{2}$$

where τ_i is constant for each $i = 1, 2, 3, \ldots, n$.

2.4 The Fourth order Runge-Kutta method based on Centroidal Mean

In this section, the fourth order Runge-Kutta method based on Centroidal Mean (RK4CeM) is used to solve a fuzzy Delay Differential Equation (2). The Centroidal Mean of two points $x_1$and x_2

is defined as $\frac{2}{3}\left(\frac{x_1^2 + x_1 x_2 + x_2^2}{x_1 + x_2}\right)$

The Runge-Kutta method based on Centroidal Mean is as follows:

$$y_{n+1} = y_n + \frac{2}{9} h\left(\frac{k_1^2 + k_1 k_2 + k_2^2}{k_1 + k_2} + \frac{k_2^2 + k_2 k_3 + k_3^2}{k_2 + k_3} + \frac{k_3^2 + k_3 k_4 + k_4^2}{k_3 + k_4}\right]$$

Where

$$k_1 = f(x_n, y_n)$$
$$k_2 = f\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_1\right)$$
$$k_3 = f\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{24}hk_1 + \tfrac{11}{24}hk_2\right)$$
$$k_4 = f\left(x_n + h, y_n + \tfrac{11}{132}hk_1 - \tfrac{25}{132}hk_2 + \tfrac{73}{66}hk_3\right)$$

3 The numerical algorithm to solve FDDE using the fourth order Ruge – Kutta method based on Centroidal Mean

In this work constant delays are considered. The delay terms are easily handled by using interpolation. For a fuzzy delay differential equation, the proposed method is defined as follows:

$$\underline{y}(x_{n+1};\alpha) = \underline{y}(x_n;\alpha) + \sum_{i=1}^{4} w_i \underline{k}_i(x;\phi(x_n;\alpha))$$

$$\overline{y}(x_{n+1};\alpha) = \overline{y}(x_n;\alpha) + \sum_{i=1}^{4} w_i \overline{k}_i(x;\phi(x_n;\alpha))$$

where c_1, c_2, c_3, and c_4 are constants and

$$\underline{k}_1(x;\phi(\mathrm{x};\alpha)) = \min\{hf(x,\phi(x),\phi(x-\tau)), \phi(\mathrm{x}) \in [\underline{\phi}(x_{k,n};\alpha), \bar{\phi}(x_{k,n};\alpha)],$$
$$\phi(x-\tau) \in [\underline{\phi}(x_{k,n}-\tau;\alpha), \bar{\phi}(x_{k.n}-\tau;\alpha)]\}$$
$$\bar{k}_1(x;\phi(x;\alpha)) = \max\{hf(x,\phi(x),\phi(x-\tau)), \phi(x) \in [\underline{\phi}(x_{k,n};\alpha), \bar{\phi}(x_{k.n};\alpha)],$$
$$\phi(x-\tau) \in [\underline{\phi}(x_{k,n}-\tau;\alpha), \bar{\phi}(x_{k.n}-\tau;\alpha)]\}$$
$$\underline{k_2}(x;\phi(x;\alpha)) = \min\left\{hf\left(x+\frac{h}{2}, \phi(x), \phi(x-\tau)\right), \phi(x) \in \left[\underline{z_1}(x_{k,n}, \underline{\phi}(x_{k,n};\alpha)), \overline{z_1}(x_{k,n}, \bar{\phi}(x_{k,n};\alpha))\right],\right.$$
$$\left.\phi(x-\tau) \in \left[\underline{z_1}(x_{k,n}-\tau, \underline{\phi}(x_{k,n}-\tau;\alpha)), \overline{z_1}(x_{k,n}-\tau, \bar{\phi}(x_{k,n}-\tau;\alpha))\right]\right\}$$

$$\overline{k_2}(x;\phi(x;\alpha)) = \max\left\{hf\left(x+\frac{h}{2},\phi(x),\phi(x-\tau)\right),\phi(x)\in\left[\underline{z_1}\left(x_{k,n},\underline{\phi}\left(x_{k,n};\alpha\right)\right),\overline{z_1}\left(x_{k,n},\bar{\phi}\left(x_{k,n};\alpha\right)\right)\right],\right.$$
$$\left.\phi(x-\tau)\in\left[\underline{z_1}\left(x_{k,n}-\tau,\underline{\phi}\left(x_{k,n}-\tau;\alpha\right)\right),\overline{z_1}\left(x_{k,n}-\tau,\bar{\phi}\left(x_{k,n}-\tau;\alpha\right)\right)\right]\right\}$$
$$\underline{k_3}(x;\phi(x;\alpha)) = \min\left\{hf\left(x+\frac{h}{2},\phi(x),\phi(x-\tau)\right),\quad \phi(x)\in\left[\underline{z_2}\left(x_{k,n},\underline{\phi}\left(x_{k,n};\alpha\right)\right),\overline{z_2}\left(x_{k,n},\bar{\phi}\left(x_{k,n};\alpha\right)\right)\right],\right.$$
$$\left.\phi(x-\tau)\in\left[\underline{z_2}\left(x_{k,n}-\tau,\underline{\phi}\left(x_{k,n}-\tau;\alpha\right)\right),\overline{z_2}\left(x_{k,n}-\tau,\bar{\phi}\left(x_{k,n}-\tau;\alpha\right)\right)\right]\right\}$$
$$\overline{k_3}(x;\phi(x;\alpha)) = \max\left\{hf\left(x+\frac{h}{2},\phi(x),\phi(x-\tau)\right),\phi(x)\in\left[\underline{z_2}\left(x_{k,n},\underline{\phi}\left(x_{k,n};\alpha\right)\right),\overline{z_2}\left(x_{k,n},\bar{\phi}\left(x_{k,n};\alpha\right)\right)\right],\right.$$
$$\left.\phi(x-\tau)\in\left[\underline{z_2}\left(x_{k,n}-\tau,\underline{\phi}\left(x_{k,n}-\tau;\alpha\right)\right),\overline{z_2}\left(x_{k,n}-\tau,\bar{\phi}\left(x_{k,n}-\tau;\alpha\right)\right)\right]\right\}$$
$$\underline{k_4}(x;\phi(x;\alpha)) = \min\left\{hf(x+h,\phi(x),\phi(x-\tau)),\phi(x)\in\left[\underline{z_3}\left(x_{k,n},\underline{\phi}\left(x_{k,n};\alpha\right)\right),\overline{z_3}\left(x_{k,n},\bar{\phi}\left(x_{k,n};\alpha\right)\right)\right],\right.$$
$$\left.\phi(x-\tau)\in\left[\underline{z_3}\left(x_{k,n}-\tau,\underline{\phi}\left(x_{k,n}-\tau;\alpha\right)\right),\overline{z_3}\left(x_{k,n}-\tau,\bar{\phi}\left(x_{k,n}-\tau;\alpha\right)\right)\right]\right\}$$
$$\overline{k_4}(x;\phi(x;\alpha)) = \max\left\{hf\left(x+\frac{h}{2},\phi(x),\phi(x-\tau)\right),\quad \phi(x)\in\left[\underline{z_3}\left(x_{k,n},\underline{\phi}\left(x_{k,n};\alpha\right)\right),\overline{z_3}\left(x_{k,n},\bar{\phi}\left(x_{k,n};\alpha\right)\right)\right],\right.$$
$$\left.\phi(x-\tau)\in\left[\underline{z_3}\left(x_{k,n}-\tau,\underline{\phi}\left(x_{k,n}-\tau;\alpha\right)\right),\overline{z_3}\left(x_{k,n}-\tau,\bar{\phi}\left(x_{k,n}-\tau;\alpha\right)\right)\right]\right\}$$

where

$$\underline{z_1}\left(x_{k,n},\underline{\phi}\left(x_{k,n};\alpha\right)\right) = \underline{\phi}\left(x_{k,n};\alpha\right)+\frac{h}{2}\underline{k_1}\left(x_{k,n},\underline{\phi}\left(x_{k,n};\alpha\right)\right)$$

$$\overline{z_1}\left(x_{k,n},\underline{\phi}\left(x_{k,n};\alpha\right)\right) = \bar{\phi}\left(x_{k,n};\alpha\right)+\frac{h}{2}\overline{k_1}\left(x_{k,n},\bar{\phi}\left(x_{k,n};\alpha\right)\right)$$

$$\underline{z_2}\left(x_{k,n},\underline{\phi}\left(x_{k,n};\alpha\right)\right) = \underline{\phi}\left(x_{k,n};\alpha\right)+h\left(\frac{1}{24}\underline{k_1}\left(x_{k,n},\underline{\phi}\left(x_{k,n};\alpha\right)\right)+\frac{11}{24}\underline{k_2}\left(x_{k,n},\underline{\phi}\left(x_{k,n};\alpha\right)\right)\right]$$

$$\overline{z_2}\left(x_{k,n},\underline{\phi}\left(x_{k,n};\alpha\right)\right) = \bar{\phi}\left(x_{k,n};\alpha\right)+h\left(\frac{1}{24}\overline{k_1}\left(x_{k,n},\bar{\phi}\left(x_{k,n};\alpha\right)\right)+\frac{11}{24}\overline{k_2}\left(x_{k,n},\bar{\phi}\left(x_{k,n};\alpha\right)\right)\right]$$

$$\begin{aligned}&\underline{z_3}\left(x_{k,n},\underline{\phi}\left(x_{k,n};\alpha\right)\right)\\&=\underline{\phi}\left(x_{k,n};\alpha\right)\\&+h\left[\frac{11}{132}\underline{k_1}\left(x_{k,n},\underline{\phi}\left(x_{k,n};\alpha\right)\right)-\frac{25}{132}\underline{k_2}\left(x_{k,n},\underline{\phi}\left(x_{k,n};\alpha\right)\right)+\frac{73}{66}\underline{k_3}\left(x_{k,n},\underline{\phi}\left(x_{k,n};\alpha\right)\right)\right]\end{aligned}$$

$$\begin{aligned}\overline{z_3}\left(x_{k,n},\bar{\phi}\left(x_{k,n};\ \alpha\right)\right)&\\&=\bar{\phi}\left(x_{k,n};\alpha\right)\\&+h\left[\frac{11}{132}\overline{k_1}\left(x_{k,n},\bar{\phi}\left(x_{k,n};\alpha\right)\right)-\frac{25}{132}\overline{k_2}\left(x_{k,n},\bar{\phi}\left(x_{k,n};\alpha\right)\right)+\frac{73}{66}\overline{k_3}\left(x_{k,n},\bar{\phi}\left(x_{k,n};\alpha\right)\right)\right]\end{aligned}$$

$$S[x,\phi(x;\alpha),\bar{\phi}(x;\alpha)] = \left[\begin{array}{l}\frac{\underline{k}_1^2(x;\phi(x;\alpha))+\underline{k_1}(x;\phi(x;\alpha))\underline{k_2}(x;\phi(x;\alpha))+\underline{k}_2^2(x;\phi(x;\alpha))}{\underline{k_1}(x;\phi(x;\alpha))+\underline{k_2}(x;\phi(x;\alpha))}+\\\frac{\underline{k}_2^2(x;\phi(x;\alpha))+\underline{k_2}(x;\phi(x;\alpha))\underline{k_3}(x;\phi(x;\alpha))+\underline{k}_3^2(x;\phi(x;\alpha))}{\underline{k_2}(x;\phi(x;\alpha))+\underline{k_3}(x;\phi(x;\alpha))}+\\\frac{\underline{k}_3^2(x;\phi(x;\alpha))+\underline{k_3}(x;\phi(x;\alpha))\underline{k_4}(x;\phi(x;\alpha))+\underline{k}_4^2(x;\phi(x;\alpha))}{\underline{k_3}(x;\phi(x;\alpha))+\underline{k_4}(x;\phi(x;\alpha))}\end{array}\right]$$

$$T[x,\phi(x;\alpha),\overline{\phi}(x;\alpha)] = \left[\begin{array}{l}\frac{\overline{k_1^2}(x;\phi(x;\alpha))+\overline{k_1}(x;\phi(x;\alpha))\overline{k_2}(x;\phi(x;\alpha))+\overline{k_2^2}(x;\phi(x;\alpha))}{\overline{k_1}(x;\phi(x;\alpha))+\overline{k_2}(x;\phi(x;\alpha))}+\\\frac{\overline{k_2^2}(x;\phi(x;\alpha))+\overline{k_2}(x;\phi(x;\alpha))\overline{k_3}(x;\phi(x;\alpha))+\overline{k_3^2}(x;\phi(x;\alpha))}{\overline{k_2}(x;\phi(x;\alpha))+\overline{k_3}(x;\phi(x;\alpha))}+\\\frac{\overline{k_3^2}(x;\phi(x;\alpha))+\overline{k_3}(x;\phi(x;\alpha))\overline{k_4}(x;\phi(x;\alpha))+\overline{k_4^2}(x;\phi(x;\alpha))}{\overline{k_3}(x;\phi(x;\alpha))+\overline{k_4}(x;\phi(x;\alpha))}\end{array}\right]$$

Then the approximate solution is given by

$\underline{y}(x_{n+1};\alpha) = \underline{\phi}(x_{n+1};\alpha) + \underline{y}(x_n - \tau;\alpha)$,

$\overline{y}(x_{n+1};\alpha) = \overline{\phi}(x_{n+1};\alpha) + \overline{y}(x_n - \tau;\alpha)$,

where

$$\underline{\phi}(x_{n+1};\alpha) = \underline{\phi}(x;\alpha) + \frac{2}{9}h\,S[x,\underline{\phi}(x;\alpha),\overline{\phi}(x;\alpha)]$$

$$\overline{\phi}(x_{n+1};\alpha) = \overline{\phi}(x_n;\alpha) + \frac{2}{9}h\,T[x,\underline{\phi}(x;\alpha),\overline{\phi}(x;\alpha)]$$

4 Numerical Example

In this section Malthusian growth model is considered to test the efficiency of the proposed method.

4.1 Fuzzy time-delay Malthusian growth model

Consider a fuzzy time-delay Malthusian growth model also known as a mathematical model of population growth based on exponential growth law. It was introduced by Thomas Robert Malthus. Regardless of population size, the population would increase at a fixed rate over a certain amount of time.

$y'(t) = ry(t-1) \quad t \geq 0,\ r > 0$

$y(t) = y_0 \quad -1 \leq t \leq 0.$

Where $y(t)$ refers to the population at time t. r is used to describe the growth rate and the initial value y_0 atriangularfuzzynumberwithcuts.Suppose the initial value

$[y_0]^\alpha = [\alpha - 1, 1 - \alpha] = [(1-\alpha)(-1,1)], \quad \alpha \in [0,1]$ and $r > 0.$

The comparison of exact and approximate solutions of the problem at t = 2 and r =1 is shown in the following table and figure. Table 1 represents the numerical solution obtained by the fourth order Runge- Kutta method based on centroidal mean.

Table 1. Numerical solution obtained by RK4CeM, Euler and exact

α	RK4CeM		Euler		Exact	
	$\underline{y}(t)$	$\overline{y}(t)$	$\underline{y}(t)$	$\overline{y}(t)$	$\underline{y}(t)$	$\overline{y}(t)$
0.0	-2.125014	2.125014	-2.112500	2.112500	-2.125000	2.125000
0.1	-1.912513	1.912513	-1.901250	1.901250	-1.912500	1.912500
0.2	-1.700011	1.700011	-1.690000	1.690000	-1.700000	1.700000
0.3	-1.487510	1.487510	-1.478750	1.478750	-1.487500	1.487500
0.4	-1.275008	1.275008	-1.267500	1.267500	-1.275000	1.275000
0.5	-1.062507	1.062507	-1.056250	1.056250	-1.062500	1.062500
0.6	-0.850006	0.850006	-0.845000	0.845000	-0.850000	0.850000
0.7	-0.637504	0.637504	-0.633750	0.633750	-0.637500	0.637500
0.8	-0.425003	0.425003	-0.422500	0.422500	-0.425000	0.425000
0.9	-0.212501	0.212501	-0.211250	0.211250	-0.212500	0.212500
1.0	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

For the purpose of comparison, the numerical solution obtained by the Euler method and the exact solution are also presented in the table. Table 2 gives the absolute error between the numerical solution obtained by the fourth order Runge-Kutta method based on the centroidal mean and exact solution also the absolute error between the Euler method and exact solution. Figure 1 represent the comparison between the exact solution and the proposed method.

The following figure gives the graphical representation of the exact solution and the approximate solution obtained by the proposed method for the Fuzzy time delay Malthusian growth model.

Table 2. Absolute error obtained for RK4CeM and Euler method

α	RK4CeM		Euler	
	$\underline{y}(t)$	$\overline{y}(t)$	$\underline{y}(t)$	$\overline{y}(t)$
0.0	1.407815e-05	1.407815e-05	1.250000e-02	1.250000e-02
0.1	1.267033e-05	1.267033e-05	1.125000e-02	1.125000e-02
0.2	1.126252e-05	1.126252e-05	1.000000e-02	1.000000e-02
0.3	9.854703e-06	9.854703e-06	8.750000e-03	8.750000e-03
0.4	8.446888e-06	8.446888e-06	7.500000e-03	7.500000e-03
0.5	7.039074e-06	7.039074e-06	6.250000e-03	6.250000e-03
0.6	5.631259e-06	5.631259e-06	5.000000e-03	5.000000e-03
0.7	4.223444e-06	4.223444e-06	3.750000e-03	3.750000e-03
0.8	2.815629e-06	2.815629e-06	2.500000e-03	2.500000e-03
0.9	1.407815e-06	1.407815e-06	1.250000e-03	1.250000e-03
1.0	0.000000e+00	0.000000e+00	0.000000e+00	0.000000e+00

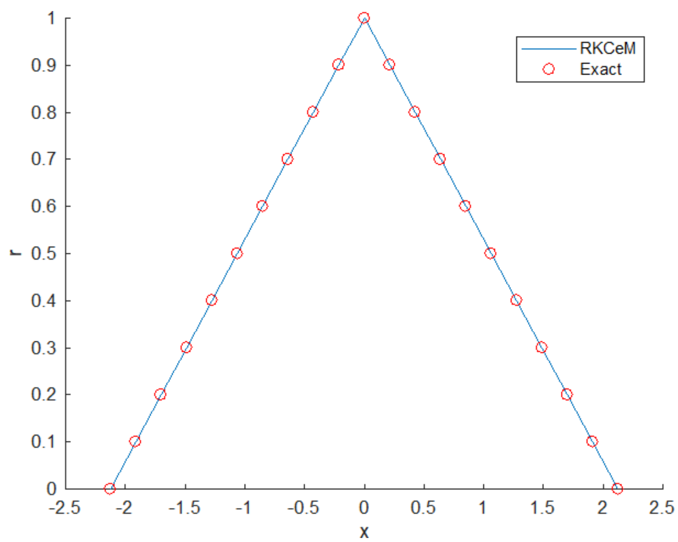

Fig 1. Comparison between the approximate solution of RK4CeM and exact values for h=0.1

5 Conclusion

In this work, the Runge-Kutta method based on Centroidal Mean is used to find the approximate solution of fuzzy time delay Malthusian growth model which is a FDDE with constant lag. The numerical algorithm is proposed with modifications to the Runge Kutta method based on the Centroidal Mean to effectively handle fuzziness and delay terms in the Fuzzy Time delay Malthusian growth model. The approximate solution obtained by the proposed method is compared with the exact solution in Table 1. It shows the numerical result converges to the exact solution To verify the accuracy of the proposed method the approximate solution is compared with the solution obtained by the Euler method. The absolute errors obtained by the proposed method and the Euler method are also compared in Table 2. It shows the proposed method gives better accuracy than the Euler method. In Figure 1 the approximate solution is overlapped with the exact solution which means the approximate solution is more accurate. This study presents an innovative algorithm that modifies the Runge-Kutta method based on the Centroidal Mean to effectively solve the Fuzzy time delay Malthusian growth model. The proposed modifications enable the method to manage the challenges posed by fuzziness and delays, with comparative analysis showing its advantages over the Euler method. The proposed method can be used for different types of fuzzy delay differential equations in the future.

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