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Cubic Structure on Soft Gamma-m-Normed Linear Space

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Abstract

Objectives: The purpose of this research is to take the lead in comprehending the fuzzy idea of cubic structure on soft Gamma-m normed linear space. According to the theory of Soft m-Normed Linear Space (SMNLS), we offer the conviction of Cauchy's sequence and convergence in cubic Soft Gamma-m Normed Linear Space (CSGMNLS). We have obtained certain results like the concept of completeness in CSGMNLS. **Methods:** In this paper we defined the soft gamma ring, soft gamma ideals and soft gamma vector space which are used to introduce the notion of soft gamma-2-normed linear space, Soft Gamma-m-Normed Linear Space (SGMNLS) and its properties. Also, the CSGMNLS can be analyzed by using the SGMNLS. **Findings:** In this research from CSGMNLS construct a norm function that satisfies the properties of SGMNLS, and additionally given that example with proof in which a sequence is cauchy sequence and convergence in SGMNLS if it is cauchy and convergence sequence in CSGMNLS. Also, provided theorem and its proof for completeness of a sequence in CSGMNLS. **Novelty:** Already gamma ring and fuzzy n-normed linear space has been defined. We introduced the concept of SGMNLS using this also initiated the CSGMNLS and some results obtained from its properties. We suggested a necessary conditions for completeness of a sequence in CSGMNLS.

Keywords: Soft Gamma ring; Soft gamma vector space; Soft gamma normed linear space; Soft m-norm; Soft m-normed linear space; 2-Normed and m-normed right soft gamma linear space

1 Introduction

In 1999, D. Molodtsov[1] spearheaded the use of soft sets to address uncertainty-filled problems in fields including economics, environmental science, medicine, and so forth. Complex problems involving probability theory cannot be resolved by classical methods, and Vagus set theory and soft set theory are specific cases of Lotif Aliasker Zadeh's[2] fuzzy sets. Intact set theory Molodtsov took various examples into consideration to demonstrate this point.

Subsequently, Maj. et al.[3] presented some useful operations on soft sets[4] and widely it has been applied to the solutions in decision-making problems[5]. H. Aktos and N. Camgmas proposed the definition of the soft group and its features[6,7]. Soft semi rings were developed and studied by Feng et al.[8] as soft ideals over a semi group that characterize generalized fuzzy ideals[9]. Initial concepts of soft rings and definitions of softsub-ring were introduced by Acar et al.[10]. Sen and Saha in 1986 have proposed the Gamma semi-group and Gamma-group[11]. The definition of a gamma ring and several novel ideas related to it, including prime and primary ideals, gamma-homomorphism, and the m-system created by Barnas[12,13], Nobusawa presented the idea of a gamma-ring, which is more broadly defined than a ring.

Gahler[14] introduced 2-metric and proposed the mathematical structure known as 2-normed spaces as a generalization of normed linear spaces Interval valued fuzzy sets make up the notation cubic set theory that Jun et al.[15,16] established[17]. The theory of intuitionistic n-normed linear space, interval valued fuzzy linear space, fuzzy n-normed linear and soft normed spaces introduced in[18,19]. Reddy, B.S.[20] proposed fuzzy-anti-n-norm on linear space. Fuzzy-anti-2-norm was investigated, and certain results were established in cubic gamma n-normed linear space[21]. The study also examined the Cauchy and convergent sequence conviction in fuzzy-anti-n-normed linear space. Vijaya Balaji[22] created left gamma linear spaces that are both 2-normed and n-normed and provided some results on it[23].

Motivated by the aforementioned theory, we present the concept of cubic structure in soft gamma-m-normed linear spaces and delineate the convergent and cauchy sequences inside them.

2 Methodology

2.1. Definition: Let U be a universe and M be a set of parameters. Let P(U) represents the power set of U and S is the non-empty subset of M. A pair (X, S) is called a soft set over U, where X is a mapping given by $X:S \to P(U)$. That is a soft set over U is a parameterized family of subsets of the universe U. For $\delta \in S$, $X(\delta)$ may be considered as the set of δ – approximate elements of the P(U) soft set (X, S).

As an illustration, let us consider the following example.

Example: Let (X, S) be the soft set which describes the simplicity of the houses which Mr. Ramaswamy is going to purchase. U is the set of all houses and M be the set of parameters {high price, simple furnishing, wooden, in low price, modern, in good looking, in bad repair}. In this case, to define a soft set means to point out the high price, modern, simple houses, and so on. It denotes that the sets $X(\delta)$ may be empty for some $\delta \in S$.

2.2. Definition: For two soft sets (X_1, S_1) and (X_2, S_2) over a common universe U, then (i) (X_1, S_1) is a soft subset of (X_2, S_2) if $S_1 \subseteq S_2$ and for all $\delta \in S$, $X_2(\delta) \subseteq X_2(\delta)$. We write $(X_1, S_1) \subset \delta\ (X_2, S_2)$ (ii) (X_2, S_2) is said to be a soft superset of (X_1, S_1) if (X_2, S_2) is a soft subset of (X_1, S_1) We denote it by $(X_1, S_1) \supset (X_2, S_2)$ (iii) If (X_1, S_1) is a soft subset of (X_2, S_2) and (X_2, S_2) is a soft subset of (X_1, S_1), then two soft sets (X_1, S_1) and (X_2, S_2) over a common universe U are equivalent.

2.3. Definition: A soft set (X, S) has a complement represented by (X^C, S),

Where $X^C:S \to P(U)$ is a mapping such that $X^C(s) = U - X(s)$ for all $s \in S$.

2.4. Definition: The union of two soft sets (X_1, S_1) and (X_2, S_2) with the common universe U is the soft set (X, S), where $S = S_1 \cup S_2$. For every $s \in S$,

$X(s) = X_1(s)$, if $s \in S_1 - S_2$

$= X_2(s)$, if $s \in S_2 - S_1$

$= X_1(s) \cup X_2(s)$, if $s \in S_1 \cap S2$.

We express it as $(X_1, S_1) \cup (X_2, S_2) = (X, S)$.

2.5. Definition: Let U is the common universe of two soft sets (X_1, S_1) and (X_2, S_2) and intersection of these two sets is the soft set (X, S), if (i) $S = S_1 \cap S_2$, (ii) $X(s) = X_1(s) \cap X_2(s)$, for all $s \in S_1 \cap S_2$. This relation is denoted by $(X_1, S_1) \cap (X_2, S_2) = (X, S)$

Remark: The fact that $(X_1, S_1) \cap (X_2, S_2) = (X, S)$ does not exist in many circumstances was highlighted as evidence that the concept of intersection of soft sets is not well-defined. For this, the following example is given.

Example : Let (S_1, M_1) and (S_2, M_2) be two soft sets, and let (S, M) be the soft set which is intersection of two sets (S_1, M_1), (S_2, M_2) where U is a set of cars; $U = \{C^1, C^2, C^3, C^4, C^5, C^6\}$ and M_1 = {Luxurious, Attractive color, Driving comfort}, and M_2 = {Fuel efficiency, Driving comfort, More safety} two parameter sets. Noticing the ε-approximate elements may differ from person to person, we assume that S_1 (Luxurious) = $\{C^2, C^3, C^5\}$, S_1 (Driving Comfort) = $\{C^1, C^2, C^4, C^6\}$, S_1 (Attractive color) = $\{C^1, C^2, C^6\}$, S_2 (Fuel efficiency) = $\{C^1, C^3, C^5\}$ S_2 (Driving comfort) = $\{C^2, C^3, C^5, C^6\}$, S_2 (more safety) = $\{C^2, C^4, C^6\}$, we have "Driving comfort" $\in M_1 \cap M_2$ then S_1 (Driving Comfort) = $\{C^1, C^2, C^4, C^6\}$, S (Driving Comfort) and S_2 (Driving comfort) = $\{C^2, C^3, C^5, C^6\}$= S_2 (Driving comfort) which is a contradiction to the fact that $(S_1, M_1) \cap (S_2, M_2)$ does not exist in many cases makes it is not possible to check the validity of some of the assertions.

2.6. Definition: Assume that M is any nonempty set with an arbitrary binary relation of σ between a member of M and an element of G. As a result, σ is a subset of (M,G) and, unless otherwise indicated by a set valued function S_G: M→P(G), which is defined as follows: S_G (m)={n ∈G/n ∝m⇔(n, m) ∈σ for all m ∈M}. Consequently, if the pair (S_G M) is a soft set over G that is generated from the relation σ.

2.7. Definition: Support of a set is indicated by supp(S_G, M) and it is defined as for any m ∈ M such that S_G (m) is nonempty, that is supp(S_G, M)={m ∈M / S_G (m) $\neq\emptyset$}.

2.8. Definition: Suppose that G is group and P(G) collection of subsets of G and let M be the set of parameters. We denote the identity element in G by the symbol e_G.

2.9. Definition: Let (S_G, M) be a soft set over G then (S_G, M) is called a soft group over G if and only if S_G (m) is a sub group of G for all m ∈ M.

Let (S_{1G}, M_1) and (S_{2G}, M_2) be two soft subgroups over G then (S_{1G}, M_1) is a soft sub group of (S_{2G}, M_2) if (i) $M_1 \subset M_2$ (ii) S_1(m) $\prec$ S_2(m) it is denoted by (S_{1G}, M_1) $\prec$ (S_{2G}, M_2).

Example: Let S={1,2,3,4,5} then S_5 is the symmetry group of all permutations with elements of S. let M be any subgroup of S_5, the subgroup $M$is cyclic and it is generated by an element of order 5, since 5 is prime number $M = \{m_e = (1), m_1 = (12354), m_2 = (13425), m_3 = (15243), m_4 = (14532)\}$. Consider the function:

$S_G^p(m_1) = \{m_e, m_1, m_2, m_3, m_4\}$, $S_G^p(m_2) = \{m_e, m_1, m_2, m_3, m_4\}$, $S_G^p(m_3) = \{m_e, m_1, m_2, m_3, m_4\}$, $S_G^p(m_4) = \{m_e, m_1, m_2, m_3, m_4\}$. *All are subgroup of* M^p.

2.10. Definition: Let (S_R, M) be a non-null soft set in R then (S_R, M) is called a soft ring over R iff S_R (M) is a sub-ring of R for all m∈ M.

Example: Let R=Z_{17}=({0,1,2,3,4,5,6,7,8,9,1},$+_{12}$,x_{12}) be a ring under modulo 17.

S_R(m)={r ∈ R /r x_{12}m=0},then the following all are soft sub-ring of R

S_R(0)=S_R(1)= S_R(5)= S_R(7)= S_R(11)={0}, S_R(2)= S_R(10)={0,6}, S_R(3)= S_R(9)={0,4,8}, S_R(4)= S_R(8)={3,6,9}, S_R(6)={0,2,4,6,8,10}, S_R(8)={0,3,8,9}.

2.11. Definition: Let S_G is soft additive abelian group (SAAG), and Γ be any additive group the mapping $\xi : s_G \times \Gamma \times s_G \to s_G$ and it is defined as $\xi(s_{g1}, \gamma, s_{g2}) = s_{g1}.\gamma.s_{g2}$ such that S_G is a soft Gamma Γ-ring if it is satisfying the following properties let for any s_{g1},s_{g2}, $s_{g3} \in S_G$ and $\gamma_1, \gamma_2 \in \Gamma$

$(S_\gamma - 1)\, \xi(s_{g1} + s_{g2}, \gamma_1, s_{g3}) = \xi(s_{g1}, \gamma_1, s_{g3}) + \xi(s_{g2}, \gamma_1, s_{g3})$

$(S_\gamma - 2)\, \xi(s_{g1}, \gamma_1 + \gamma_1, s_{g2}) = \xi(s_{g1}, \gamma_1, s_{g2}) + \xi(s_{g1}, \gamma_2, s_{g2})$

$(S_\gamma - 3)\, \xi(s_{g1}, \gamma_1, s_{g2} + s_{g3}) = \xi(s_{g1}, \gamma_1, s_{g2}) + \xi(s_{g1}, \gamma_1, s_{g3})$

$(S_\gamma - 4)\, \xi((s_{g1}, \gamma_1, s_{g2}), \gamma_2, s_{g3}) = \xi(s_{g1}, \gamma_1, (s_{g2}, \gamma_2, s_{g3}))$

2.12. Definition: Let S^i_G be any subset of the soft Gamma-ring S_G is right ideal or left ideal of S_G if S^i_G is abelian additive soft subgroup of S_G and $S_G\gamma S^i{}_G = \{s_g\gamma s^i{}_g / s_g \in S_G, s^i{}_g \in S^i{}_G \, and \, \gamma \in \Gamma\}$

or $S^i{}_G\gamma S_G = \left\{\frac{s^i{}_g\gamma s_g}{s_g} \in S_G, s^i{}_g \in S^i{}_G \, and \, \gamma \in \Gamma\right\}$ is contained in S^i_G. if S^i_G is both a left and right soft ideal of S_G and S^i_G is called an ideal or two sided ideal of S_G.

2.13. Definition: If the soft Gamma-ring S_D contains an identity element and its only non-zero ideal itself then it represents a division soft Gamma-ring.

2.14: Definition: Let V be a vector space over the F⊆ℝ field, and let M be the real number set as the parameter set. Let a soft set over V be (S_V, M). If exactly one m∈M such that V(m)={v}, then the soft set (S_V, M) is a soft vector. In other words, let V is a vector space in the field F⊆ℝ, and let M be the real number set as the parameter set. Let S_V be a soft set over (V, M). If S_V(m) is a vector subspace of V for any m∈M, then S_V be the soft vector space or soft linear space of V in F.

Example: suppose R is an Euclidean n-dimensional space over ℝ, let D={1,2,3,…….,r} be the set of parameters let S_V:D→P($\mathbb{R}^n$) be defined as follows S_V(v)={v∈$\mathbb{R}^n$, k^{th} co-ordinate of S_V is zero, k=1,2,3,…,n}, $s_v^{(k)}$=(1,1,1,…..0-k^{th},1,1,….1)∈$\mathbb{R}^n$, k=1,2,3,…,r then $s_v^{(k)}$ is soft vector.

2.15. Definition: Let (S_V,+) be a soft abelian group and let S_D be a soft division Gamma-ring with identity element and the function $\psi : s_D \times \Gamma \times s_V \to s_V$ and is defined as $\psi(s_v, \gamma, s_d) = s_v.\gamma.s_d$ then S_V be a right soft Gamma-vector space over S_D if the following properties holds for all $s_{v1}, s_{v2} \in S_V$, $s_{d1}, s_{d2} \in S_D$ and $\gamma_1, \gamma_2 \in \Gamma$

$(S_V - 1)\, \psi(s_{v1} + s_{v2}, \gamma, s_{d1}) = \psi(s_{v1}, \gamma, s_{d1}) + \psi(s_{v2}, \gamma, s_{d1})$

$(S_V - 2)\, \psi(s_{v1}, \gamma, s_{d1} + s_{d2}) = \psi(s_{v1}, \gamma, s_{d1}) + \psi(s_{v1}, \gamma, s_{d2})$

$(S_V - 3)\, \psi(s_{v1}, \gamma_1, (s_{d1}, \gamma_2, s_{d2})) = \psi((s_{v1}, \gamma_1, s_{d1}), \gamma_2, s_{d2})$

$(S_V - 4)\, \psi(s_{v1}, \gamma, 1) = s_{v1}, \, for \, some \, \gamma \in \Gamma$

The elements s_{v1}, s_{v2} are called soft vectors in S_V,$s_{d1}, s_{d2}$are called soft scalars in S_D.

2.16. Definition: Let S_V be a is a right soft Gamma-vector space over S_D for any $\gamma \in \Gamma$ then set of vectors $\{s^t{}_{vj}/j\ is\ positive\ integer\}$ is called a Gamma-independent over S_D, if for each finite subset of vectors $(s^t{}_{v1}, s^t{}_{v2}, s^t{}_{v3}, ..., s^t{}_{vm-1}, s^t{}_{vm}\}$ such that $s^t{}_{v1}\gamma s_{d1} + s^t{}_{v2}\gamma s_{d2} + s^t{}_{v3}\gamma s_{d3} +, ... + s^t{}_{vm-1}\gamma s_{dm-1} = 0 \Leftrightarrow s^t{}_{v1} = s^t{}_{v2} = s^t{}_{v3} = ... = s^t{}_{vm-1} = s^t{}_{vm} = 0$

2.17. Definition: Let S_V is a linear space over a soft field F a real valued function

$\|, ..., \|$: $S_V \times S_V \times S_V ..... \times S_V (m\ times) \to \{0,1\}$ and it has the following properties

($S_N{}^m$-1) $\| s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm} \| = 0$ if and only if $s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}$ are linearly independent over F.

($S_N{}^m$-2) $\| s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm} \|$ is invariant under any permutation $s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}$.

$(S_N^m - 3)$ $\| s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, ks_{vm} \| = k \| s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm} \|$

($S_N{}^m$-4) $\| s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm} + s'_{vm} \| \le \| s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm} \| + \| s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s'_{vm} \|$ the pair

$(S_V, \| ., ..., . \|)$is referred to as the soft m-normed linear space and is called the soft m-norm on S_V.

2.18. Definition: A sequence $\{s_{vk}\}_{k=1}^{\infty}$ in a SMNLS $(S_V, \| ., ..., . \|)$ is said to convergence to $s_v \in S_V$ if. $\lim\limits_{k\to\infty} \| s_{v1}, s_{v2}, s_{v3}, ..., s_{vk-1}, s_{vk} - s_v \| = 0$

A Cauchy sequence is defined as $\{s_{vk}\}_{k=1}^{\infty}$ in a SMNLS $((S_V, \| ., ..., . \|)$ if $\lim_{k,p\to\infty} \| s_{v1}, s_{v2}, s_{v3}, ..., s_{vk-}1, s_{vk} - s_{vp} \| = 0$.

2.19. Definition: If every Cauchy sequence in a SMNLS $(S_V, \| ., ..., . \|)\|)$ is convergent, then the space is a complete.

2.20. Definition: A binary operation $S_\Delta : [0,1] \times [0,1] \to [0,1]$ is continuous triangular norm if it satisfies the following axioms for every t_1, t_2, t_3, & $t_4 \in$ [0,1]

$(\Delta - 1)$ S_Δ is associative and commutative,

$(\Delta - 2)$ S_Δ is continuous,

$(\Delta - 3) t_1 S_\Delta 1 = t_1$,

$(\Delta - 4)$ $t_1 S_\Delta t_2 \le t_3 S_\Delta t_4\ whenever\ t_1 \le t_3\ and\ t_2 \le t_4$.

2.21. Definition: A binary operation $S_{\Delta^c} : [0,1] \times [0,1] \to [0,1]$ is continuous triangular co-norm if it satisfies the following axioms for every t_1, t_2, t_3, & $t_4 \in$ [0,1]

$(\Delta^c - 1) S_{\Delta^c}$ is associative and commutative,

$(\Delta^c - 1)$ S_{Δ^c} is continuous,

$(\Delta^c - 3) t_1 S_{\Delta^c} 1 = t_1$,

$(\Delta^c - 4)$ $t_1 S_{\Delta^c} t_2 \le t_3 S_{\Delta^c} t_4\ whenever\ t_1 \le t_3\ and\ t_2 \le t_4$.

Example: For any t_1, t_2, $t_3 \in$ [0,1] then $t_1 S_\Delta t_2 = min\{t_1, t_2\}\ and\ t_1 S_{\Delta^c} t_2 = max\{t_1, t_2\}$.

2.22. Definition: Let S_V linear space over a soft field F. A soft subset N_S of S_VxS_V x……S_VxS_V(m-times)xℝ and the pair (S_V, N_S) is called SMNLS. A soft m-norm on S_V if and only if

$(S_M - 1)$ $N_S(s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}, s_t) = 0\ for\ all\ s_t \le 0 and\ s_t \in \mathbb{R}$

$(S_M - 2)$ $N_S(s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}, s_t) = 1\ if\ and\ only\ if s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}$ $are\ linearly\ dependent\ if\ for\ all\ s_t \ge 0\ and\ s_t \in \mathbb{R}$.

$(S_M - 3)$ $N_S(s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}, s_t)\ is\ invariant\ under\ any\ permutation\ of\ s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}$.

$(S_M - 4)$ $N_S(s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_\rho s_{vm}, s_t) = N_S(s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}, \frac{s_t}{|s_\rho|}), s_\rho \ne 0$

$(S_M - 5)$ $N_S(s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm} + S'_{vm}, s_{t1} + s_{t2}) \ge$ $min\{N_S(s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}, s_{t1}), N_S(s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s'_{vm}, s_{t2})\}$

$(S_M - 6)$ $N_S(s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}, s_t)$

is a left continuous and non-decreasing function of$s_t \in \mathbb{R}\ such\ that\ \lim\limits_{m\to\infty} N_S(s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}, s_t) = 1$.

Example: let $(S_V, \| ., ..., . \|)$ be a SMNLS define

$N_S(s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}, s_t) = \frac{s_t - \| s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm} \|}{s_t + \| s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm} \|}$

= 1, if $s_t > 0$ and $s_t \in \mathbb{R}$ for all $s_{vj} \in S_V$

=0, if $s_t \le 0$ and $s_t \in \mathbb{R}$, for all $s_{vj} \in S_V$. Then (S_V, N_S) is a SMNLS.

2.23. Definition: Let S_V linear space over a soft field F a soft subset $N_S{}^c$ of S_VxS_V x…….S_VxS_V(m-times)xℝ and the pair (S_V, $N_S{}^c$) is called soft anti m-normed linear space and S_VxS_V x…….S_VxS_V(m-times)xℝ is called as a soft anti m-norm on S_V if and only if

$(S_{M^c} - 1)$ $N_S c(s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}, s_t) = 1\ for\ all s_t \le 0\ and\ s_t \in \mathbb{R}$

$(S_{M^c} - 2)$ $N_{SC}(s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}, s_t) = 0\ if\ and\ only\ if$

$s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}\ are\ linearly\ dependent\ if\ for\ all$

$s_t > 0\ with\ s_t \in \mathbb{R}$

$(S_{M^C}-3)\, N_{S}c(s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}, s_t)\, is\, invariant\, under\, any\, permutation\, of\, s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}.$
$(S_{M^C}-4)\, N_{S^C}(s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_\rho s_{vm}, s_t) = N_{S^C}(s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}, \frac{s_t}{|s_\rho|}), s_\rho \neq 0$

$(S_{M^C}-5)\, N_{S^C}(s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}+s'_{vm}, s_{t1}+s_{t2}) \leq$
$max\{N_{S^C}(s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}, s_{t1}),\, N_{S^C}(s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s'_{vm}, s_{t2})\}$

$(S_{M^C}-6) N_{S^C}(s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}, s_t)$ is a non-decreasing and right continuous function s_t ∈ℝ such that $\lim_{m\to\infty} N_{S^C}(s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}, s_t) = 0$.

Example: let $(S_V, \| ., .., . \|)$be a SMNLS define

$N_{S^c}(s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}, s_t) = \frac{2\|s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}\|}{s_t + \|s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}\|}$=0, if $s_t > 0$with s_t ∈ℝ, for all $s_{vj} \in S_V$=1, if $s_t \leq 0$ with s_t ∈ℝ, for all $s_{vj} \in S_V$

Then (S_V, N_{S^c}) is a soft anti m-normed linear space.

3 Results and Discussions

3.1. Definition: Let S_V be a Gamma linear space in S_D any real valued function $\| .., .. \|: S_V \times S_V \to [0, \infty)$ is a soft Gamma-2-normed linear space over S_D and is represented as $(S_V, \| ., .., . \|)$ if each of the following properties is met for all $s_{v1}, s_{v2} \in S_V$, $s_{d1}, s_{d2} \in S_D$ and $\gamma \in \Gamma$

$(S_\gamma{}^2\text{-}1)$ $\| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2} \| = 0 \Leftrightarrow s_{v1}, s_{v2}$linearly independent over S_D

$(S^2_\gamma - 2)$ $\| s_{v1}\gamma s_{d1}, \tau(s_{v2}\gamma s_{d2}) \| = |\tau| \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2} \|\, for\, any\, \tau \in \Gamma$

$(S^2_\gamma - 3)$ $\| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2} + s_{v3}\gamma s_{d3} \| \leq \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2} \| + \| s_{v1}\gamma s_{d1}, s_{v3}\gamma s_{d3} \|$

3.2. Definition: Let S_V is a right Gamma linear space in S_D a real valued function$\| ..., ... \|$: $S_V \times S_V ....S_V \times S_V(m times) \to [0, \infty)$ is referred to be soft right Gamma-m-normed linear space over S_D if it meets the following criteria, for any $s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm} \in S_V$,$s_{d1}, s_{d2}, s_{d3}, ..., s_{dm-1}, s_{dm} \in S_D$ and $\gamma \in \Gamma$, It is denoted by $(S_V, \| ., ..., . \|)$

$(S_\Upsilon{}^m\text{-}1)$ $\| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} \| = 0 \Leftrightarrow s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}$ linearly independent over S_D.

$(S_\Upsilon{}^m\text{-}2)$ $\| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} \|$is invariant under any permutation of $s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}$.

$(S_\Upsilon{}^m\text{-}3)$ $\begin{array}{l}\| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, \tau(s_{vm}\gamma s_{dm}) \| \\ = |\tau| \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} \|\, for\, any\, \tau \in \Gamma\end{array}$.

$(S^m_\Upsilon - 4)$ $\| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm} + s_{vm}\gamma s_{dm} \| \leq$
$\| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} \| +$
$\| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm} \|$

In the similar way, we define soft left Gamma m-normed linear space over S_D.

3.3. Definition: Let S_V is a right or left Gamma linear space in S_D and a real valued function $\| ..., ... \|$: $S_V \times S_V ....S_V \times S_V(m times) \to [0, \infty)$ is said to be soft Gamma m-normed linear space (SGMNLS) over S_D if it is either soft right Gamma m-normed linear space or soft left Gamma-m-normed linear space over S_D.

3.4. Definition: Let S be a non-empty set, a cubic set C in a set S is structure $C = \{< s, \mu_C(s), \sigma(s) > / s \in S\}$ it is elliptically denoted by $C =< \mu_C, \sigma >$where $\mu_C \in [\mu_{C^-}, \mu_{C^+}]$ is an interval valued fuzzy set in S and $\sigma : S \to \{0, 1\}$ is soft set in S.

3.5. Definition: Let S_V is a soft linear space in a field F and (S_V, μ) be the interval valued soft linear space and (S_V, μ') be a soft linear space of S_V. A cubic set $C =< \mu, \mu' >$ in S_V is represents a cubic linear space of S_V if for every $s_{v1}, s_{v2} \in S_V$ and π_1, π_2∈F

(i) $\mu(\pi_1 s_{v1} S_\Delta \pi_2 s_{v2}) \geq min\{\pi_1 s_{v1}, \pi_2 s_{v2}\}$

(ii) $\mu'(\pi_1 s_{v1} S_{\Delta^c} \pi_2 s_{v2}) \leq max\{\pi_1 s_{v1}, \pi_2 s_{v2}\}$

3.6. Definition: In a SGMNLS, $(S_V, \| ., ..., . \|)$ a sequence $\{s_{vk}\gamma s_{dk}\}_{k=1}^\infty$ is convergence to $s_v\gamma s_d$, for any $s_v\gamma s_d \in S_V$ if $\lim_{k\to\infty} \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vk-1}\gamma s_{dk-1}, s_{vk}\gamma s_{dk} - s_v\gamma s_d \| = 0$.

3.7 Definition: A sequence $\{s_{vk}\gamma s_{dk}\}_{k=1}^\infty$ in SGMNLS, $(S_V, \| ., ..., . \|)$ is a Cauchy sequence if
$\lim_{k,l\to\infty} \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vk-1}\gamma s_{dk-1}, s_{vk}\gamma s_{dk} - s_{vl}\gamma s_{dl} \| = 0$.

3.8 Definition: If every Cauchy sequence in a SGMNLS $(S_V, \| ., ..., . \|)$ is convergent, the space is complete.

3.9. Definition: Let S_V be a SGMNLS over S_D a real valued function $N_\gamma : S_V \times S_V \times S_V ..... \times S_V(m times) \to \{0, 1\}$ and $N_{\gamma^c} : S_V \times S_V \times S_V ..... \times S_V(m times) \to \{0, 1\}$ is soft set and an interval valued fuzzy set, and the cubic structure $(S_V, N_\gamma, N_{\gamma^c})$ is a cubic soft Gamma m-normed linear space and it is elliptically CSGMNLS is called if it meets the following requirements, for any $s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm} \in S_V$, $s_{d1}, s_{d2}, s_{d3}, ..., s_{dm-1}, s_{dm} \in S_D$ and $\gamma \in \Gamma$

$(S_{M^\gamma}-1)\ N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_t) = 1$
$(S_{M^\gamma}-2) N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_t) = 1$ if and only if $s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}$ are linearly dependent.
$(S_{M^\gamma}-3)\ N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_t)$is invariant under any permutation of $s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}$
$(S_{M^\gamma}-4)\ N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_p(s_{vm}\gamma s_{dm}), s_t) = N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, \frac{s_t}{|s_\rho|}), s_\rho \neq 0\, and\, s_\rho \in \Gamma$
$(S_{M^\gamma}-5)\ N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} + s'_{vm}\gamma s_{dm}, s_{t1}+s_{t2}) \geq \{N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_{t1})$.
$S_\Delta N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm}, s_{t2})\}$
$(S_{M^\gamma}-6)\ N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_t)$ is non-increasing and right continuous of function $s_t \in \mathbb{R}$such that $\lim_{m\to\infty} N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_t) = 1$
$(S_{M^\gamma}-7)\ N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_t) = 0.$
$(S_{M^\gamma}-8)\ N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_t) = 0$
if and only of $s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}$ are linearly dependent.
$(S_{M^\gamma}-9)\ N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_t)$is invariant under any permutation of $s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}$.
$(S_{M^\gamma}-10)\ N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_p(s_{vm}\gamma s_{dm}),\ s_t) = N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, \frac{s_t}{|s_\rho|}), s_\rho \neq 0\, and\, s_\rho \in \Gamma$
$(S_{M^\gamma}-11)\ N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} + s'_{vm}\gamma s_{dm}, s_{t1}+s_{t2})$
$\leq \{N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_{t1})$
$S_{\Delta^C} N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm}, s_{t2})\}$
$(S_{M^\gamma}-12)\ N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_t)$
is left continuous and non-decreasing function of $s_t \in \mathbb{R}$such that
$\lim_{m\to\infty} N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_t) = 0.$

3.10. Definition: Let $(S_V, \|\ .,....,.\ \|)$be a SGMNLS over S_V define that For any $t_1, t_2 \in [0,1]$ then $t_1 S_\Delta t_2 = min\{t_1, t_2\} and t_1 S_{\Delta^c} t_2 = max\{t_1, t_2\}$. Also define

$$N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_t) = \frac{s_t - \|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \cdots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}{s_t + \|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \cdots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}$$, and

$$N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_t) = \frac{2\|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \cdots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}{s_t + \|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \cdots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}$$

Then we prove that the cubic structure $(S_V, N_\gamma, N_{\gamma^c})$ is a CSGMNLS

$(S_{M^\gamma}-1)$ for every $s_t > 0$, with $s_t \in\mathbb{R}$, such that

$$N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_t) = 0$$
$$\Leftrightarrow \frac{s_t - \|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \cdots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}{s_t + \|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \cdots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|} = 1$$

$(S_{M^\gamma}-2)\ N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_t) = 1$

$$\Leftrightarrow [s_t - \|\ s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\ \|]$$
$$= [s_t + \|\ s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\ \|]$$
$$\Leftrightarrow \|\ s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\ \|] = 0$$

$\Leftrightarrow s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}$ are linearly dependent.

$(S_{M^\gamma}-3)\ N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_t)$

$$= \frac{s_t - \|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \cdots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}{s_t + \|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \cdots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}$$
$$\Leftrightarrow \frac{s_t - \|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \cdots, s_{vm}\gamma s_{dm}, s_{vm-1}\gamma s_{dm-1}\|}{s_t + \|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \cdots, s_{vm}\gamma s_{dm}, s_{vm-1}\gamma s_{dm-1}\|} =$$
$$N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm}\gamma s_{dm}, s_{vm-1}\gamma s_{dm-1}, s_t) = \ldots\ldots\ldots so\ on,$$

This proves the invariance under any permutation of $s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}$.

$(S_{M^\gamma}-4)$ Now we consider LHS

$$N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_\rho(s_{vm}\gamma s_{dm}), s_t)$$
$$= \frac{s_t - \|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \cdots, s_{vm-1}\gamma s_{dm-1}, s_\rho(s_{vm}\gamma s_{dm})\|}{s_t + \|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \cdots, s_{vm-1}\gamma s_{dm-1}, s_\rho(s_{vm}\gamma s_{dm})\|}$$

$\Leftrightarrow \frac{s_t - s_\rho \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}{s_t + s_\rho \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}$

$\Leftrightarrow \frac{\frac{s_t}{s_\rho} - \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}{\frac{s_t}{s_\rho} + \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}$

$N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, \frac{s_t}{|s_\rho|}) = RHS,\ s_\rho \in \Gamma$

$(S_{M^\gamma} - 5)$ Without loss of generality, we consider

$N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm}, s_{t2}) \leq$
$N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_{t1})$

$\Leftrightarrow \frac{s_{t2} - \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm}\|}{s_{t2} + \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm}\|}$
$\leq \frac{s_{t1} - \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}{s_{t1} + \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}$

$\Leftrightarrow [s_{t2} - \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm} \|]$
$[s_{t1} + \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} \|]$
$\leq [s_{t1} - \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}]$
$[s_{t2} + \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm} \|]$

$\Leftrightarrow s_{t1}s_{t2} - s_{t1} \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm} \|$
$+ s_{t2} \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} \|$
$- \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} \|$
$. \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm} \|$
$\leq s_{t1}s_{t2} + s_{t1} \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm} \|$
$- s_{t2} \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} \|$
$- \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} \|$
$. \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm} \|$

$\Leftrightarrow \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} \|$
$\leq \frac{s_{t1}}{s_{t2}} \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm} \|$

$\Leftrightarrow \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} \|$
$+ \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm} \|$
$\leq (\frac{s_{t1}}{s_{t2}} + 1) \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm} \|$
$= (\frac{s_{t1}+s_{t2}}{s_{t2}}) \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm} \|$

$\Leftrightarrow \frac{\| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} + s'_{vm}\gamma s_{dm}\|}{s_{t1}+s_{t2}}$
$\leq \frac{\| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm}\|}{s_{t2}}$

$\Leftrightarrow 1 + \frac{\| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} + s'_{vm}\gamma s_{dm}\|}{s_{t1}+s_{t2}}$
$\leq 1 + \frac{\| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm}\|}{s_{t2}}$

$\Leftrightarrow \frac{s_{t1}+s_{t2}+\| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} + s'_{vm}\gamma s_{dm}\|}{s_{t1}+s_{t2}}$
$\leq \frac{s_{t2}+\| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm}\|}{s_{t2}}$

$\Leftrightarrow \frac{(s_{t1}+s_{t2}) - \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} + s'_{vm}\gamma s_{dm}\|}{(s_{t1}+s_{t2}) + \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} + s'_{vm}\gamma s_{dm}\|}$
$\geq \frac{s_{t2} - \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm}\|}{s_{t2} + \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm}\|}$

$N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, +s'_{vm}\gamma s_{dm},\ s_{t1}+s_{t2}) \geq$
$min\{N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_{t1}),$
$N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm}, s_{t2})\}$

$(S_{M^\gamma} - 6)$ To prove right continuity, suppose that

$s_{t2} > s_{t1} > 0 with s_{t1}, s_{t2} \in [0,1)$

$\frac{s_{t2} - \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}{s_{t2} + \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}$
$- \frac{s_{t1} - \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}{s_{t1} + \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}$
$= \frac{(s_{t2}-s_{t1}) \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}{(s_{t2} + \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|)(s_{t1} + \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|)} \geq 0$

$\Leftrightarrow \frac{s_{t2} - \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}{s_{t2} + \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}$
$\geq \frac{s_{t1} - \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}{s_{t1} + \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}$

$\Leftrightarrow N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_{t2})$
$\geq N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_{t1})$

Hence $N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, \ldots, s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_t)$

is right continuous and non-increasing function of $s_t \in \mathbb{R}$

$\Rightarrow \lim_{m\to\infty} N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_t)$

$= \lim_{m\to\infty} \frac{s_t - \|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}{s_t + \|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|} = 1$

$(S_{M^\gamma} - 7)$ Obviously

$N_{\gamma c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_t) = 0$

$(S_{M^\gamma} - 8)$ we have

$N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_t)$

$= \frac{2\|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}{s_t + \|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}$

Such that

$N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_t) = 0$

$\Leftrightarrow \frac{2\|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}{s_t + \|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|} = 0$

$\Leftrightarrow \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} \| = 0$

Hence, $s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}$ are linearly dependent.

$(S_{M^\gamma} - 9)$ we have

$N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_t) =$

$\frac{2\|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}{s_t + \|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}$

$= \frac{2\|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm}\gamma s_{dm}, s_{vm-1}\gamma s_{dm-1}\|}{s_t + \|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm}\gamma s_{dm}, s_{vm-1}\gamma s_{dm-1}\|}$

$N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm}\gamma s_{dm}, s_{vm-1}\gamma s_{dm-1}, s_t) =$ ....so on

Hence,

$N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm}\gamma s_{dm}, s_{vm-1}\gamma s_{dm-1}, s_t)$

is invariant under any permutation of

$s_{v1}, s_{v2}, s_{v3}, ..., s_{vm-1}, s_{vm}$

$(S_{M^\gamma} - 10)$ We have

$N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_\rho(s_{vm}\gamma s_{dm}), s_t)$

$= \frac{2\|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_\rho(s_{vm}\gamma s_{dm})\|}{s_t + \|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_\rho(s_{vm}\gamma s_{dm})\|}$

$\Leftrightarrow \frac{s_\rho 2\|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}{s_t + s_\rho\|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}$

$\Leftrightarrow \frac{2\|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}{\frac{s_t}{s_\rho} + \|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}$

$= N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, \frac{s_t}{|s_\rho|}), s_\rho \in \Gamma$

$(S_{M^\gamma} - 11)$ Without loss of generality, we assume that

$N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}, s_{t1})$

$\leq N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm}, s_{t2})$

$\Leftrightarrow \frac{2\|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}{s_{t1} + \|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}\|}$

$\leq \frac{2\|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm}\|}{s_{t2} + \|s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm}\|}$

$\Leftrightarrow [s_{t2} + \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm} \|]$

$[2 \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} \|]$

$\leq [s_{t1} + \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm}]$

$[2 \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm} \|]$

$\Leftrightarrow s_{t2} \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} \| +$

$\| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm} \|$

$. \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} \|$

$\Leftrightarrow s_{t1} \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm} \| +$

$\| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm} \|$

$. \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} \|$

$$\begin{aligned} &\Leftrightarrow s_{t2} \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} \| \\ &\leq s_{t1} \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s'_{vm}\gamma s_{dm} \| \end{aligned} \tag{1}$$

we have

$\frac{2\|s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},\ldots,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}+s'_{vm}\gamma s_{dm}\|}{(s_{t1}+s_{t2})+\|s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},\ldots,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}+s'_{vm}\gamma s_{dm}\|}$
$-\frac{2\|s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},\ldots,s_{vm-1}\gamma s_{dm-1},s'_{vm}\gamma s_{dm}\|}{s_{t2}+\|s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},\ldots,s_{vm-1}\gamma s_{dm-1},s'_{vm}\gamma s_{dm}\|}$
$\frac{2\|s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},\ldots,s_{vm-1}\gamma s_{dm-1},s'_{vm}\gamma s_{dm}\|}{s_{t2}+\|s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},\ldots,s_{vm-1}\gamma s_{dm-1},s'_{vm}\gamma s_{dm}\|}$
$\Leftrightarrow \frac{s_{t2}\|s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},\ldots,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}\|-s_{t1}\|s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},\ldots,s_{vm-1}\gamma s_{dm-1},s'_{vm}\gamma s_{dm}\|}{[(s_{t1}+s_{t2})+\|s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},\ldots,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}+s'_{vm}\gamma s_{dm}\|][s_{t2}+\|s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},\ldots,s_{vm-1}\gamma s_{dm-1},s'_{vm}\gamma s_{dm}\|]}$
≤ 0

By Equation (1)

$\frac{2\|s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},\ldots,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}+s'_{vm}\gamma s_{dm}\|}{(s_{t1}+s_{t2})+\|s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},\ldots,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}+s'_{vm}\gamma s_{dm}\|}$
$\leq \frac{2\|s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},\ldots,s_{vm-1}\gamma s_{dm-1},s'_{vm}\gamma s_{dm}\|}{s_{t2}+\|s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},\ldots,s_{vm-1}\gamma s_{dm-1},s'_{vm}\gamma s_{dm}\|}$

$N_{\gamma^c}(s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},...,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}+s'_{vm}\gamma s_{dm},s_{t1}+s_{t2}) \leq$
$max\{N_{\gamma^c}(s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},...,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm},s_{t1}),$
$N_{\gamma^c}(s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},...,s_{vm-1}\gamma s_{dm-1},s'_{vm}\gamma s_{dm},s_{t2})\}$

$(S_{M^\gamma}-12)$ we have clearly

$N_{\gamma^c}(s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},...,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm},s_t)$
$=\frac{2\|s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},\ldots,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}\|}{s_t+\|s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},\ldots,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}\|}$

is non-decreasing left continuous function of s_t.

$\lim_{m\to\infty} N_{\gamma^c}(s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},...,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm},s_t)=0$
$\Leftrightarrow \lim_{m\to\infty} \frac{2\|s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},\ldots,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}\|}{s_t+\|s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},\ldots,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}\|}=0$

3.11. Definition: A sequence $\{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$ in CSGMNLS $(S_V, N_\gamma, N_{\gamma^c})$ is said to be convergent into $s_v\gamma s_d$ if for every given $w\in(0,1)$,$s_t>0$, there exist an integer n ∈ℕ such that

$N_\gamma(s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},...,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}-s_v\gamma s_d,s_t)>1-w$ and
$N_{\gamma^c}(s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},...,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}-s_v\gamma s_d,s_t)<w\, for\, all\, m\geq n$

Theorem 3.1 .1:

Let a sequence $\{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$ in CSGMNLS $(S_V, N_\gamma, N_{\gamma^c})$ is convergent into $s_v\gamma s_d$⇔

(i) $N_\gamma(s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},...,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}-s_v\gamma s_d,s_t)=1\, as\, m\to\infty$
(ii) $N_{\gamma^c}(s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},...,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}-s_v\gamma s_d,s_t)=0\, as\, m\to\infty$.

Proof : Let a sequence $\{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$in CSGMNLS $(S_V, N_\gamma, N_{\gamma^c})$ is convergent to $s_v\gamma s_d$ and fix $s_t>0$ then for every given $w\in(0,1)$ there exist an integer n ∈ ℕ such that $N_\gamma(s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},...,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}-s_v\gamma s_d,s_t)>1-w$

$N_{\gamma^c}(s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},...,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}-s_v\gamma s_d,s_t)<w\, for\, all\, m\geq n.$

Therefore, $1-N_\gamma(s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},...,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}-s_v\gamma s_d,s_t)<w,$
and $N_{\gamma^c}(s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},...,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}-s_v\gamma s_d,s_t)<w$

Hence (i) $N_\gamma(s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},...,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}-s_v\gamma s_d,s_t)\to 1\, as\, m\to\infty$ and
(ii) $N_{\gamma^c}(s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},...,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}-s_v\gamma s_d,s_t)\to 0\, as\, m\to\infty$

Conversely, we have for each $s_t>0$ such that
$N_\gamma(s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},...,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}-s_v\gamma s_d,s_t)\to 1\, as\, m\to\infty$, and
$N_{\gamma^c}(s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},...,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}-s_v\gamma s_d,s_t)\to 0\, as\, m\to\infty$
then if for every given $w\in(0,1)$, $s_t>0$, there exist an integer n ∈ℕ such that
$1-N_\gamma(s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},...,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}-s_v\gamma s_d,s_t)<w$
$\Leftrightarrow N_\gamma(s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},...,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}-s_v\gamma s_d,s_t)>1-w$. and
$N_{\gamma^c}(s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},...,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}-s_v\gamma s_d,s_t)<w\, for\, all\, m\geq n.$

Hence, a sequence $\{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$in $(S_V, N_\gamma, N_{\gamma^c})$ CSGMNLS is s convergence into $s_v\gamma s_d$.

3.12. Definition: A sequence $\{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$in CSGMNLS $(S_V, N_\gamma, N_{\gamma^c})$ is a Cauchy sequence if for a given every $\theta\in(0,1)$ and $s_t>0$, there is an integer n ∈ℕ such that

$N_\gamma(s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},...,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}-s_{vl}\gamma s_{dl},s_t)>1-\theta$
and
$N_{\gamma^c}(s_{v1}\gamma s_{d1},s_{v2}\gamma s_{d2},s_{v3}\gamma s_{d3},...,s_{vm-1}\gamma s_{dm-1},s_{vm}\gamma s_{dm}-s_{vl}\gamma s_{dl},s_t)<\theta\, for\, all\, l,m\geq n$

Theorem 3.1.2 :

In CSGMNLS $(S_V, N_\gamma, N_{\gamma^c})$ every convergent sequence is Cauchy sequence.

Proof: Let a sequence $\{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$in CSGMNLS $(S_V, N_\gamma, N_{\gamma^c})$ and is s convergence to $s_v\gamma s_d$in

$(S_V, N_\gamma, N_{\gamma^c})$. If $s_t > 0$ and $\theta \in (0,1)$. Consider $w \in (0,1)$ such that $(1-w)S_\Delta(1-w) > 1-\theta abndwS_{\Delta^c}w < \theta$
Since a sequence $\{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$ in $(S_V, N_\gamma, N_{\gamma^c})$ and it is convergence to $s_v\gamma s_d$.
There is an integer n $\in \mathbb{N}$ such that
$N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_v\gamma s_d, \frac{s_t}{2}) > 1-w$. And
$N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_v\gamma s_d, \frac{s_t}{2}) < w$, for all, m $\geq$ n.
$N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_{vl}\gamma s_{dl}, s_t) =$
$N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_v\gamma s_d + s_v\gamma s_d - s_{vl}\gamma s_{dl}, \frac{s_t}{2} + \frac{s_t}{2})$
$\geq N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_v\gamma s_d, \frac{s_t}{2})$
$N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} + s_v\gamma s_d - s_{vl}\gamma s_{dl}, \frac{s_t}{2})$
We have $N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_{vl}\gamma s_{dl}, s_t) =$
$N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_v\gamma s_d + s_v\gamma s_d - s_{vl}\gamma s_{dl}, \frac{s_t}{2} + \frac{s_t}{2}) \geq N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_v\gamma s_d, \frac{s_t}{2})S_\Delta$
$N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} + s_v\gamma s_d - s_{vl}\gamma s_{dl}, \frac{s_t}{2}) > (1-w)S_\Delta(1-w) > 1-\theta$, for all l, m$\geq$nw. And we have
$N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_{vl}\gamma s_{dl}, s_t) =$
$N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_v\gamma s_d + s_v\gamma s_d - s_{vl}\gamma s_{dl}, \frac{s_t}{2} + \frac{s_t}{2})$
$\leq N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_v\gamma s_d, \frac{s_t}{2})S_{\triangle^c}$
$N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} + s_v\gamma s_d - s_{vl}\gamma s_{dl}, \frac{s_t}{2}) < wS_{\Delta^c}w < \theta$,for all 1, m$\geq$n.
Therefore, a sequence $\{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$ in CSGMNLS $(S_V, N_\gamma, N_{\gamma^c})$ is a Cauchy sequence.

3.13. Definition: A CSGMNLS $(S_V, N_\gamma, N_{\gamma^c})$ is called complete if every Cauchy sequence is convergent in it.

Remark: In following example we prove that there may exist Cauchy sequence in a $(S_V, N_\gamma, N_{\gamma^c})$ which is not convergent.

Example: Let a CSGMNLS $(S_V, N_\gamma, N_{\gamma^c})$ as in the previous example Let $\{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$ be sequence in $(S_V, N_\gamma, N_{\gamma^c})$ then

(i)$\{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$ is a Cauchy sequence in $(S_V, \| ., ..., . \|) \Leftrightarrow \{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$ is Cauchy sequence in $(S_V, N_\gamma, N_{\gamma^c})$
(ii)$\{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$ is a convergence sequence in $(S_V, \| ., ..., . \|) \Leftrightarrow \{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$ is convergence sequence $(S_V, N_\gamma, N_{\gamma^c})$.

Proof: (i) Suppose $\{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$ is a Cauchy sequence in $(S_V, \| ., ..., . \|)$
$\Leftrightarrow \lim_{m,l\to\infty} \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_{vl}\gamma s_{dl} \| = 0$
$\Leftrightarrow \lim_{m,l\to\infty} N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_{vl}\gamma s_{dl}, s_t)$
$\Leftrightarrow \lim_{m,l\to\infty} \frac{s_t - \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_{vl}\gamma s_{dl} \|}{s_t + \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_{vl}\gamma s_{dl} \|} = 1$
$\Leftrightarrow N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_{vl}\gamma s_{dl}, s_t) \to 1\, as\, m, l \to \infty$
$\Leftrightarrow N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_{vl}\gamma s_{dl}, s_t) > 1-w, w \in (0,1)\, for\, all\, l, m > n$
And $\lim_{m,l\to\infty} N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_{vl}\gamma s_{dl}, s_t)$
$\Leftrightarrow \lim_{m,l\to\infty} \frac{2\| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_{vl}\gamma s_{dl} \|}{s_t + \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_{vl}\gamma s_{dl} \|} = 0$
$\Leftrightarrow N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_{vl}\gamma s_{dl}, s_t) \to 0\, as\, m, l \to \infty$
$\Leftrightarrow N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_{vl}\gamma s_{dl}, s_t) < w, w \in (0,1)\, for\, all\, l, m > n$
Hence, $\{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$ is Cauchy sequence in $(S_V, N_\gamma, N_{\gamma^c})$
(ii) Suppose $\{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$ is a convergent sequence in $(S_V, \| ., ..., . \|)$
$\Leftrightarrow \lim_{m\to\infty} \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_v\gamma s_d \| = 0$
$\Leftrightarrow \lim_{m\to\infty} N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_v\gamma s_d, s_t)$
$\Leftrightarrow \lim_{m\to\infty} \frac{s_t - \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_v\gamma s_d \|}{s_t + \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_v\gamma s_d \|} = 1$
$\Leftrightarrow N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_v\gamma s_d, s_t) \to 1\, as\, m \to \infty$
$\Leftrightarrow N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_v\gamma s_d, s_t) > 1-w, w \in (0,1)\, for\, all\, m > n$ and
$\lim_{m\to\infty} N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_v\gamma s_d, s_t)$
$\Leftrightarrow \lim_{m\to\infty} \frac{2\| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_v\gamma s_d \|}{s_t + \| s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_v\gamma s_d \|} = 0$
$\Leftrightarrow N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_v\gamma s_d, s_t) \to 0\, as\, m \to \infty$
$\Leftrightarrow N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_v\gamma s_d, s_t) < w, w \in (0,1)\, for\, all\, m > n$
Hence, $\{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$ is convergence sequence in $(S_V, N_\gamma, N_{\gamma^c})$
Thus, if there is a soft m-normed right Gamma-linear space

$(S_V, \| \cdot, ..., \cdot \|)$is not complete then the CSGMNLS such a crisp soft m-norm on an incomplete CSGMNLS S_V is an incomplete cubic soft m-normed right Gamma-linear space.

Theorem 3.1.3:

A CSGMNLS $(S_V, N_\gamma, N_{\gamma^c})$ is complete if in which every cauchy sequence has a convergence sub sequence.

Proof : A sequence $\{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$in CSGMNLS $(S_V, N_\gamma, N_{\gamma^c})$ and $\{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$ be a subsequence of $\{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$ that convergence to $s_v\gamma s_d$ in $(S_V, N_\gamma, N_{\gamma^c})$. We need to prove that $\{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$is convergence to $s_v\gamma s_d$if $s_t > 0$and $\theta \in (0,1)$. Choose $w \in (0,1)$ such that

$(1-w)S_\Delta(1-w) > 1-\theta and w S_{\Delta^c} w < \theta$

Since a sequence $\{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$is Cauchy sequence in $(S_V, N_\gamma, N_{\gamma^c})$. There exist an integer n $\in \mathbb{N}$ such that

$N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_{vl}\gamma s_{dl}, \frac{s_t}{2}) > 1-w, w \in (0,1)\, for\, all\, l, m > n$ and

$N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vm}\gamma s_{dm} - s_{vl}\gamma s_{dl}, \frac{s_t}{2}) < w, w \in (0,1)\, for\, all\, l, m > n$

Also, we have $\{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$is convergent to $s_v\gamma s_d$ There is a positive integer j> n such that

$N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vmj}\gamma s_{dmj} - s_v\gamma s_d, \frac{s_t}{2}) > 1-w\, and$

$N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vmj}\gamma s_{dmj} - s_v\gamma s_d, \frac{s_t}{2}) < w$

Now $N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vmj}\gamma s_{dmj} - s_v\gamma s_d, s_t) =$

$$\left.\begin{array}{l} N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vmj}\gamma s_{dmj} - s_v\gamma s_d, \frac{s_t}{2} + \frac{s_t}{2}) \\ \geq [N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vmj}\gamma s_{dmj} - s_v\gamma s_d, \frac{s_t}{2})] \\ S_\Delta[N_\gamma(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vmj}\gamma s_{dmj} - s_v\gamma s_d, \frac{s_t}{2})] \\ > (1-w)S_\Delta(1-w) > 1-\theta \end{array}\right. \text{, for all j} \geq \text{n. And}$$

$N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vmj}\gamma s_{dmj} - s_v\gamma s_d, s_t) =$

$$\left.\begin{array}{l} N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vmj}\gamma s_{dmj} - s_v\gamma s_d, \frac{s_t}{2} + \frac{s_t}{2}) \\ \leq [N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vmj}\gamma s_{dmj} - s_v\gamma s_d, \frac{s_t}{2})] \\ S_{\Delta^c}[N_{\gamma^c}(s_{v1}\gamma s_{d1}, s_{v2}\gamma s_{d2}, s_{v3}\gamma s_{d3}, ..., s_{vm-1}\gamma s_{dm-1}, s_{vmj}\gamma s_{dmj} - s_v\gamma s_d, \frac{s_t}{2})] \\ < w S_{\Delta^c} w < \theta \end{array}\right. \text{, for all } j \geq n.$$

Therefore, a sequence $\{s_{vk}\gamma s_{dk}\}_{k=1}^{\infty}$ is convergent to $s_v\gamma s_d$ in cubic soft Gamma m-normed linear space $(S_V, N_\gamma, N_{\gamma^c})$ and hence it is complete.

4 Conclusion

This work introduces the concept of soft gamma ring, soft gamma vector space. Using this notion, the SGMNLS and CSGMNLS are introduced. It presents detailed properties with generalization theory of soft-m-normed linear space and obtained the results on cauchy and convergent sequence in CSGMNLS. It has even provided theorem of completeness in CSGMNLS. This work can be extended to Banach spaces by introducing the notions of completeness in CSGMNLS.

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