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Inventory Optimization Model with Quadratic Demand Pattern and Non-instantaneous Deterioration Rate with Mixed Cash and Advance Payment Scheme

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Abstract

Objectives: To develop a mathematical model for cost minimization by examining a scenario involving a seasonal product retail market, evaluating the outcome, and providing solutions to unanswered queries of existing models are the highlights of this present work. **Methods:** 1. Using dynamic differential equations with initial and boundary conditions, we mathematically model a retail market scenario for perishable commodities and determine the retailer's total cost. 2. We consider a quadratic function of time to precisely describe the maintenance cost and demand function. Also, a combination of cash and advance payment schemes is implemented in our model. 3. Our model was numerically formulated using a dataset deduced from earlier research. We have included a sensitivity analysis and a numerical example with a graphical depiction of the total cost function that highlights the most and least influential factors. 4. Based on the Hessian matrix condition and the calculus derivative approach, we developed an algorithm to calculate the optimal cost. **Findings:** This model provides retailers with insights for effective stock management while optimizing the cost of retailing perishable commodities with non-linear demand patterns. Sensitivity analysis showed that the two factors that have the biggest impact on retailers' overall costs are non-linear demand and holding cost parameters. The smaller optimal inventory cycle time and the large value of time at which a shortage happens raise the overall cost. Additionally, because of higher demand, the time-dependent deterioration rate has the least impact on overall cost. Additionally, a three-dimensional graph of cost minimization is offered to illustrate the convexity of total cost. **Novelty:** Implementation of Quadratic time-dependent demand and non-linear holding costs under a mixed cash advance payment structure, along with time-dependent deterioration and partial backlog shortages.

Keywords: Quadratic Demand pattern; Noninstantaneous decay rate; Cash and Advance Payment scheme; Partial Backlog Shortages

1 Introduction

Over the past few decades, applying mathematical concepts to real-life problems, such as inventory control, has been advancing rapidly. It is crucial to manage inventory efficiently to prevent losses in the enterprise's profit margin or reduce the total cost. Sustainable inventory management for perishable goods has become an interesting study topic due to several factors, based on economic or environmental conditions, and hence impact different market demand determinants. For industries that transport or fly their products from another country or region, the lead time duration is crucial. Inventory is affected by these situations because it takes time for the products to arrive from far away. Although labor and machinery can lower the investment required, the firm and the market system may collapse if they lack access to raw materials and provided items. Also, stocks of the product or items indicate a block of investment without a profit. Thus, stock carrying costs, damage during storage, and devaluation have a significant impact on firm profit mechanisms. Moreover, the examination of an inventory system would not be complete without acknowledging the influence of degradation on the inventory. Every inventory system needs to take into account degradation as a crucial element, and this factor should be included in the analysis. Despite the appeal of these characteristics to numerous suppliers, manufacturers, and retailers, the potential adverse impacts of stock deterioration on the practitioner's profits cannot be ignored. The initial inventory model for items undergoing exponential decay was created by Ghare and Shrader[(1)]. Several studies like[(2–5)], have delved into these impacts through the use of different decay rates depending on the characteristics of the goods. One of these is instant damage, which is the harm that occurs to items while in storage. Conversely, non-instantaneous damage refers to a delay in the progression of the deterioration process i.e., deterioration may commence after a specific duration and then escalate. The decay of various perishable items, such as grains, seeds, fruits, and vegetables, happens gradually and swiftly as they are stored over time. This underscores the significance of examining the factor of deterioration.

Momena et al.[(6)] analyzed the optimal retail strategy by considering selling price and advertisement frequency-dependent demand. In this study, they introduce two different warehousing scenarios to find which one is best for the tenure of owned and rented warehouses. Shah et al.[(7)] suggested an inventory control plan for products that deteriorate slowly over time and have multivariate probabilistic demand by optimizing the selling price, economic order quantity, cycle time, and preservation technique investment. In this study, authors test three trade credit financing policy scenarios for net profit optimization. Similar to this, it is simple to access a wide range of scholarly studies on inventory models, including credit payment systems. A system of advance payments guarantees both timely payment and delivery of the products. Studies like[(8)] have also used mixed cash and advance payment scheme inventory models with backorder difficulties in the relevant literature that is recently available. In their study, the authors employed linear price-dependent demand alongside an advanced payment method in their analysis and stated that introducing non-linear time-sensitive demand would be a more suitable approach in real-world scenarios and could act as an extension of their current model. Thus, the consequences of employing a hybrid cash and advance payment scheme in conjunction with non-linear holding costs under a partial backlog event of the shortage have not received much thought.

The list below outlines the contribution of this paper by addressing the following queries:

- What possible relationship exists between the various parameters and how they affect the overall cost of the suggested inventory model for seasonal goods?
- The present study examines the potential significance of advance payment schemes for efficient financial management and seamless organizational operations.
- The current study explores the enormous influence that non-linear time-dependent demand might have on a retailer's overall costs.
- While using a hybrid cash and advance payment plan for seasonal products, what should be the strategic actions that the management may implement to reduce costs associated with this inventory scenario?
- What strategic measures retailers will take if there is no prepayment plan for the model scenario?

This study investigates the inventory problem for perishable products with non-linear demand rates and holding expenses. We have tried to answer the above-generated queries in our newly developed model. The presentation of this present research work has been arranged in the following manner: Section 1 includes an introduction and a literature review of the previous studies. We have described the methodology in Section 2. Section 3 contributes to the results and the discussion includes numerical examples, sensitivity analysis, and discussions of the implications for management. Finally, Section 4 offers the conclusion of the study.

1.1 Literature Review of Inventory Model with Advance Payment Scheme

A system of making payments in advance offers a level of flexibility for retailers to effectively interact with customers, even though the amount of capital available may be somewhat reduced compared to usual. Additionally, with advance payment, retailers must ensure the timely completion of purchases, since a portion of the purchase cost has already been transferred to the supplier's account. This requirement can assist in making firm decisions when procuring items from suppliers. The literature on inventory management extensively investigates a variety of payment schemes. Many researchers like[9–12] have worked on various kinds of payment mechanisms. Recently, Das et al.[13] Developed an inventory model in which three payment methods of credit, cash, and advance are compared and the vendors' per-unit profits are examined. The rate at which the product is consumed depends not only on how long it takes to process various payment methods but also on the cost and level of freshness of the commodity. In the framework of an order-based prepayment and discount regulation, Khan et al.[14] created an inventory model to examine the company's pricing strategy, inventory management, and prepayment policies to enhance profit margins. Adopting a pragmatic approach, their study incorporates a model that views customer demand as a multiplicatively separable power function, which effectively integrates product pricing dynamics and storage duration considerations. A model was developed by Khan et al.[15] which explores the most advantageous prepayment installment arrangement tailored for practitioners, considering a setting where both unit selling price and storage duration jointly influence customer consumption patterns. By incorporating unit holding costs modeled as a power function relative to storage time, it aims to furnish practitioners with strategic frameworks for pricing strategies, payment structuring, and inventory replenishment choices, thereby furnishing invaluable insights conducive to fostering profitability and ensuring sustainable business operations. Moreover, authors in[16] created an inventory model with two storage spaces and advance payment under three different scenarios, considering the decay in owned and rented warehouses. Table 1 represents the comparison of our study with existing studies.

Table 1. Comparison of the present work with the existing literature

Authors	Demand rate	Non-Instantaneous	Instantaneous	Partial Back-log	Holding Cost	Advance payment scheme	Solution Procedure
Liao et al.[17]	Constant	Yes	No	No	Linear	Yes	Mathematically
Saha et al.[18]	Price Time-dependent	No	Yes	Yes	Constant	No	Mathematically
Mashud[19]	Price and stock dependent	No	Yes	No	Constant	No	Soft Computing, mathematically
Shaikh et al.[20]	Price and advertisement dependent	Yes	No	Yes	Constant	Yes	Mathematically

Continued on next page

Table 1 continued

Duary et al.[21]	Selling price, time, and advertisement dependent	No	Yes	Yes	Constant	Yes	Mathematically
Panda et al.[22]	Price and stock-dependent demand	No	Yes	Yes	Constant	No	Mathematically
Duary et al.[23]	Non-linear price-dependent demand	Yes	No	Yes	Constant	Yes	Mathematically
Rahman et al.[24]	Linear Stock depended	No	Yes	Constant	Constant	Yes	Mathematically
This Paper	Non-Linear Time depended	Yes	No	Yes, time-dependent	Non-linear time-dependent	Yes	Mathematically

2 Methodology

2.1 Assumptions:

- Single non-instantaneous perishable product is considered.
- Lead time is constant i.e., L, and the planning horizon is infinite, there is neither replacement nor repair is permitted for the products during the planning horizon.
- As the decay is non-instantaneous, the decay rate is given by a linear function associated with storage time i.e. $\psi(t)$=θt, where $0 < \theta < 1$.
- Demand rate is deterministic and considered as follows:

D(t) = $\begin{cases} D_1(t) & if 0 \le t \le t_1 \\ D_2(t) & if t_1 < t \le T \end{cases}$, where $D_1(t) = a + bt + ct^2$ and $D_2(t) = a + bt$.

- The parameters a, b, and c, respectively, stand for the initial demand in units, the rising demand rate, and the change in demand rate.
- Allowed shortages and backlog to exist partially and the backlogged parameter is a function associated with waiting time given by , where $\delta > 0$ and $(T - t)$ is the customer's waiting time.
- Initially, there is no effect of deterioration in the storage space during the time interval$[0, t_d]$. The factor of deterioration comes into the picture over time $t = t_d$.
- The carrying cost varies as a square of the amount of time in the storage space and is given by $h(t) = h + \alpha t^2$, $\alpha > 0$

2.2 Notations:

- T**:** Inventory Cycle Time (time unit)
- t_1**:** Time at which shortage occurred (time unit)
- I_1**:** Level of Inventory during the interval $[0, t_d]$
- I_2**:** Level of Inventory during the interval $[t_d, t_1]$
- $D(t)$**:** Demand Rate
- δ**:** constant backlogging parameter $0 \le \delta \le 1$
- t_1^***:** Optimal time for shortage
- T^***:** Optimal inventory cycle time
- Q^***:** Optimum order quantity
- A: Ordering Cost
- *a,b,c*: Parameters of demand ($a > 0, b, c$ are non-zero and constant)
- c_p**:** Per unit acquisition price
- h**:** Carrying cost to hold a unit item (constant)
- α**:** The time-sensitive parameter in carrying cost

- c_2**:** Opportunity cost
- c_1**:** Shortage cost
- θ**:** Deterioration Rate $(0 < \theta < 1)$
- t_d**:** Time at which deterioration starts Decay commencing time
- R**:** Maximum Backlogged quantity
- S**:** Quantity of Stock present at the beginning of the inventory cycle
- L**:** Permitted time to finish the prepayment
- n: The number of prepayment segments
- β**:** The portion of the acquisition price used for the prepayment
- TC: Total cost for the inventory procedure

2.3 Problem Definition:

Consider an ordering scenario in which a retailer orders $(S+R)$ item units initially by using an advance payment concept. In this payment scheme, the retailer completes his required payment by paying a portion of the purchase price, i.e., β, to the supplier L months in advance during the allotted time L in n identical segments at identical intervals, and the remaining $(1-\beta)$ portion is paid at the moment of shipment receipt. Following the arrival of the shipment, the retailer store immediately fulfilled all back-order quantities R, resulting in a reduction in the current number of remaining quantities to S.

The following diagram in Figure 1 depicts how inventory moves through the retailer's storage area:

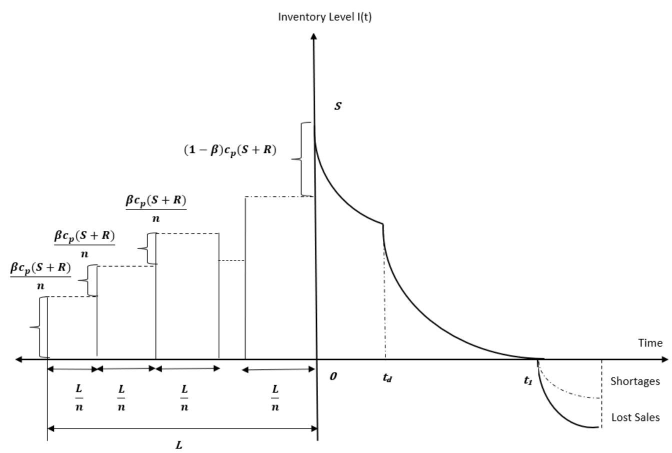

Fig 1. Flow of Inventory [8] during the cycle time [0, T]

Thus, the rate at which the level of inventory changes at any instant in time t in $[0, T]$ must satisfy the differential equation below:

$$\frac{dI_1}{dt} = -D_1(t), \quad when \quad 0 \le t \le t_d \tag{1}$$

$$\frac{dI_2}{dt} = -D_1(t) - \psi(t) I_2(t), \quad when \quad t_d \le t \le t_1 \tag{2}$$

$$\frac{dI_3}{dt} = -\left(\frac{D_2(t)}{1+\delta(T-t)}\right), \quad when \quad t_1 \le t \le T \tag{3}$$

with the boundary conditions as

$I_1(t) = S \quad when \quad t = 0, I_2(t) = 0 \quad when \quad t = t_1$ and $I_3(t) = -R \quad when \quad t = T$

After solving Equations (1), (2) and (3), we get

$$I_1(t) = S - at - \frac{bt^2}{2} - \frac{ct^3}{3} \tag{4}$$

$$I_2(t) = e^{-\frac{\theta t^2}{2}} \int_t^{t_1} e^{-\frac{\theta u^2}{2}} (a + bu + cu^2) du \tag{5}$$

$$I_3(t) = \frac{bt}{\delta} + \frac{(b + a\delta + bT\delta)(Log(1 + \delta(T - t))}{\delta^2} - \frac{bT}{\delta} - R \tag{6}$$

As the level of inventory $I(t)$ maintains continuity at $t = t_d$ and $t = t_1$, one may readily determine the maximum level of inventory S, and the maximum backlog quantity R, by taking advantage of the concept of continuity at these points.

$$\begin{aligned} S = [&a(t_1 - t_d) + b\frac{(t_1^2 - t_d^2)}{2} + c\frac{(t_1^3 - t_d^3)}{3} + \frac{1}{6}a(t_1^3 - t_d^3)\theta + \frac{1}{8}b(t_1^4 - t_d^4)\theta + \frac{1}{10}c(t_1^5 - t_d^5)\theta + \frac{1}{40}a(t_1^5 - t_d^5)\theta^2 \\ &+ \frac{1}{48}b(t_1^6 - t_d^6)\theta^2 + \frac{1}{56}c(t_1^7 - t_d^7)\theta^2 + \frac{1}{10}c(t_1^5 - t_d^5)\theta + \frac{1}{40}a(t_1^5 - t_d^5)\theta^2 + \frac{1}{48}b(t_1^6 - t_d^6)\theta^2 + \frac{1}{56}c(t_1^7 - t_d^7)\theta^2 \\ &+ \frac{1}{336}a(t_1^7 - t_d^7)\theta^3 + \frac{1}{384}b(t_1^8 - t_d^8)\theta^3 + \frac{1}{432}c(t_1^9 - t_d^9)\theta^3]e^{-\frac{\theta v^2}{2}} + av + b\frac{v^2}{2} + c\frac{v^3}{3}] \end{aligned} \tag{7}$$

and

$$R = \frac{bt_1}{\delta} + \frac{(b + a\delta + bT\delta)(Log[1 + \delta(T - t_1)]}{\delta^2} - \frac{bT}{\delta} \tag{8}$$

2.4 Associated Costs

2.4.1 Ordering Cost

Ordering costs are expenses incurred while purchasing goods from the supplier to replenish the inventory. Regardless of the size of the order, a corporation must suffer certain costs every time it places one.

$OC = A$

2.4.2 Purchase Cost

Essentially, this represents an inventory's nominal cost. It is the price paid for purchasing from a supplier. A unit's cost is fixed, although the price may change as the number purchased rises or falls.

$PC = c_p(S + R)$

2.4.3 Holding Cost

The expense connected with keeping a product in inventory is referred to as carrying cost, sometimes known as holding cost. It is proportional to the amount of inventory and the time taken to hold that inventory. The overall holding cost in this inventory system consists of two divisions, namely, H_1 i.e., the cost to carry goods during $[0, t_d]$ and H_2 which is the cost of carrying goods during $[t_d, t_1]$. Now, H_1 is given as

$$H_1 = \int_0^{t_d} (h + \alpha t^2) I_1(t) dt = \int_0^{t_d} (h + \alpha t^2) \left(S - at - \frac{bt^2}{2} - \frac{ct^3}{3}\right) dt \tag{9}$$

Again, by utilizing the expansion of the exponential function, the holding cost during $[t_d, t_1]$ is

$$\begin{aligned}
&H_2=\int_{t_d}^{t_1}(h+\alpha t^2)I_2(t)dt=-\frac{1}{2}ah(t_1^2-t_d^2)-\frac{1}{6}bh(t_1^3-t_d^3)\\
&-\frac{1}{12}ch(t_1^4-t_d^4)+ah(t_1-t_d)t_1+\frac{1}{2}bh(t_1-t_d)t_1^2+\frac{1}{3}ch(t_1-t_d)t_1^3\\
&-\frac{1}{4}a(t_1^4-t_d^4)\alpha-\frac{1}{10}b(t_1^5-t_d^5)\alpha-\frac{1}{18}c(t_1^6-t_d^6)\alpha+\frac{1}{3}a(t_1^3-t_d^3)t_1\alpha\\
&+\frac{1}{6}b(t_1^3-t_d^3)t_1^2\alpha+\frac{1}{9}c(t_1^3-t_d^3)t_1^3\alpha+\frac{1}{12}ah(t_2^4-t_d^4)\theta+\frac{1}{40}bh(t_1^5-t_d^5)\theta\\
&+\frac{1}{90}ch(t_1^6-t_d^6)\theta-\frac{1}{6}ah(t_1^3-t_d^3)t_1\theta-\frac{1}{12}bh(t_1^3-t_d^3)t_1^2\theta+\frac{1}{6}ah(t_1-t_d)t_1^3\theta\\
&-\frac{1}{18}ch(t_1^3-t_d^3)t_1^3\theta+\frac{1}{8}bh(t_1-t_d)t_1^4\theta+\frac{1}{10}ch(t_1-t_d)t_1^5\theta+\frac{1}{18}a(t_1^6-t_d^6)\alpha\theta\\
&+\frac{1}{56}b(t_1^7-t_d^7)\alpha\theta+\frac{1}{120}c(t_1^8-t_d^8)\alpha\theta-\frac{1}{10}a(t_1^5-t_d^5)t_1\alpha\theta-\frac{1}{20}b(t_1^5-t_d^5)t_1^2\alpha\theta\\
&+\frac{1}{18}a(t_1^3-t_d^3)t_1^3\alpha\theta-\frac{1}{30}c(t_1^5-t_d^5)t_1^3\alpha\theta+\frac{1}{24}b(t_1^3-t_d^3)t_1^4\alpha\theta+\frac{1}{30}c(t_1^3-t_d^3)t_1^5\alpha\theta\\
&-\frac{1}{90}ah(t_1^6-t_d^6)\theta^2-\frac{1}{336}bh(t_1^7-t_d^7)\theta^2-\frac{1}{840}ch(t_1^8-t_d^8)\theta^2+\frac{1}{40}ah(t_1^5-t_d^5)t_1\theta^2\\
&+\frac{1}{80}bh(t_1^5-t_d^5)t_1^2\theta^2-\frac{1}{36}ah(t_1^3-t_d^3)t_1^3\theta^2+\frac{1}{120}ch(t_1^5-t_d^5)t_1^3\theta^2-\frac{1}{48}bh(t_1^3-t_d^3)t_1^4\theta^2\\
&+\frac{1}{40}ah(t_1-t_d)t_1^5\theta^2-\frac{1}{60}ch(t_1^3-t_d^3)t_1^5\theta^2+\frac{1}{48}bh(t_1-t_d)t_1^6\theta^2+\frac{1}{56}ch(t_1-t_d)t_1^7\theta^2\\
&-\frac{1}{120}a(t_1^8-t_d^8)\alpha\theta^2-\frac{1}{432}b(t_1^9-t_d^9)\alpha\theta^2-\frac{c(t_1^{10}-t_d^{10})\alpha\theta^2}{1050}+\frac{1}{56}a(t_1^7-t_d^7)t_1\alpha\theta^2\\
&+\frac{1}{112}b(t_1^7-t_d^7)t_1^2\alpha\theta^2-\frac{1}{60}a(t_1^5-t_d^5)t_1^3\alpha\theta^2+\frac{1}{168}c(t_1^7-t_d^7)t_1^3\alpha\theta^2-\frac{1}{80}b(t_1^5-t_d^5)t_1^4\alpha\theta^2\\
&+\frac{1}{120}a(t_1^3-t_d^3)t_1^5\alpha\theta^2-\frac{1}{100}c(t_1^5-t_d^5)t_1^5\alpha\theta^2+\frac{1}{144}b(t_1^3-t_d^3)t_1^6\alpha\theta^2+\frac{1}{168}c(t_1^3-t_d^3)t_1^7\alpha\theta^2
\end{aligned} \tag{10}$$

Now, the total cost of holding all products for a single replenishment is

$$HC=h\left(t\right)\left(\int_0^{t_d}I_1\left(t\right)dt+\int_{t_d}^{t_1}I_2\left(t\right)\right)dt \tag{11}$$

2.4.4 Shortage Cost

The expense incurred when clients are not being serviced is known as the stockout cost. These costs imply shortages which lead to a loss in potential sales, and customers' goodwill would be damaged.

As the stock level downs to zero at $t=t_1$, a stock-out situation soon follows, and shortages develop as a result of this circumstance depending on when the next shipment arrives during $[t_1, T]$. Hence, the shortage cost for a single replenishment is

$$\begin{aligned}
SC&=-c_1\int_{t_1}^{T}I_3(t)dt\\
&=-c_1\left[\frac{b}{2\delta}(T^2-t_1^2)+\frac{b+a\delta+bT\delta}{\delta^2}\left\{\delta\left(T^2-Tt_1-\frac{T^2-t_1^2}{2}\right)\right.\right.\\
&\left.\left.-\frac{\delta^2}{2}\left(\frac{T^2(T^2-t_1^2)}{2}+\frac{T^3-t_1^3}{3}-T(T^2-t_2^2)\right)\right\}-\frac{b}{\delta}T(T-t_1)-R(T-t_1)\right]
\end{aligned} \tag{12}$$

2.4.5 Lost Sale Cost

When a business fails to satisfy the consumers' purchasing needs, it incurs a cost known as the cost of lost sales which is given by

$$\begin{aligned}
LSC&=c_2\int_{t_1}^{T}D_2(t)\left(1-\frac{1}{1+\delta(T-t)}\right)dt\\
&=c_2\left[a(T-t_1)+\frac{b(T^2-t_1^2)}{2}+\frac{b(T-t_1)}{\delta}-\frac{bLog(1-t_1\delta+T\delta)}{\delta^2}\right.\\
&\left.-\frac{aLog(1-t_1\delta+T\delta}{\delta}-\frac{bTLog(1-t_1\delta+T\delta)}{\delta}\right]
\end{aligned} \tag{13}$$

The cost of the prepaid amount for a single refill can be easily determined from Figure 1. So, the capital expense is therefore

$$I_c\left[\beta c_3(S+R)\frac{L}{n}(1+2+3+\ldots.+n)\right]=\frac{n+1}{2n}I_cL\beta c_3(S+R) \tag{14}$$

The total cost for the practitioner during the completion of the entire inventory cycle is given by the sum of the ordering cost A, the purchase cost PC, the holding cost HC, the stock-out cost SC, the lost sale cost LSC, and the capital cost. Consequently, the whole cyclic cost is

$A+\left(1+\frac{n+1}{2n}I_cL\beta\right)c_3(S+R)+HC+SC+LSC$

2.4.6 Optimality Conditions of Total Cost

Lastly, the cost of the retailer per unit time is as follows:

$$TC(t_1,T)=\frac{1}{T}\left(A+\left(1+\frac{n+1}{2n}I_cL\beta\right)c_3(S+R)+HC+SC+LSC\right) \tag{15}$$

The objective function is given as

Minimize TC(t_1,T)

$t_1>0$, T >0 and T-$t_1>0$

Now, to reduce the practitioner's cost TC(t_1, T), the highest priority is to acquire the optimum values of t_1 and T (say $t_1^*$and T^*). For the convexity of $TC(t_1,T)$ the optimal solution must satisfy the following conditions which are given by Equations (16), (17), (18), (19) and (20).

$$\frac{\partial TC}{\partial t_1}=0, \tag{16}$$

$$\frac{\partial TC}{\partial T}=0, \tag{17}$$

$$\frac{\partial^2 TC}{\partial t_1^2}|_{(t_1^*,T^*)}>0, \tag{18}$$

$$\frac{\partial^2 TC}{\partial T^2}|_{(t_1^*,T^*)}>0, \tag{19}$$

and

$$\left\{\frac{\partial^2 TC}{\partial t_1^2}\frac{\partial^2 TC}{\partial T^2}-\left(\frac{\partial^2 TC}{\partial t_1\partial T}\right)^2\right\}|_{(t_1^*,T^*)}>0 \tag{20}$$

Table 2. Algorithm for proposed model's optimal solution

Step 1	Initialize the values to the parameters such as A, a, b, c, c_3, c_1, c_2, h, α, v, θ, L, I_3, n, β, δ.
Step 2	Calculate TC from Equation (15) in terms of t_1 and T.
Step 3	Solve Equations (16) and (17) and find the critical values for the decision variable t_1 and T.
Step 4	Check whether the values obtained in Step 3 satisfy Equations (18), (19) and (20).
Step 5	If both conditions are met, then set, and proceed to the next step otherwise, if the results are not feasible, and proceed to Step 1.
Step 6	Using these optimal values find S and R from Equations (7) and (8).
Step 7	From Equation (15) find the total cost which gives the optimum cost.

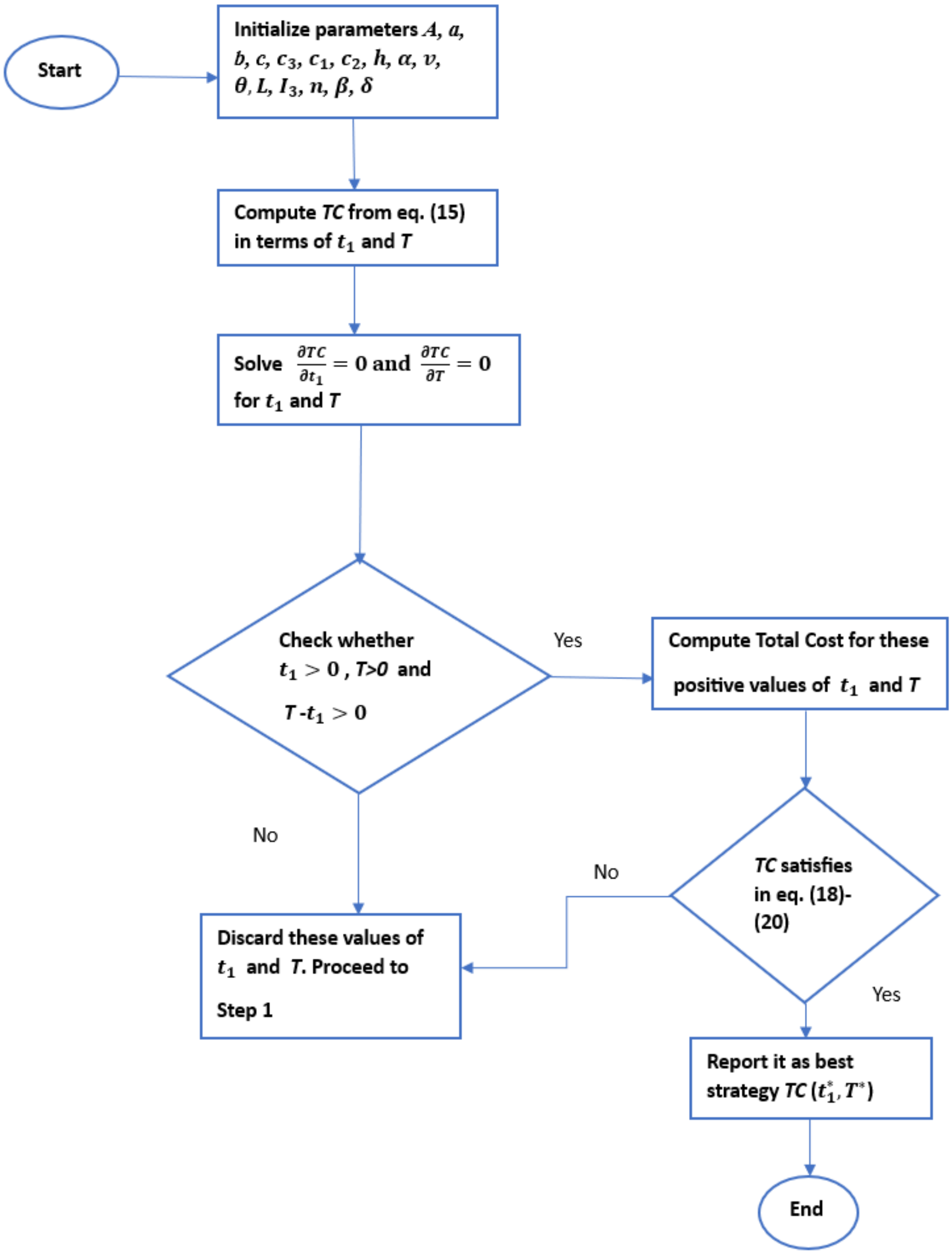

Fig 2. Flow chart illustrating the suggested model's step-by-step solution

3 Results and Discussion

3.1 Numerical Examples

Numerical Example 1

Consider the inventory system using the appropriate units for the following parameters:

$A = \$\ 2000/order$, $a = 125 units/month$, $b = 12.5$, $c = 1.25$, $c_3 = \$\ 10/unit$, $c_1 = \$\ 4/unit/month$, $c_2 = \$\ 8/unit/month$, $h = \$\ 0.5/unit/month$, $\alpha = 20/unit/(month)^2 v = 0.5$, $\theta = 0.8$, $L = 0.5\ month$, $I_3 = \$\ 0.01/month$, $n = 3$, $\beta = 0.6$, $\delta = 0.04$. We use Mathematica 11.3 software to find the optimum solution to this problem. The optimal value for inventory cycle time is $T^* = 2.224\ month$, and the optimal time for the shortage is given by $t_1^* = 0.7615\ month$. The optimum value of the average cost function corresponding to these optimal values is given by $TC = 2661.229\ \$$. We have used the MATHEMATICA 11.3 and data from the previous study as a source to understand the phenomenon. Figure 3 illustrates the practitioner's cyclic cost's joint convexity against ${t_1}^*$ and T^*.

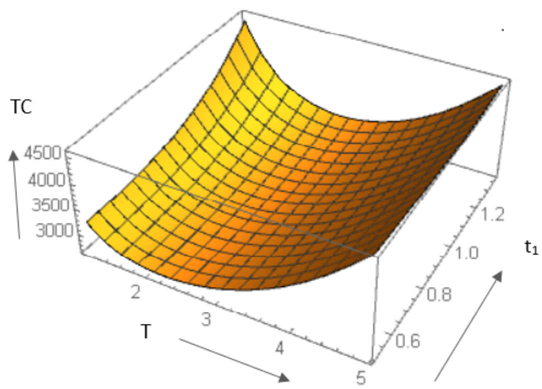

Fig 3. Cost Minimization Graph

Here, we found that 2.224 months is the optimal inventory cycle time T* according to the current study. The optimal replenishment is lower than that of the previous study[8] because we take into account non-linear demand, which makes the considered demand always bigger than the demand in that study. On the other hand,[8] has a greater ideal stock-in duration (t_1*). Seasonal demand or even EOQ may be the cause of this feature. When considering the power up to 7 degrees, which yields more accurate findings, the power of expansion of the exponential function during the holding cost calculation also affects this. Since greater powers tend to be zero, so we ignore them.

Numerical Example 2 (Without Advance Payment Scheme)

Considering the same input parameters as we consider after removing the advance payment factors in Numerical Example 1. The optimal value for inventory cycle time is $T^* = 2.181\ month$, and the optimal time for the shortage is given by $t_1^* = 0.7175\ month$. The optimum value of the average cost function corresponding to these optimal values is given by $TC = 3344\ \$$.

Comparative study for with and without advance payment scheme

We also study our proposed model scenario without an advance payment plan in a Numerical Example 2. Here, we see that the retailer's total costs rise when the advance payment system factor is eliminated. A comparison of with and without an early payment scheme between the Total Cost of the proposed scenario is shown in Table 3 and graphically in Figure 4.

Table 3. Comparison between the results of the model with and without the advance payment scheme

S. No.	Optimal Values	With Advance Payment	Without Advance Payment
1	${t_1}^*$	0.7615	0.7175
2	T^*	2.224	2.181
3	TC^*	2661.229	3344

Some managerial insights are also obtained for the benefit of the retailer from this comparison which are as follows:

- It's evident from the results of the numerical analysis that expenses of the proposed model without advance payment are larger than the case when we include the advance payment scheme for the model scenario. This happens because the retailer benefits from being able to pay advance in multiple installments, which removes the requirement to pay the full amount at once. But in return, depending on how many installments he makes, he will have to pay interest, which will lower his total business expenses.
- The lack of an advance payment plan causes inventory shortages earlier and leads to a lower inventory cycle time (T^*). So, retailers might delay purchases to use funds elsewhere, shortening the time items spend in inventory before sale and limiting the available funds.
- Since, we considered the non-linear demand, which may result in insufficient stock and earlier shortages for the retailers. The alignment of replenishment with actual sales becomes more challenging without the predictability that advance payments can provide, potentially leading to early stockouts.
- Retailers may reduce order quantities to minimize the risk of holding deteriorating stock, leading to more frequent reordering and shorter inventory cycles.

Therefore, in both cases, the non-linear nature of demand and time-dependent holding costs complicate inventory management, making it crucial for retailers to have robust forecasting and inventory optimization strategies. Without the benefits of advance payments, such as priority treatment from suppliers and potential discounts, retailers need to be more flexible and receptive to shifts in customer needs to avoid the negative impacts on inventory cycle time and the timing of shortages.

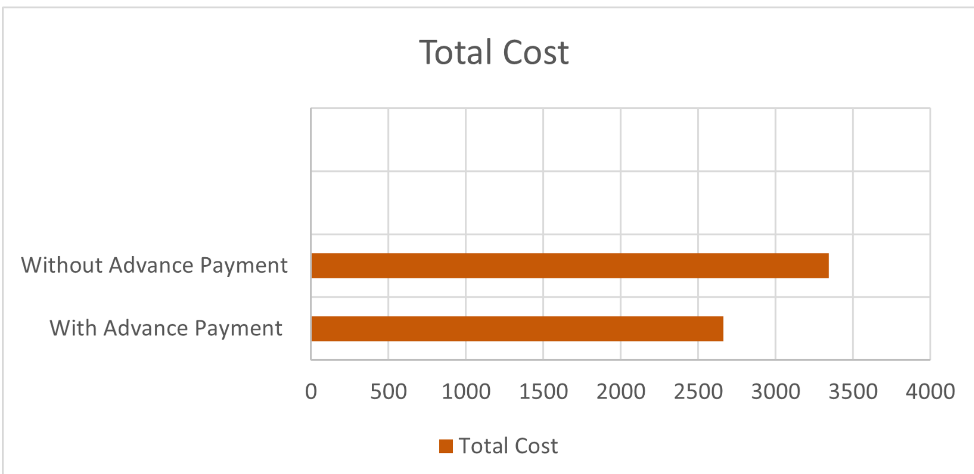

Fig 4. Comparison of Total Cost with and without advance payment scheme

3.2 Sensitivity Analysis and Managerial Insights

To prepare the analysis, the known parameters were changed from -20% to +20%. One parameter has been altered at a time during the studies, with the remaining parameters remaining constant at their initial values. Observations 1 through 6 provide the actual findings of this analysis. To inspect the impact of the changes in the known inventory parameters on the optimal policy, post-optimality analyses are executed for the numerical example mentioned above. It investigates the effects of various input parameters on the ideal solutions for the above-mentioned inventory system. The following insights are formed as per the effect of different parameters on t_1^* and T^* as shown in Figure 8.

Observation 1:

An increase in ordering cost A tends to increase in holding cost. As the holding cost per unit item increases (c_1), it leads to the fact that the item remains in inventory for some extra time, and the time t_1 at which shortage occurs gets delayed, which then increases total cost (TC) as shown in Figure 5 and Figure 6.

Observation 2:

Also, the increase in the initial demand parameter a, leads to the early occurrence of shortage during the cycle time T as shown in Figure 8**(a)** and Figure 8 **(b).** The impact of a company's product demand on its inventory cycle can be significant

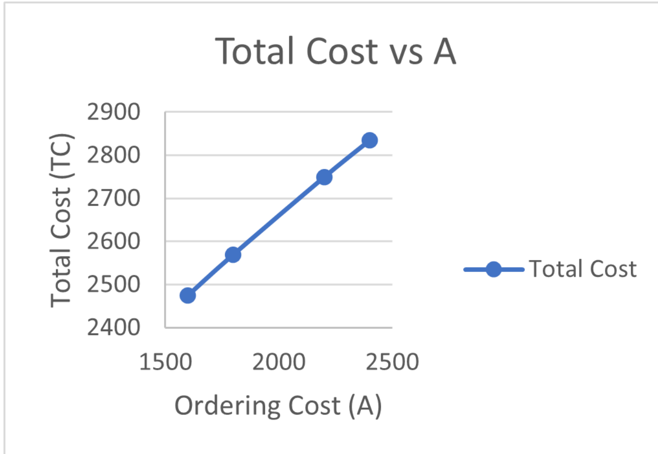

Fig 5. Graph of Total Cost vs Ordering Cost

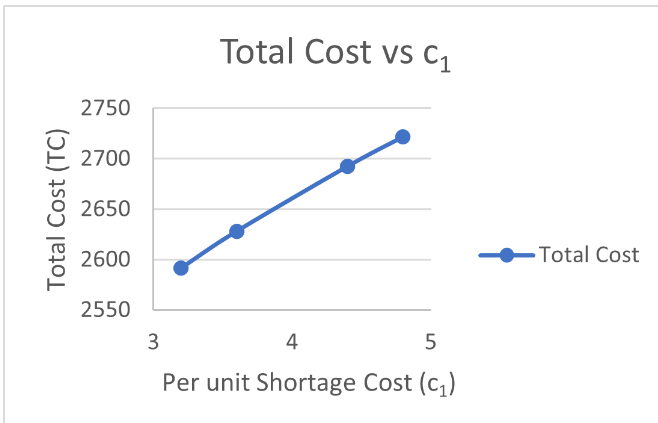

Fig 6. Graph of Total Cost vs c $_1$

since more demand can result in more frequent inventory cycles and improved inventory management.

Observation 3:

As we can observe in Figure 7 case of an increase in t_d from 0.4 to 0.6 means the time at which deterioration of the product increases leads to the fact that the product deteriorates less leading to a decrease in total cost from 2668 to 2654, depending on reduced waste and reduced inventory holding costs.

Observation 4:

With the increase in α i.e. the rate of change of holding cost increases from 16 to 24 with time, the retailer needs to increase the facility of storage space so that products don't deteriorate. As a result of this, the items are in good condition and likely to get sold quickly. Hence, results in an increase in holding per unit cost from 94.93 to 99.16.

Observation 5:

As the deterioration rate θ increases, the items that are fresh enough to sell decrease, which leads to the early arrival of shortage i.e., t_1 decreases from 0.77 to 0.74 for fresh items and hence, decreases the inventory cycle time T. This justifies the reason behind the result of the sensitivity effect of θ on t_1 and T.

Observation 6:

Total cost (TC^*) for the retailer rises with an increase in the values of parameters A, a, b, c_1, c_2, α, and θ. Also, the optimal total cost falls with an increase in the value of t_d.

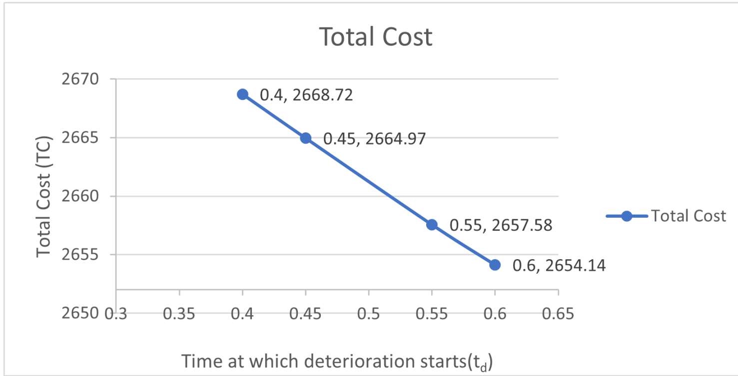

Fig 7. Impact of change in t_d on Total Cost

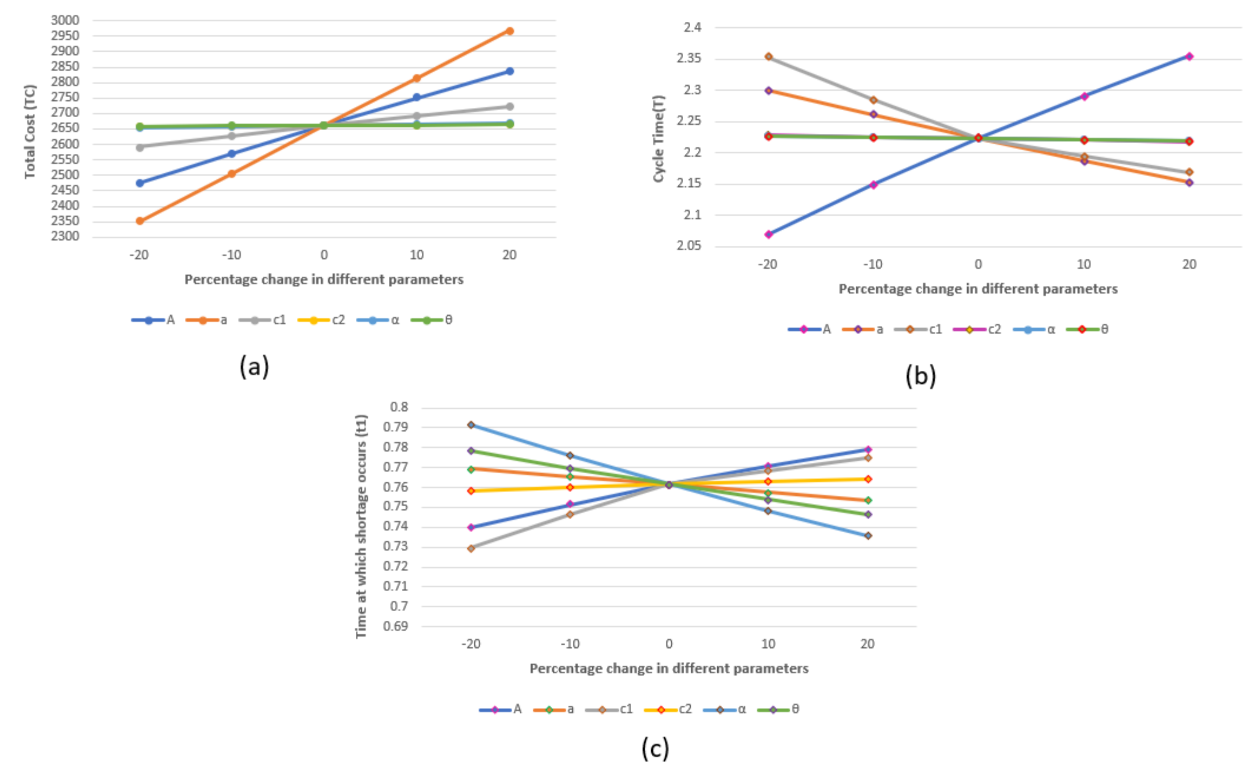

Fig 8. Sensitivity effect of different parameter on Total Cost (TC), Inventory cycle time (T) and Time at which shortage occurs (t_1)

4 Conclusion

Sustainable inventory management techniques for both time-dependent and non-instantaneous deteriorating products are examined in this study, and a mixed payment scheme of advance and cash is implemented. We have applied the idea of an advance payment plan, which aids the vendor to avoid the issue of lack of capital. The analysis revealed that the initial demand parameter, ordering cost, and holding cost parameter all had a considerable impact on the practitioner's final cost which has been observed from the sensitive analysis in Observations 1-6. As a result, it is beneficial to take into consideration the expenses of keeping extra seasonal inventory against the costs of addressing shifting demand. In addition, a vendor should utilize the product's prior history to generate cheap order costs when demand for an item falls and hence results in minimizing the total cost for the retailer.

The proposed model can be improved using Fuzzy techniques to address the uncertainty of demand and preservation technology to reduce the deterioration rate to make the model more realistic. Also, we can try to adopt soft computing techniques to find the optimum solution for decision variables which is global.

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