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RESEARCH ARTICLE

On Certain Applications via $(\varphi, \wp)$ -Suzuki type Fixed Point Results in A_b-Metric Spaces

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Abstract

Objective: In this study, we establish the existence and uniqueness of the common coupled fixed point theorem on complete A_b- metric space. **Method:** We have used the $(\varphi, \wp)$ - type Suzuki contraction on two mappings. **Findings:** For two mappings on the complete A_b-Metric Space, we were able to obtain and prove a unique common coupled fixed point theorem and justify it with a suitable example. **Novelty:** A unique common coupled fixed point is obtained by using$(\varphi, \wp)$ -type Suzuki contraction on two mappings and application of Suzuki contractive fixed point solutions of $(\varphi, \wp)$ - type to integral equations in the setup of A_b-Metric Space.

Subject classification for mathematics in 2000. 54E50, 47H10, and 54H25.
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1 Introduction

A stunning fusion of geometry, topology, and analysis is seen in fixed point theory. Due to its numerous applications in approximation theory, non-linear analysis, medical innovation, biomechanics, integral, and impulsive differential equations, problem solving, homotopy theory, and algorithms, it has been playing a significant role.

Generalized versions of the core discoveries of Banach and Edelstein were recently shown by Suzuki [(1)] which sparked a great deal of interest in this area [(2–4)]. The notion of b- Metric Space (MS) was proposed [(5)] in 1989. Later, many authors generalized this and introduced the new Metric Space. M. Abbas et al. [(6)] presented the idea of n-tuple Metric Space and examined its topological characteristics in 2015. A_b-Metric Space is a generalized version of n-tuple Metric Space, as first proposed by M. Ughade et al. [(7)]. Then, in partially ordered A_b-MS, K. Ravibabu et al. [(8)] obtained unique coupled common fixed point (UCCFP) theorems. Guo and Lakshmikantham [(9)] generated the notion of a coupled fixed point (CFP) in 1987. Later, Bhaskar and Lakshmikantham [(10)] employing a weak contractive type assumption, established a novel fixed point theorem for a mixed monotone mapping on a Metric Space equipped with partial order. Several authors [(11–16)] studied the coupled fixed-point results.

This paper is to provide unique coupled common fixed point theorems in the set-up of A_b-Metric Space for$(\varphi,\wp)$-type contractions and $(\varphi,\wp)$-type Suzuki contractive mapping. Additionally, we can provide appropriate examples and an application to integral equations.

2 Methodology

This section includes a few definitions, examples, and lemmas that are necessary to provide our primary findings.

Definition 2.1 [7]: Consider $\mathfrak{J}$ as a non-empty set, & $\vartheta \geq 1$ as a real number. An A_b-metric on $\mathfrak{J}$ is defined as a mapping $\tilde{n}_b : \mathfrak{J}^n \to [0,\infty)$ that meets the following constraints for every $æ_Z, ß \in I$

z = 1,2,3,…n.

$$(\tilde{n}_b 1) \quad \tilde{n}_b\left(æ_1, æ_2, \ldots\ldots.., æ_{n-1}, æ_n\right) \geq 0$$

$$(\tilde{n}_b 2) \quad \tilde{n}_b\left(æ_1, æ_2, \ldots\ldots.., æ_{n-1}, æ_n\right) = 0 \Leftrightarrow æ_1 = æ_2 = \cdots\cdots = æ_{n-1} = æ_n$$

$$(\tilde{n}_b 3) \quad \tilde{n}_b\left(æ_1, æ_2, \ldots\ldots.., æ_{n-1}, æ_n\right) \leq \vartheta \begin{pmatrix} \tilde{n}_b(æ_1, æ_1, \ldots\ldots\ldots.(æ_1)_{n-1}, ß) \\ +\tilde{n}_b(æ_2, æ_2, \ldots\ldots\ldots.., (æ_2)_{n-1}, ß) \\ +\ldots..+\tilde{n}_b(æ_{n-1}, æ_{n-1}, \ldots\ldots\ldots.., (æ_{n-1})_{n-1}, ß) \\ +\tilde{n}_b(æ_n, æ_n, \ldots\ldots.., (æ_n)_{n-1}, ß) \end{pmatrix}$$

Then $(\mathfrak{J}, \tilde{n}_b)$ is called an A_b- Metric Space

Definition 2.2 [7]**:** A metric space $(\mathfrak{J}, \tilde{n}_b)$ is symmetric if

$\tilde{n}_b\left(æ\,,\,æ, \ldots, (æ)_{n-1}, œ\right) = \tilde{n}_b\left(œ\,œ \ldots, (œ)_{n-1,}, æ\right) \forall æ, œ \in \mathfrak{J}$

Definition 2.3 [7]**:**Let $(\mathfrak{J}, \tilde{n}_b)$ denote a A_b-Metric Space Then, if $æ \in \mathfrak{J}, \delta > 0$, we define the open ball $B\tilde{n}_b(æ, \delta)$ and closed ball $B\tilde{n}_b[æ, \delta]$ with center æ and radius δ as follows:

$B\tilde{n}_b(æ, \delta) = \{œ \in \mathfrak{J} : \tilde{n}_b\left(œ, œ, \ldots, (œ)_{n-1}, æ\right) < \delta\}$, and

$B\tilde{n}_b[æ, \delta] = \{œ \in \mathfrak{J} : \tilde{n}_b\left(œ, œ, \ldots, (œ)_{n-1}, æ\right) \leq \delta\}$.

Lemma 2.4 [7]**: In a** A_b-Metric Space, we have

1. $\tilde{n}_b\left(æ,\, æ, \ldots, (æ)_{n-1}, œ\right) \leq \vartheta\, \tilde{n}_b\left(œ, œ, \ldots, (œ)_{n-1}, æ\right)$

2.$\tilde{n}_b\left(æ, æ, \ldots, (æ)_{n-1}, ß\right) \leq \vartheta\ (n-1)\ \tilde{n}_b\left(æ, æ, \ldots, (æ)_{n-1}, œ\right) + \vartheta^2 \tilde{n}_b\left(œ, œ, \ldots, (œ)_{n-1}, ß\right)$

Definition 2.5 [7]: Let $(\mathfrak{J}, \tilde{n}_b)$ be a Metric Space. A sequence $(æ_z\}$ $in\ \mathfrak{J}$ is defined as follows:

1. If $n_0 \in N$exists such that $\tilde{n}_b\left(æ_z, æ_z, \ldots, (æ_z)_{n-1}, æ_1\right) < \grave{0}$ to every $l, z \geq n_0$ then $(æ_z\}$ is $\tilde{n}_b$- Cauchy sequence.

2. We denote $\lim\limits_{z\to\infty} æ_z = æ$. $\tilde{n}_b$ convergent to the point $æ \in \mathfrak{J}$ if, $\forall \in > 0$ there exists a positive integer n_0 such that $\tilde{n}_b\left(æ_z\,,\, æ_z, \ldots, (æ_z)_{n-1}, æ\right) < \in$ to every $z \geq n_0$

3. In a metric space $(\mathfrak{J}, \tilde{n}_b)$ if all $\tilde{n}_b$- Cauchy sequences in $\mathfrak{J}$ are $\tilde{n}_b$-convergent, then A_b is said to be complete.

Lemma 2.6: Assuming that$\{æ_k\}$is $\tilde{n}_b$-convergent to æ and $\{œ_k\}$ is $\tilde{n}_b$-convergent to œ &$(\mathfrak{J}, \tilde{n}_b)$ is the A_b-metric space with $\vartheta \geq 1$ we have

$$\begin{aligned} (i)\ \tfrac{1}{\vartheta^2}\tilde{n}_b\left(æ, æ, \ldots, (æ)_{n-1}, œ\right) &\leq \lim_{z\to\infty} inf \tilde{n}_b\left(æ_z, æ_z, \ldots, (æ_z)_{n-1}, œ_z\right) \\ &\leq \lim_{z\to\infty} sup \tilde{n}_b\left(æ_z, æ_z, \ldots, (æ_z)_{n-1}, œ_z\right) \\ &\leq \vartheta^2 \tilde{n}_b\left(æ, æ, \ldots, (æ)_{n-1}, œ\right) \end{aligned}$$

In particular, if $œ_z$ = œ is a constant, then

$$\begin{aligned} (ii)\ \tfrac{1}{\vartheta^2}\tilde{n}_b\left(æ, æ, \cdots, (æ)_{n-1}, œ\right) &\leq \lim_{z\to\infty} inf \tilde{n}_b\left(æ_z, æ_z, \ldots, (æ_z)_{n-1}, œ\right) \\ &\leq \lim_{z\to\infty} sup \tilde{n}_b\left(æ_z, æ_z, \ldots, (æ_z)_{n-1}, œ\right) \\ &\leq \vartheta^2 \tilde{n}_b\left(æ, æ, \ldots, (æ)_{n-1}, œ\right) \end{aligned}$$

We consider the following to obtain our results.

3 Results and Discussion

Definition 3.1: Assume $(\mathfrak{J},\tilde{n}_b)$ be the A_b-Metric Space, a mapping $F:\mathfrak{J}^2\rightarrow\mathfrak{J}$. Then (æ,œ) is referred to as a coupled fixed point of Fif $F\,(æ,œ)=æ$, $F(œ,æ)\ =\ œ$, for $æ,œ\in\mathfrak{J}$

Definition 3.2: Assume that $F:\mathfrak{J}^2\rightarrow\mathfrak{J}$ and $f:\mathfrak{J}\rightarrow\mathfrak{J}$ be two mappings where $(\mathfrak{J},\tilde{n}_b)$ is the A_b-Metric space

1. If $F\,(æ,œ)=fæ,\ F(œ,æ)=fœ$,, then an element (æ,œ) is considered a coupled coincident point of F & f
2. A Coupled common point of F & f is defined as an element (æ,œ) if
$F\,(æ,œ)=fæ=æ,\ F(œ,æ)=fœ\ =\ œ,$
3. If $f(F\,(æ\,,œ))\ =\ F(fæ\,,fœ)$ and $=F(fœ,fæ)$ then for all $œ,æ\in\mathfrak{J}$, then there exists a pair $(F,f)\ni F\,(æ\,,œ)=fæ\,,\ F(œ,æ)=fœ\,\textit{then}\ (F,f)\ \textit{is called}\ \omega-\textit{compatible}$

Definition 3.3:$(\mathfrak{J},\tilde{n}_b)$ be a A_b- metric space & $F:\mathfrak{J}\times\mathfrak{J}\rightarrow\mathfrak{J}$ & $f:\mathfrak{J}\rightarrow\mathfrak{J}$ be two mappings

(i_0) The pair (F,f) is supposed to be a $(\varphi,\,\wp)$-type contraction, if there exist two functions $\varphi\in\Theta$ & $\wp\in\Omega$ such that

$$\forall æ,œ,ß,d\in\mathfrak{J},\ \tilde{n}_b\left(F\,(æ,œ)\,,F\,(æ,œ)\,,...F\,(æ,œ)\,,F\,(ß,d)\right)>0$$

$$\Rightarrow\wp\left((n-1)^3\,\vartheta^6\tilde{n}_b\left(F\,(æ,œ)\,,F\,(æ,œ)\,,...,F\,(æ,œ)\,,F\,(ß,d)\right)\right)\leq\wp\left(M\,(æ,œ,ß,d)\right)-\varphi\left(M\,(æ,œ,ß,d)\right)\tag{3.1}$$

(i_1) The pair (F,f) is supposed to be a $(\varphi,\,\wp)$-type Suzuki contraction, if There exists two functions $\varphi\in\Theta$and $\wp\in\Omega$ such that$\forall,æ,œ,ß,\partial,\in I$ with$F\,(æ,œ)\neq fæ$,$F\,(œ,\ æ)\neq fœ$ and $f\,æ\neq fœ$

$$\frac{1}{n\vartheta^2}\tilde{n}_b\left(fæ,fæ,...,(fæ)_{n-1},F(æ,œ)\right)\leq max\begin{pmatrix}\tilde{n}_b(fæ,fæ,...,(fæ)_{n-1},fß),\\\tilde{n}_b(fœ,fœ,...,(fœ)_{n-1},fd),\\\tilde{n}_b(fß,fß,...(fß)_{n-1},F(ß,d)),\\\tilde{n}_b(fd,fd,...,(fd)_{n-1},F(d,ß))\end{pmatrix}\ implies$$

$$\wp\left((n-1)^3\,\vartheta^6\tilde{n}_b\left(F\,(æ,œ)\,,F\,(æ,œ)\,,...,F\,(æ,œ)\,,F\,(ß,d)\right)\right)\leq\wp\left(M\,(æ,œ,ß,d)\right)-\varphi\left(M\,(æ,œ,ß,d)\right)\tag{3.2}$$

Where

$$M(æ,œ,ß,d)=max\begin{pmatrix}\tilde{n}_b(fæ,fæ,...,(fæ)_{n-1},fß),\\\tilde{n}_b(fœ,fœ,...,(fœ)_{n-1},fd),\\\tilde{n}_b(fæ,fæ,...,(fæ)_{n-1},F(æ,œ)),\\\tilde{n}_b(fœ,fœ,...,(fœ)_{n-1},F(œ,æ)),\\\tilde{n}_b(fß,fß,...,(fß)_{n-1},F(ß,d)),\\\tilde{n}_b(fd,fd,...,(fd)_{n-1},F(d,ß)),\\\tilde{n}_b(fß,fß,...,(fß)_{n-1},F(æ,œ)),\\\tilde{n}_b(fd,fd,...,(fd)_{n-1},F(œ,æ)),\\\frac{1}{(n-1)\vartheta^3}\tilde{n}_b(fæ,fæ,...,(fæ)_{n-1},F(ß,d)),\\\frac{1}{(n-1)\vartheta^3}\tilde{n}_b(fœ,fœ,....,(fœ)_{n-1},F(d,ß))\end{pmatrix}$$

And $\Omega=\{\wp/\wp:[0,\infty)\rightarrow[0,\infty)\}$ & $\Theta=\{\varphi/\varphi:[0,\infty)\rightarrow[0,\infty)\}$ are the functions satisfying the following conditions.

$(i_0)\wp$ is continuous, monotonically non-decreasing, and φ is lower semi-continuous.

$(i_1)\wp\,(t)=0=\varphi\,(t)$ if and only if $t=0$

Lemma 3.4: Assume that $\{æ_z\}$ is a sequence in $\mathfrak{J}$ such that $\lim\limits_{z\rightarrow\infty}\ \tilde{n}_b\left(æ_z,æ_z,...,æ_{z+1}\right)=0$

Let $(\mathfrak{J},\tilde{n}_b)$ be a A_b-Metric Space with coefficient $\vartheta\geq1$. There exist $\in>0$ & two sequences say $\left(æ_{w_k}\right\}$ & $\left(æ_{z_k}\right\}$ of positive integers such that the below instances hold if $\{x_z\}$ is not a ϱ_b-Cauchy sequence

$$(i)\ \grave{0}\leq\lim_{k\rightarrow\infty}inf\tilde{n}_b\left(æ_{w_k},æ_{w_k},...,æ_{z_k}\right)\leq\lim_{k\rightarrow\infty}sup\tilde{n}_b\left(æ_{w_k},æ_{w_k},...,æ_{z_k}\right)\leq(n-1)\,\vartheta\grave{0}$$

$$(ii)\ \frac{\grave{0}}{(n-1)\,\vartheta}\leq\lim_{k\rightarrow\infty}inf\tilde{n}_b\left(æ_{w_{k+1}},æ_{w_{k+1}},...,æ_{z_k}\right)\leq\lim_{k\rightarrow\infty}sup\tilde{n}_b\left(æ_{w_{k+1}},æ_{w_{k+1}},...,æ_{z_k}\right)\leq(n-1)^2\,\vartheta^2\grave{0}$$

$$(iii)\ \frac{\grave{0}}{(n-1)\vartheta} \leq \lim_{k\to\infty} inf\tilde{n}_b\left(æ_{w_k}, æ_{w_k}, ..., æ_{z_{k+1}}\right) \leq \lim_{k\to\infty} sup\tilde{n}_b\left(æ_{w_k}, æ_{w_k}, ..., æ_{z_{k+1}}\right) \leq (n-1)^2\vartheta^2\grave{0}$$

$$(iv)\ \frac{\grave{0}}{(n-1)\vartheta^3} \leq \lim_{k\to\infty} inf\tilde{n}_b\left(æ_{w_{k+1}}, æ_{w_{k+1}}, ..., æ_{z_{k+1}}\right) \leq \lim_{k\to\infty} sup\tilde{n}_b\left(æ_{w_{k+1}}, æ_{w_{k+1}}, ..., æ_{z_{k+1}}\right) \leq (n-1)^2\vartheta^4\grave{0}$$

Proof. Suppose $\{æ_z\}$ is not a $\tilde{n}_b$-Cauchy sequence, then there exists an $\in > 0$ & two sequences$\left\{æ_{w_k}\right\}$ & $\left\{æ_{z_k}\right\}$ (say) of positive integers such that $z_k > w_k > k$,

$$\tilde{n}_b\left(æ_{w_k}, æ_{w_k}, ..., æ_{z_{k-1}}\right) < \grave{0}\ and\ \tilde{n}_b\left(æ_{w_k}, æ_{w_k}, ..., æ_{z_k}\right) \geq \grave{0} \tag{3.3}$$

Using the fact that $\lim_{z\to\infty} \tilde{n}_b(æ_z, æ_z, ..., æ_{z+1}) = 0$ and (Equation (3.3)), we have that

$$\grave{0} \leq \tilde{n}_b\left(æ_{w_k}, æ_{w_k}, ..., æ_{z_k}\right) \leq (n-1)\vartheta\tilde{n}_b\left(æ_{w_k}, æ_{w_k}, ..., æ_{z_{k-1}}\right) + \vartheta^2\tilde{n}_b\left(æ_{z_{k-1}}, æ_{z_{k-1}}, ..., æ_{z_k}\right)$$
$$\leq (n-1)\vartheta\grave{0} + \vartheta^2\tilde{n}_b\left(æ_{z_{k-1}}, æ_{z_{k-1}}, ..., æ_{z_k}\right)$$

Clearly, we have that

$$\grave{0} \leq \lim_{k\to\infty} inf\tilde{n}_b\left(æ_{w_k}, æ_{w_k}, ..., æ_{z_k}\right) \leq \lim_{k\to\infty} sup\tilde{n}_b\left(æ_{w_k}, æ_{w_k}, ..., æ_{z_k}\right) \leq (n-1)\vartheta\grave{0}$$

Moreover, we have that

$$\grave{0} \leq \tilde{n}_b\left(æ_{w_k}, æ_{w_k}, ..., æ_{z_k}\right)$$

$$\leq (n-1)\vartheta\tilde{n}_b\left(æ_{w_k}, æ_{w_k}, ..., æ_{w_{k+1}}\right) + \vartheta^2\tilde{n}_b\left(æ_{w_{k+1}}, æ_{w_{k+1}}, ..., æ_{z_k}\right)$$

$$\leq (n-1)\vartheta\tilde{n}_b\left(æ_{w_k}, æ_{w_k}, ..., æ_{w_{k+1}}\right) + (n-1)\vartheta^3\tilde{n}_b\left(æ_{w_{k+1}}, æ_{w_{k+1}}, ..., æ_{z_{k+1}}\right)$$
$$+\vartheta^4\tilde{n}_b\left(æ_{z_{k+1}}, æ_{z_{k+1}}, ..., æ_{z_k}\right) \quad \text{and}$$
$$\tilde{n}_b\left(æ_{w_{k+1}}, æ_{w_{k+1}}, ..., æ_{z_{k+1}}\right) \leq (n-1)\vartheta\tilde{n}_b\left(æ_{w_{k+1}}, æ_{w_{k+1}}, ..., æ_{w_k}\right) + \vartheta^2\tilde{n}_b\left(æ_{w_k}, æ_{w_k}, ..., æ_{z_{k+1}}\right)$$
$$\leq (n-1)\vartheta\tilde{n}_b\left(æ_{w_{k+1}}, æ_{w_{k+1}}, ..., æ_{w_k}\right) + (n-1)\vartheta^3\tilde{n}_b\left(æ_{w_k}, æ_{w_k}, ..., æ_{z_k}\right)$$
$$+\vartheta^4\tilde{n}_b\left(æ_{z_k}, æ_{z_k}, ..., æ_{z_{k+1}}\right)$$
$$\leq (n-1)\vartheta\tilde{n}_b\left(æ_{w_{k+1}}, æ_{w_{k+1}}, ..., æ_{w_k}\right) + (n-1)^2\vartheta^4\tilde{n}_b\left(æ_{w_k}, æ_{w_k}, ..., æ_{z_{k-1}}\right)$$
$$+(n-1)\vartheta^5\tilde{n}_b\left(æ_{z_{k-1}}, æ_{z_{k-1}}, ..., æ_{z_k}\right) + \vartheta^4\tilde{n}_b\left(æ_{z_k}, æ_{z_k}, ..., æ_{z_{k+1}}\right)$$

By considering the lower limit in the first inequality as $k \to \infty$, the upper limit in the second inequality as $k \to \infty$ we arrive at

$$\frac{\grave{0}}{(n-1)\vartheta^3} \leq \lim_{k\to\infty} inf\tilde{n}_b\left(æ_{w_{k+1}}, æ_{w_{k+1}}, \cdots, æ_{z_{k+1}}\right) \leq \lim_{k\to\infty} sup\tilde{n}_b\left(æ_{w_{k+1}}, æ_{w_{k+1}}, \cdots, æ_{z_{k+1}}\right) \leq (n-1)^2\vartheta^4\grave{0}$$

Additionally, we have that

$$\grave{0} \leq \tilde{n}_b\left(æ_{w_k}, æ_{w_k}, ..., æ_{z_k}\right) \quad \leq \quad (n-1)\vartheta\tilde{n}_b\left(æ_{w_k}, æ_{w_k}, ..., æ_{z_{k+1}}\right) + \vartheta^2\tilde{n}_b\left(æ_{z_{k+1}}, æ_{z_{k+1}}, ..., æ_{z_k}\right)$$

And

$$\tilde{n}_b\left(æ_{w_k}, æ_{w_k}, ..., æ_{z_{k+1}}\right)$$

$$\leq (n-1)\vartheta\tilde{n}_b\left(æ_{w_k},æ_{w_k},...,æ_{z_k}\right)+\vartheta^2\tilde{n}_b\left(æ_{z_k},æ_{z_k},...,æ_{z_{k+1}}\right)$$

$$\leq (n-1)^2\vartheta^2\tilde{n}_b\left(æ_{w_k},æ_{w_k},...,æ_{z_{k-1}}\right)+\vartheta^3\tilde{n}_b\left(æ_{z_{k-1}},æ_{z_{k-1}},...,æ_{z_k}\right)$$

$+\vartheta^2\tilde{n}_b\left(æ_{z_k},æ_{z_k},...,æ_{z_{k+1}}\right)$

We also have that

$$\frac{\grave{0}}{(n-1)\vartheta}\leq \lim_{k\to\infty} inf\tilde{n}_b\left(æ_{w_k},æ_{w_k},...,æ_{z_{k+1}}\right)\leq \lim_{k\to\infty} sup\tilde{n}_b\left(æ_{w_k},æ_{w_k},...,æ_{z_{k+1}}\right)\leq (n-1)^2\vartheta^2\grave{0}$$

Applying a comparable method, we determine that

$$\frac{\grave{0}}{(n-1)\vartheta}\leq \lim_{k\to\infty} inf\tilde{n}_b\left(æ_{w_{k+1}},æ_{w_{k+1}},...,æ_{z_k}\right)\leq \lim_{k\to\infty} sup\tilde{n}_b\left(æ_{w_{k+1}},æ_{w_{k+1}},...,æ_{z_k}\right)\leq (n-1)^2\vartheta^2\grave{0}.$$

Theorem 3.5: Let $(\mathfrak{J},\tilde{n}_b)$ be a A_b-Metric Space and $F:\mathfrak{J}^2\to\mathfrak{J}$ & $F:\mathfrak{J}\to\mathfrak{J}$ be two mappings that satisfy $(\varphi,\wp)$-type suzuki contraction with

a)$F\left(\mathfrak{J}^2\right)\subseteq f\left(\mathfrak{J}\right)$ & $f\left(\mathfrak{J}\right)$ is complete,

b) The pair (F, f) right is ω -compatible.

Then there is unique coupled common fixed point of F & f in$\mathfrak{J}$

Proof: Suppose $æ_0, œ_0\in\mathfrak{J}$, then from (a), we establish the sequences$\{æ_z\},\{œ_z\}\,in\,\mathfrak{J}\,as$

$$F(æ_z,œ_z)=fæ_{z+1},F(œ_z,æ_z)\ =fœ_{z+1}\,where\,z=0,1,2.....$$

Case (i): Assume that

$$fæ_z\neq fæ_{z+1}\ or\ fœ_z\neq fœ_{z+1}\forall z. \tag{3.4}$$

Since,

$$\frac{1}{n\vartheta^2}\,\tilde{n}_b\left(fæ_0,fæ_0,\cdots,F(æ_0,œ_0)\right)=\frac{1}{n\vartheta^2}\tilde{n}_b\left(fæ_0,fæ_0,\cdots,fæ_1\right)$$

$$\leq\tilde{n}_b\left(fæ_0,fæ_0,\cdots,fæ_1\right)$$

$$\leq max\begin{Bmatrix}\tilde{n}_b\left(fæ_0,fæ_0,\cdots,fæ_1\right),\\ \tilde{n}_b\left(fœ_0,fœ_0,\cdots,fœ_1\right),\\ \tilde{n}_b\left(fæ_0,fæ_0,\cdots,F(æ_0,œ_0)\right),\\ \tilde{n}_b\left(fœ_0,fœ_0,\cdots,F(œ_0,æ_0)\right),\end{Bmatrix}$$

Then from (Equation (3.2)), we get

$$\wp\left((n-1)^3\vartheta^6\tilde{n}_b(fæ_1,fæ_1,...,fæ_2)\right)=\wp\left((n-1)^3\vartheta^6\tilde{n}_b\left(F(æ_0,œ_0),F(æ_0,œ_0),...,F(æ_1,œ_1)\right)\right)$$

$$\leq\ \wp(M(æ_0,œ_0,æ_1,œ_1))-\varphi(M(æ_0,œ_0,æ_1,œ_1))\qquad\leq\ \wp\left(max\begin{Bmatrix}\tilde{n}_b\left(fæ_0,fæ_0,...,fæ_1\right),\\ \tilde{n}_b\left(fœ_0,fœ_0,...,fœ_1\right),\\ \tilde{n}_b\left(fæ_1,fæ_1,...,fæ_2\right),\\ \tilde{n}_b\left(fœ_1,fœ_1,...,fœ_2\right)\end{Bmatrix}\right)-$$

$$\varphi\left(max\begin{Bmatrix}\tilde{n}_b\left(fæ_0,fæ_0,...,fæ_1\right),\\ \tilde{n}_b\left(fœ_0,fœ_0,...,fœ_1\right),\\ \tilde{n}_b\left(fæ_1,fæ_1,..,fæ_2\right),\\ \tilde{n}_b\left(fœ_1,fœ_1,...,fœ_2\right)\end{Bmatrix}\right)$$

$$\leq\wp\left(max\begin{Bmatrix}\tilde{n}_b\left(fæ_0,fæ_0,...,fæ_1\right),\\ \tilde{n}_b\left(fœ_0,fœ_0,...,fœ_1\right),\\ \tilde{n}_b\left(fæ_1,fæ_1,...,fæ_2\right),\\ \tilde{n}_b\left(fœ_1,fœ_1,...,fœ_2\right)\end{Bmatrix}\right)$$

Because of

$$M(æ_0, œ_0, æ_1, œ_1) = max \left\{ \begin{array}{c} \tilde{n}_b\left(fæ_0, fæ_0, ..., fæ_1\right), \tilde{n}_b\left(fœ_0, fœ_0, ..., fœ_1\right), \\ \tilde{n}_b\left(fæ_0, fæ_0, ..., F(æ_0, œ_0)\right), \tilde{n}_b\left(fœ_0, fœ_0, ..., F(œ_0, æ_0)\right), \\ \tilde{n}_b\left(fæ_1, fæ_1, ..., F(æ_1, œ_1)\right), \tilde{n}_b\left(fœ_1, fœ_1, ..., F(œ_1, æ_1)\right), \\ \tilde{n}_b\left(fæ_1, fæ_1, ..., F(æ_0, œ_0)\right), \tilde{n}_b\left(fœ_1, fœ_1, ..., F(œ_0, æ_0)\right), \\ \frac{1}{(n-1)\vartheta^3}\tilde{n}_b\left(fæ_0, fæ_0, ..., F(æ_1, œ_1)\right), \\ \frac{1}{(n-1)\vartheta^3}\tilde{n}_b\left(fœ_0, fœ_0, ..., F(œ_1, æ_1)\right), \end{array} \right\}$$

$$\leq max \left\{ \begin{array}{c} \tilde{n}_b\left(fæ_0, fæ_0, ..., fæ_1\right), \tilde{n}_b\left(fœ_0, fœ_0, ..., fœ_1\right), \\ \tilde{n}_b\left(fæ_1, fæ_1, ..., fæ_2\right), \tilde{n}_b\left(fœ_1, fœ_1, ..., fœ_2\right) \end{array} \right\}$$

From definition of $\wp$, we have

$$\tilde{n}_b(fæ_1, fæ_1, ..., fæ_2) \leq \frac{1}{(n-1)^3\vartheta^6} max \left\{ \begin{array}{c} \tilde{n}_b\left(fæ_0, fæ_0, ..., fæ_1\right), \\ \tilde{n}_b\left(fœ_0, fœ_0, ..., fœ_1\right), \\ \tilde{n}_b\left(fæ_1, fæ_1, ..., fæ_2\right), \\ \tilde{n}_b\left(fœ_1, fœ_1, ..., fœ_2\right) \end{array} \right\} \tag{3.5}$$

Similarly, we can prove that

$$\tilde{n}_b(fœ_1, fœ_1, ..., fœ_2) \leq \frac{1}{(n-1)^3\vartheta^6} max \left\{ \begin{array}{c} \tilde{n}_b\left(fæ_0, fæ_0, ..., fæ_1\right), \\ \tilde{n}_b\left(fœ_0, fœ_0, ..., fœ_1\right), \\ \tilde{n}_b\left(fæ_1, fæ_1, ..., fæ_2\right), \\ \tilde{n}_b\left(fœ_1, fœ_1, ..., fœ_2\right) \end{array} \right\} \tag{3.6}$$

From (Equations (3.5) and (3.6)) one can conclude

$$max \left\{ \begin{array}{c} \tilde{n}_b\left(fæ_1, fæ_1, ..., fæ_2\right), \\ \tilde{n}_b\left(fœ_1, fœ_1, ..., fœ_2\right) \end{array} \right\} \leq \frac{1}{(n-1)^3\vartheta^6} max \left\{ \begin{array}{c} \tilde{n}_b\left(fæ_0, fæ_0, ..., fæ_1\right), \\ \tilde{n}_b\left(fœ_0, fœ_0, ..., fœ_1\right), \\ \tilde{n}_b\left(fæ_1, fæ_1, ..., fæ_2\right), \\ \tilde{n}_b\left(fœ_1, fœ_1, ..., fœ_2\right) \end{array} \right\} \tag{3.7}$$

If $max \left\{ \begin{array}{c} \tilde{n}_b\left(fæ_0, fæ_0, ..., fæ_1\right), \\ \tilde{n}_b\left(fœ_0, fœ_0, ..., fœ_1\right) \end{array} \right\} \leq max \left\{ \begin{array}{c} \tilde{n}_b\left(fæ_1, fæ_1, ..., fæ_2\right), \\ \tilde{n}_b\left(fœ_1, fœ_1, ..., fœ_2\right) \end{array} \right\}$.

Then from (Equation (3.7)), we have

$$max \left\{ \begin{array}{c} \tilde{n}_b(fæ_1, fæ_1, ..., fæ_2), \\ \tilde{n}_b(fœ_1, fœ_1, ..., fœ_2) \end{array} \right\} \leq \frac{1}{(n-1)^3\vartheta^6} max \left\{ \begin{array}{c} \tilde{n}_b\left(fæ_1, fæ_1, ..., fæ_2\right), \\ \tilde{n}_b\left(fœ_1, fœ_1, ..., fœ_2\right) \end{array} \right\}$$

This is a contradiction. Hence, from (Equation (3.7)), we have

$$max \left\{ \begin{array}{c} \tilde{n}_b(fæ_1, fæ_1, ..., fæ_2), \\ \tilde{n}_b(fœ_1, fœ_1, ..., fœ_2) \end{array} \right\} \leq \frac{1}{(n-1)^3\vartheta^6} max \left\{ \begin{array}{c} \tilde{n}_b\left(fæ_0, fæ_0, ..., fæ_1\right), \\ \tilde{n}_b\left(fœ_0, fœ_0, ..., fœ_1\right) \end{array} \right\}.$$

Continuing in this way, we get

$$max \left\{ \begin{array}{c} \tilde{n}_b\left(fæ_z, fæ_z, ..., fæ_{z+1}\right), \\ \tilde{n}_b\left(fœ_z, fœ_z, ..., fœ_{z+1}\right) \end{array} \right\} \leq \frac{1}{(n-1)^3\vartheta^6} max \left\{ \begin{array}{c} \tilde{n}_b\left(fæ_{z-1}, fæ_{z-1}, ..., fæ_z\right), \\ \tilde{n}_b\left(fœ_{z-1}, fœ_{z-1}, ..., fœ_z\right) \end{array} \right\}$$

$$\leq \frac{1}{((n-1)^3\vartheta^6)^2} max \left\{ \begin{array}{c} \tilde{n}_b\left(fæ_{z-2}, fæ_{z-2}, ..., fæ_{z-1}\right), \\ \tilde{n}_b\left(fœ_{z-2}, fœ_{z-2}, ..., fœ_{z-1}\right) \end{array} \right\}$$

$$\vdots$$

$$\leq \frac{1}{((n-1)^3\vartheta^6)^p} max \left\{ \begin{array}{c} \tilde{n}_b\left(fæ_0, fæ_0, ..., fæ_1\right), \\ \tilde{n}_b\left(fœ_0, fœ_0, ..., fœ_1\right) \end{array} \right\}. \to 0 \; as \; z \to \infty$$

Therefore,

$$\lim_{z\to\infty} \tilde{n}_b\left(fæ_z, fæ_z, ..., fæ_{z+1}\right) = 0 \; and \; \lim_{z\to\infty} \tilde{n}_b\left(fœ_z, fœ_z, ..., fœ_{z+1}\right) = 0 \tag{3.8}$$

The $\tilde{n}_b$- cauchy sequence $\{f\ae_z\}$ will now be demonstrated. Assuming that$\{f\ae_z\}$ is not a $\tilde{n}_b$-Cauchy sequence, then $\tilde{n}_b\left(f\ae_{w_k}, f\ae_{w_k},...,f\ae_{z_k}\right) \geq \in$ and sequences of positive integers$\{z_k\}$ and$\{w_k\}$ where $z_k \geq w_k \geq k$. We can select $\{z_k\}$ as the smallest positive integer for each $k > 0$, corresponding to w_k such that $\tilde{n}_b\left(f\ae_{w_k}, f\ae_{w_k},...,f\ae_{z_k}\right) \geq \in$, We can select $z_0 \in N \cup \{0\}$ such that $\tilde{n}_b\left(f\ae_{w_k}, f\ae_{w_k},...,f\ae_{z_{k-1}}\right) < \in$

$$\frac{1}{n\vartheta^2}\tilde{n}_b\left(f\ae_{w_k}, f\ae_{w_k},...,F(\ae_{w_k},\oe_{w_k})\right) < \frac{\in}{n\vartheta^2} < \in \leq \tilde{n}_b(f\ae_{w_k}, f\ae_{w_k},...,f\ae_{z_k})$$

$$\leq max\left\{\begin{array}{c}\tilde{n}_b(f\ae_{w_k}, f\ae_{w_k},...,f\ae_{z_k}),\\ \tilde{n}_b(f\oe_{w_k}, f\oe_{w_k},...,f\oe_{z_k}),\\ \tilde{n}_b\left(f\ae_{w_k}, f\ae_{w_k},...,F(\ae_{w_k},\oe_{w_k})\right),\\ \tilde{n}_b\left(f\oe_{w_k}, f\oe_{w_k},...,F(\oe_{w_k},\ae_{w_k})\right)\end{array}\right\}.$$

Then, from (Equation (3.2)), we can get

$$\wp\left((n-1)^3\vartheta^6\tilde{n}_b\left(f\ae_{w_{k+1}}, f\ae_{w_{k+1}},...,f\ae_{z_{k+1}}\right)\right) = \wp\left((n-1)^3\vartheta^6\tilde{n}_b\left(F\left(\ae_{w_k},\oe_{w_k}\right), F\left(\ae_{w_k},\oe_{w_k}\right),...,F\left(\ae_{z_k},\oe_{z_k}\right)\right)\right)$$

$$\leq \wp\left(M\left(\ae_{w_k},\oe_{w_k},\ae_{z_k},\oe_{z_k}\right)\right) - \varphi\left(M\left(\ae_{w_k},\oe_{w_k},\ae_{z_k},\oe_{z_k}\right)\right) \tag{3.9}$$

Where $M(\ae_{w_k},\oe_{w_k},\ae_{z_k},\oe_{z_k})$

$$= max\left\{\begin{array}{c}\tilde{n}_b\left(f\ae_{w_k}, f\ae_{w_k},...,f\ae_{z_k}\right), \tilde{n}_b\left(f\oe_{w_k}, f\oe_{w_k},...,f\oe_{z_k}\right),\\ \tilde{n}_b\left(f\ae_{w_k}, f\ae_{w_k},...,F(\ae_{w_k},\oe_{w_k})\right), \tilde{n}_b\left(f\oe_{w_k}, f\oe_{w_k},...,F(\oe_{w_k},x_{w_k})\right),\\ \tilde{n}_b\left(f\ae_{z_k}, f\ae_{z_k},...,F(\ae_{z_k},\oe_{z_k})\right), \tilde{n}_b\left(f\oe_{z_k}, f\oe_{z_k},...,F(\oe_{z_k},\ae_{z_k})\right),\\ \tilde{n}_b\left(f\ae_{z_k}, f\ae_{z_k},...,F(\ae_{w_k},\oe_{w_k})\right), \tilde{n}_b\left(f\oe_{z_k}, f\oe_{z_k},...,F(\oe_{w_k},\ae_{w_k})\right),\\ \frac{1}{(n-1)\vartheta^3}\tilde{n}_b\left(f\ae_{w_k}, f\ae_{w_k},...,F(\ae_{z_k},\oe_{z_k})\right),\\ \frac{1}{(n-1)\vartheta^3}\tilde{n}_b\left(f\oe_{w_k}, f\oe_{w_k},...,F(\oe_{z_k},\ae_{z_k})\right)\end{array}\right\}$$

$$\leq max\left\{\begin{array}{c}\tilde{n}_b\left(f\ae_{w_k}, f\ae_{w_k},...,f\ae_{z_k}\right), \tilde{n}_b\left(f\oe_{w_k}, f\oe_{w_k},...,f\oe_{z_k}\right),\\ \tilde{n}_b\left(f\ae_{w_k}, f\ae_{w_k},...,f\ae_{w_{k+1}}\right), \tilde{n}_b\left(f\oe_{w_k}, f\oe_{w_k},...,f\oe_{w_{k+1}}\right),\\ \tilde{n}_b\left(f\ae_{z_k}, f\ae_{z_k},...,f\ae_{z_{k+1}}\right), \tilde{n}_b\left(f\oe_{z_k}, f\oe_{z_k},...,f\oe_{z_{k+1}}\right),\\ \tilde{n}_b\left(f\ae_{z_k}, f\ae_{z_k},...,f\ae_{w_{k+1}}\right), \tilde{n}_b\left(f\oe_{z_k}, f\oe_{z_k},...,f\oe_{w_{k+1}}\right)\end{array}\right\}$$

Using (Equation (3.4)) and (Equation (3.8)), we have that

$$\grave{0} \leq \lim_{k\to\infty} sup\, M(\ae_{w_k},\oe_{w_k},\ae_{z_k},\oe_{z_k})$$

$$= \lim_{k\to\infty} sup\, max\left\{\begin{array}{c}\tilde{n}_b\left(f\ae_{w_k}, f\ae_{w_k},...,f\ae_{z_k}\right), \tilde{n}_b\left(f\oe_{w_k}, f\oe_{w_k},...,f\oe_{z_k}\right),\\ \tilde{n}_b\left(f\ae_{w_k}, f\ae_{w_k},...,f\ae_{w_{k+1}}\right), \tilde{n}_b\left(f\oe_{w_k}, f\oe_{w_k},...,f\oe_{w_{k+1}}\right),\\ \tilde{n}_b\left(f\ae_{z_k}, f\ae_{z_k},...,f\ae_{z_{k+1}}\right), \tilde{n}_b\left(f\oe_{z_k}, f\oe_{z_k},...,f\oe_{z_{k+1}}\right),\\ \tilde{n}_b\left(f\ae_{z_k}, f\ae_{z_k},...,f\ae_{w_{k+1}}\right), \tilde{n}_b\left(f\oe_{z_k}, f\oe_{z_k},...,f\oe_{w_{k+1}}\right)\end{array}\right\}$$

$$\leq \{(n-1)\vartheta\grave{0}, (n-1)\vartheta\grave{0}, 0, 0, 0, 0, (n-1)^2\vartheta^3\grave{0}, (n-1)^2\vartheta^3\grave{0}\} = (n-1)^2\vartheta^3\grave{0}$$

So that by using (Equation (3.4)) and (Equation (3.9)) becomes,

$$\wp\left((n-1)^2\vartheta^3\grave{0}\right) = \wp\left((n-1)^3\vartheta^6\frac{\grave{0}}{(n-1)\vartheta^3}\right)$$

$$\leq \wp\left((n-1)^3 \vartheta^6 \lim_{z\to\infty} sup\, \tilde{n}_b\left(fæ_{w_{k+1}}, fæ_{w_{k+1}}, ..., fæ_{z_{k+1}}\right)\right)$$
$$\leq \wp\left((n-1)^3 \vartheta^6 \lim_{z\to\infty} sup\, \tilde{n}_b\left(F\left(æ_{w_k}, œ_{w_k}\right), F\left(æ_{w_k}, œ_{w_k}\right), ..., F\left(æ_{z_k}, œ_{z_k}\right)\right)\right)$$
$$\leq \lim_{z\to\infty} sup \wp\left((n-1)^3 \vartheta^6 \tilde{n}_b\left(F\left(æ_{w_k}, œ_{w_k}\right), F\left(æ_{w_k}, œ_{w_k}\right), ..., F\left(æ_{z_k}, œ_{z_k}\right)\right)\right)$$
$$\leq \wp\left((n-1)^3 \vartheta^6 \grave{0}\right) - \varphi\left((n-1)^3 \vartheta^6 \grave{0}\right)$$
$$< \wp\left((n-1)^3 \vartheta^6 \grave{0}\right).$$

It is a contradiction. Thus, in $f(\mathfrak{J})$, $\{fæ_z\}$ is a Cauchy sequence. In a similar manner, we may verify $\{fœ_z\}$ in $f(\mathfrak{J})$ is a Cauchy sequence. There exists α, β and d , n in $\mathfrak{J}$ such that $f(\mathfrak{J})$ is complete. $\lim_{z\to\infty} fæ_z = \alpha = fd$, $\lim_{z\to\infty} fœ_z = \beta = fn$

Since $fæ_z \to \alpha$, $fœ_z \to \beta$ for infinitely many z

Now we claim that for $fæ_z \neq \alpha$, $fœ_z \neq \beta$

$$max\begin{Bmatrix} \tilde{n}_b(fd, fd, ..., F(ß, Z)), \\ \tilde{n}_b(fn, fn, ..., F(Z, ß)) \end{Bmatrix} \leq max\begin{Bmatrix} \tilde{n}_b(fd, fd, ..., fß), \tilde{n}_b(fn, fn, ..., fZ), \\ \tilde{n}_b(fß, fß, ..., F(ß, Z)), \tilde{n}_b(fZ, fZ, ..., F(Z, ß)) \end{Bmatrix}$$

For all $ß, Z \in \mathfrak{J}$ with $fd \neq fß, fn \neq fZ$

Let $ß, Z \in \mathfrak{J}$ with $fd \neq fß, fn \neq fZ$. Then There exists a positive integer z_0 $such\ that\ for\ z \geq z_0$, suppose that

$$\tilde{n}_b(fæ_z, fæ_z, ..., fß) < \frac{1}{n\vartheta^2}\tilde{n}_b(fæ_z, fæ_z, ..., fæ_{z+1})$$

Or

$$\tilde{n}_b(fæ_{z+1}, fæ_{z+1}, ..., fß) < \frac{1}{n\vartheta^2}\tilde{n}_b(fæ_{z+1}, fæ_{z+1}, ..., fæ_{z+2})$$

And

$$\tilde{n}_b(fœ_z, fœ_z, ..., fZ) < \frac{1}{n\vartheta^2}\tilde{n}_b(fœ_z, fœ_z, ..., fœ_{z+1})$$

Or

$$\tilde{n}_b(fœ_{z+1}, fœ_{z+1}, ..., fZ) < \frac{1}{n\vartheta^2}\tilde{n}_b(fœ_{z+1}, fœ_{z+1}, ..., fœ_{z+2})$$

Then, using the fact that

$\tilde{n}_b(fæ_{z+1}, fæ_{z+1}, ..., fæ_{z+2}) \leq \tilde{n}_b(fæ_z, fæ_z, ..., fæ_{z+1})$ $we\ have$

$$\tilde{n}_b(fæ_z, fæ_z, ..., F(æ_z, œ_z)) = \tilde{n}_b(fæ_z, fæ_z, ..., fæ_{z+1})$$

$$\leq (n-1)\vartheta\tilde{n}_b(fæ_z, fæ_z, ..., fß) + \vartheta\tilde{n}_b(fæ_{z+1}, fæ_{z+1}, ..., fß)$$

$$\leq \frac{(n-1)}{n\vartheta}\tilde{n}_b(fæ_z, fæ_z, ..., fæ_{z+1})$$

$$+\frac{1}{(n-1)\vartheta}\tilde{n}_b(fæ_{z+1}, fæ_{z+1}, ..., fæ_{z+2})$$

$$\leq \frac{1}{\vartheta}\left(\frac{n-1}{n} + \frac{1}{n-1}\right)\tilde{n}_b(fæ_z, fæ_z, ..., fæ_{z+1})$$

$$< \tilde{n}_b(fæ_z, fæ_z, ..., F(æ_z, œ_z))$$

Since the inequality mentioned above is contradictory, we must have that

$$\frac{1}{n\vartheta^2}\tilde{n}_b\left(f æ_z, f æ_z, ..., f æ_{z+1}\right) \leq \tilde{n}_b\left(f æ_z, f æ_z, ..., f ß\right) \leq max\left\{\begin{matrix} \tilde{n}_b\left(f æ_z, f æ_z, ..., f ß\right), \\ \tilde{n}_b\left(f œ_z, f œ_z, ..., f Z\right), \\ \tilde{n}_b\left(f æ_z, f æ_z, ..., F(æ_z, œ_z)\right), \\ \tilde{n}_b\left(f œ_z, f œ_z, ..., F(œ_z, æ_z)\right) \end{matrix}\right\}$$

Or

$$\frac{1}{n\vartheta^2}\tilde{n}_b\left(f æ_{z+1}, f æ_{z+1}, ..., f æ_{z+2}\right) \leq \tilde{n}_b\left(f æ_{z+1}, f æ_{z+1}, ..., f ß\right) \leq max\left\{\begin{matrix} \tilde{n}_b\left(f æ_{z+1}, f æ_{z+1}, ..., f ß\right), \\ \tilde{n}_b\left(f œ_{z+1}, f œ_{z+1}, ..., f Z\right), \\ \tilde{n}_b\left(f æ_{z+1}, f æ_{z+1}, ..., F(æ_{z+1}, œ_{z+1})\right), \\ \tilde{n}_b\left(f œ_{z+1}, f œ_{z+1}, ..., F(œ_{z+1}, æ_{z+1})\right) \end{matrix}\right\}$$

Hence, from (Equation (3.2)), we have

$$\wp\left(\tilde{n}_b\left(F\left(æ_z, œ_z\right), F\left(æ_z, œ_z\right), ..., F\left(ß, Z\right)\right)\right) \leq \wp\left((n-1)^3\vartheta^6\tilde{n}_b\left(F\left(æ_z, œ_z\right), F\left(æ_z, œ_z\right), ..., F\left(ß, Z\right)\right)\right)$$

$$\leq \wp\left(M\left(æ_z, œ_z, ß, Z\right)\right) - \varphi\left(M\left(æ_z, œ_z, ß, Z\right)\right)$$

$$\leq \wp\left(M\left(æ_z, œ_z, ß, Z\right)\right)$$

where

$$M(æ_z, œ_z, ß, Z) = max\left\{\begin{matrix} \tilde{n}_b\left(f æ_z, f æ_z, ..., f ß\right), \tilde{n}_b\left(f œ_z, f œ_z, ..., f Z\right), \\ \tilde{n}_b\left(f æ_z, f æ_z, ..., F(æ_z, œ_z)\right), \tilde{n}_b\left(f œ_z, f œ_z, ..., F(œ_z, æ_z)\right), \\ \tilde{n}_b\left(f ß, f ß, ..., F(ß, Z)\right), \tilde{n}_b\left(f Z, f Z ..., F(Z, ß)\right), \\ \tilde{n}_b\left(f ß, f ß, ..., F(æ_z, œ_z)\right), \tilde{n}_b\left(f Z, f Z, ..., F(œ_z, æ_z)\right), \\ \frac{1}{(n-1)\vartheta^3}\tilde{n}_b\left(f æ_z, f æ_z, ..., F(x, y)\right), \frac{1}{(n-1)\vartheta^3}\tilde{n}_b\left(f œ_z, f œ_z, ..., F(y, x)\right) \end{matrix}\right\}$$

By using the notion of ⊠ and letting $z \to \infty$in the inequality above, we obtain

$$\tilde{n}_b(fd, fd, ..., F(ß, Z)) \leq max\left\{\begin{matrix} \tilde{n}_b\left(fd, fd, ..., f ß\right), \tilde{n}_b\left(fn, fn, ..., f Z\right), \\ \tilde{n}_b\left(f ß, f ß, ..., F(ß, Z)\right), \tilde{n}_b\left(f Z, f Z, ..., F(Z, ß)\right) \end{matrix}\right\}$$

Similarly, we can show that

$$\tilde{n}_b(fn, fn, ..., F(Z, ß)) \leq max\left(\begin{matrix} \tilde{n}_b\left(fd, fd, ..., f ß\right), \tilde{n}_b\left(fn, fn, ..., f Z\right), \\ \tilde{n}_b\left(f ß, f ß, ..., F(ß, Z)\right), \tilde{n}_b\left(f Z, f Z, ..., F(Z, ß)\right) \end{matrix}\right\}$$

Thus

$$max\left\{\begin{matrix} \tilde{n}_b(fd, fd, ..., F(ß, Z)), \\ \tilde{n}_b(fn, fn, ..., F(Z, ß)) \end{matrix}\right\} \leq max\left\{\begin{matrix} \tilde{n}_b\left(fd, fd, ..., f ß\right), \tilde{n}_b\left(fn, fn, ..., f Z\right), \\ \tilde{n}_b\left(f ß, f ß, ..., F(ß, Z)\right), \tilde{n}_b\left(f Z, f Z, ..., F(Z, ß)\right) \end{matrix}\right\}$$

Hence, the claim. Now consider

$$\tilde{n}_b\left(f ß, f ß, ..., F(ß, Z)\right) \leq (n-1)\vartheta^2\tilde{n}_b\left(fd, fd, ..., f ß\right) + \vartheta^2\tilde{n}_b\left(fd, fd, ..., F(ß, Z)\right)$$

$$\leq n\vartheta^2 max\left\{\begin{matrix} \tilde{n}_b\left(fd, fd, ..., f ß\right), \tilde{n}_b\left(fn, fn, ..., f Z\right), \\ \tilde{n}_b\left(f ß, f ß, ..., F(ß, Z)\right), \tilde{n}_b\left(f Z, f Z, F(Z, ß)\right) \end{matrix}\right\}.$$

Thus

$$\frac{1}{n\vartheta^2}\tilde{n}_b\left(f ß, f ß, ..., F(ß, Z)\right) \leq max\left\{\begin{matrix} \tilde{n}_b\left(fd, fd, ..., f ß\right), \tilde{n}_b\left(fn, fn, ..., f Z\right), \\ \tilde{n}_b\left(f ß, f ß, ..., F(ß, Z)\right), \tilde{n}_b\left(f Z, f Z, ..., F(Z, ß)\right) \end{matrix}\right\}.$$

Hence, from (Equation (3.2)), we have

$$\wp\left(\tilde{n}_b\left(F(d,n),F(d,n),...,F(ß,Z)\right)\right)\leq\wp\left((n-1)^3\vartheta^6\tilde{n}_b\left(F(d,n),F(d,n),...,F(ß,Z)\right)\right)$$

$$\leq\wp(M(d,n,ß,Z))-\varphi(M(d,n,ß,Z))$$

$$\leq\wp\left(max\left\{\begin{matrix}\tilde{n}_b\left(fd,fd,...,fß\right),\ \tilde{n}_b\left(fn,fn,...,fZ\right),\\ \tilde{n}_b\left(fd,fd,...,F(d,n)\right),\ \tilde{n}_b\left(fn,fn,...,F(n,d)\right),\\ \tilde{n}_b\left(fß,fß,...,F(ß,Z)\right),\ \tilde{n}_b\left(fZ,fZ,...,F(Z,ß)\right),\\ \tilde{n}_b\left(fß,fß,...,F(d,n)\right),\ \tilde{n}_b\left(fZ,fZ,...,F(n,d)\right)\end{matrix}\right\}\right)$$

By the definition of$\wp$, we have that

$$\tilde{n}_b\left(F(d,n),F(d,n),...,F(ß,Z)\right)\ \leq max\left\{\begin{matrix}\tilde{n}_b\left(fd,fd,...,fß\right),\ \tilde{n}_b\left(fn,fn,\cdots,fZ\right),\\ \tilde{n}_b\left(fd,fd,...,F(d,n)\right),\ \tilde{n}_b\left(fn,fn,\cdots,F(n,d)\right),\\ \tilde{n}_b\left(fß,fß,...,F(ß,Z)\right),\ \tilde{n}_b\left(fZ,fZ,\cdots,F(Z,ß)\right),\\ \tilde{n}_b\left(fß,fß,...,F(d,n)\right),\ \tilde{n}_b\left(fZ,fZ,\cdots,F(n,d)\right)\end{matrix}\right\}$$

Now, from **Lemma (2.6)**, we have

$$\tfrac{1}{\vartheta^2}\tilde{n}_b\left(fd,fd,...,(fd)_{n-1},F\left(d,n\right)\right)\quad\leq\quad\lim_{z\to\infty}inf\tilde{n}_b\left(fæ_{z+1},fæ_{z+1},...,F\left(d,n\right)\right)$$

Now from (Equation (3.2)) and applying $\wp$ on both sides, we have that

$$\wp\left((n-1)^3\vartheta^4\tilde{n}_b\left(fd,fd,...,F\left(d,n\right)\right)\right)\leq\lim_{z\to\infty}inf\wp\left((n-1)^3\vartheta^6\tilde{n}_b\left(fæ_{z+1},fæ_{z+1},...,F\left(d,n\right)\right)\right)\leq$$

$$\lim_{z\to\infty}inf\wp\left((n-1)^3\vartheta^6\tilde{n}_b\left(F\left(æ_z,œ_z\right),F\left(æ_z,œ_z\right),...,F\left(d,n\right)\right)\right)$$

$$\leq\lim_{z\to\infty}inf\wp\left(M\left(æ_z,œ_z,d,n\right)\right)-\lim_{z\to\infty}inf\varphi\left(M\left(æ_z,œ_z,d,n\right)\right)$$

Here $\lim_{k\to\infty}infM(æ_z,œ_z,d,n)$

$$=\lim_{k\to\infty}inf\ max\left\{\begin{matrix}\tilde{n}_b\left(fæ_z,fæ_z,...,fd\right),\ \tilde{n}_b\left(fœ_z,fœ_z,...,fn\right),\\ \tilde{n}_b\left(fæ_z,fæ_z,\cdots,F(æ_z,œ_z)\right),\ \tilde{n}_b\left(fœ_z,fœ_z,...,F(œ_z,æ_z)\right),\\ \tilde{n}_b\left(fd,fd,...,F(d,n)\right),...,\tilde{n}_b\left(fn,fn,...,F(n,d)\right),\\ \tilde{n}_b\left(fd,fd,...,F(æ_z,œ_z)\right),\ \tilde{n}_b\left(fn,fn,..,F(œ_z,æ_z)\right),\\ \frac{1}{(n-1)\vartheta^3}\tilde{n}_b\left(fæ_z,fæ_z,...,F(d,n)\right),\\ \frac{1}{(n-1)\vartheta^3}\tilde{n}_b\left(fœ_z,fœ_z,...,F(n,d)\right)\end{matrix}\right\}$$

$$\leq\lim_{k\to\infty}sup\ max\left\{\begin{matrix}\tilde{n}_b\left(fd,fd,...,F(d,n)\right),\ \tilde{n}_b\left(fn,fn,...,F(n,d)\right),\\ \frac{1}{(n-1)\vartheta^3}\tilde{n}_b\left(fæ_z,fæ_z,...,F(d,n)\right),\\ \frac{1}{(n-1)\vartheta^3}\tilde{n}_b\left(fœ_z,fœ_z,...,F(n,d)\right)\end{matrix}\right\}$$

$$\leq max\left\{\tilde{n}_b\ \ (fd,fd,\cdots,F(d,n)),\ \tilde{n}_b\left(fn,fn,...,F(n,d)\right)\ \right\}$$

Thus

$$\wp\left((n-1)^3\vartheta^4\tilde{n}_b(fd,fd,...,F(d,n))\right)\leq\wp\left(max\left\{\begin{matrix}\tilde{n}_b\left(fd,fd,...,F(d,n)\right),\\ \tilde{n}_b\left(fn,fn,...,F(n,d)\right)\end{matrix}\right\}\right)$$

$$-\lim_{z\to\infty}inf\varphi(M(æ_z,œ_z,d,n))$$

$$\leq\wp\left(max\left\{\begin{matrix}\tilde{n}_b\left(fd,fd,...,F(d,n)\right),\\ \tilde{n}_b\left(fn,fn,...,F(n,d)\right)\end{matrix}\right\}\right)$$

by the definition of $\wp$, one can get that

$$\tilde{n}_b(fd, fd, ..., F(d,n)) \le \frac{1}{(n-1)^3\vartheta^4} max \begin{Bmatrix} \tilde{n}_b(fd, fd, ..., F(d,n)), \\ \tilde{n}_b(fn, fn, ..., F(n,d)) \end{Bmatrix}$$

Similarly, we can have

$$\tilde{n}_b(fn, fn, ..., F(n,d)) \le \frac{1}{(n-1)^3\vartheta^4} max \begin{Bmatrix} \tilde{n}_b(fd, fd, ..., F(d,n)), \\ \tilde{n}_b(fn, fn, ..., F(n,d)) \end{Bmatrix}$$

Thus

$$max \begin{Bmatrix} \tilde{n}_b(fd, fd, ..., F(d,n)), \\ \tilde{n}_b(fn, fn, ..., F(n,d)), \end{Bmatrix} \le \frac{1}{(n-1)^3\vartheta^4} max \begin{Bmatrix} \tilde{n}_b(fd, fd, ..., F(d,n)), \\ \tilde{n}_b(fn, fn, ..., F(n,d)) \end{Bmatrix}.$$

In order for $F(d,n) = fd$ and $F(n,d) = fn$ to be true. As a result the coupled coincidence point of F & $f$is (d,n) given that the pair (F,f) is ω -compatible, so

$$f\alpha = f^2d = f(F(d,n)) = F(fd, fn) = F(\alpha,\beta) \ and \ f\beta = f^2n = f(F(n,d)) = F(fn, fd) = F(\beta,\alpha) \tag{3.10}$$

$$Now \ \frac{1}{n\vartheta^2}\tilde{n}_b(f\alpha, f\alpha, ..., F(\alpha,\beta)) = 0 \le max \begin{Bmatrix} \tilde{n}_b(fd, fd, ..., f\alpha), \tilde{n}_b(fn, fn, ..., f\beta), \\ \tilde{n}_b(f\alpha, f\alpha, ..., F(\alpha,\beta)), \tilde{n}_b(f\beta, f\beta, ..., F(\beta,\alpha)) \end{Bmatrix}.$$

From (Equation (3.2)), There is

$$\wp((n-1)^3\vartheta^6\tilde{n}_b(f\alpha, f\alpha, ..., fd)) = \wp((n-1)^3\vartheta^6\tilde{n}_b(F(\alpha,\beta), F(\alpha,\beta), ..., F(d,n)))$$

$$\le \wp(M(\alpha,\beta,d,n)) - \varphi(M(\alpha,\beta,d,n))$$

$$\le \wp(M(\alpha,\beta,d,n))$$

$$\le \wp\left(max \begin{Bmatrix} \tilde{n}_b(f\alpha, f\alpha, ..., fd), \tilde{n}_b(f\beta, f\beta, ..., fn), \\ \tilde{n}_b(f\alpha, f\alpha, ..., F(\alpha,\beta)), \tilde{n}_b(f\beta, f\beta, ..., F(\beta,\alpha)), \\ \tilde{n}_b(fd, fd, ..., F(d,n)), \tilde{n}_b(fn, fn, .., F(n,d)), \\ \tilde{n}_b(fd, fd, ..., F(\alpha,\beta)), \tilde{n}_b(fn, fn, ..., F(\beta,\alpha)), \\ \frac{1}{(n-1)\vartheta^3}\tilde{n}_b(f\alpha, f\alpha, ..., F(d,n)), \\ \frac{1}{(n-1)\vartheta^3}\tilde{n}_b(f\beta, f\beta, ..., F(n,d)) \end{Bmatrix} \right)$$

$$\le \wp(max(\ b\tilde{n}_b(f\alpha, f\alpha, ..., fd), b\tilde{n}_b(f\beta, f\beta, ..., fn) \ \}).$$

From the property of$\wp$, It has

$$\tilde{n}_b(f\alpha, f\alpha, ..., fd) \le \frac{1}{(n-1)^3\vartheta^5} max(\ \tilde{n}_b(f\alpha, f\alpha, ..., fd), \tilde{n}_b(f\beta, f\beta, ..., fn) \ \}.$$

Similarly,

$$\tilde{n}_b(f\beta, f\beta, ..., fn) \le \frac{1}{(n-1)^3\vartheta^5} max(\ \tilde{n}_b(f\alpha, f\alpha, ..., fd), \tilde{n}_b(f\beta, f\beta, ..., fn) \ \}.$$

Thus

$$max\left(\ \tilde{n}_b\left(f\alpha,f\alpha,...,fd\right),\tilde{n}_b\left(f\beta,f\beta,...,fn\right)\ \right\} \le \frac{1}{(n-1)^3\vartheta^5}max\left\{\begin{matrix}\tilde{n}_b\left(f\alpha,f\alpha,...,fd\right),\\ \tilde{n}_b\left(f\beta,f\beta,...,fn\right)\end{matrix}\right\}.$$

Therefore, the expressions $\beta = fn = f\beta, \alpha = fd = f\alpha$. Therefore, (α,β) is a Coupled Common Fixed Point of F and $f$as determined by (Equation (3.10)), we will demonstrate the unique coupled common fixed point in $\mathfrak{J}$ in the sections that follow. Assume that

F, f has another coupled fixed point$\left(\alpha^{'},\beta^{'}\right)$. Now think about,

$$\frac{1}{n\vartheta^2}\tilde{n}_b\left(f\alpha,f\alpha,...,F\left(\alpha,\beta\right)\right) = 0 \le max\left\{\begin{matrix}\tilde{n}_b\left(f\alpha,f\alpha,...,f\alpha^{'}\right),\tilde{n}_b\left(f\beta,f\beta,...,f\beta^{'}\right),\\ \tilde{n}_b\left(f\alpha,f\alpha,...,F\left(\alpha,\beta\right)\right),\tilde{n}_b\left(f\beta,f\beta,...,F\left(\beta,\alpha\right)\right)\end{matrix}\right\}$$

by (Equation (3.2)), we have

$$\wp\left((n-1)^3\vartheta^6\tilde{n}_b(\alpha,\alpha,...,\alpha^{'})\right) = \wp\left((n-1)^3\vartheta^6\tilde{n}_b\left(F(\alpha,\beta),F(\alpha,\beta),...,F(\alpha^{'},\beta^{'})\right)\right)$$

$$\le \wp\left(max\left\{\tilde{n}_b\left(\alpha,\alpha,...,\alpha^{'}\right),\tilde{n}_b\left(\beta,\beta,...,\beta^{'}\right)\right\}\right).$$

Then we have

$$\tilde{n}_b(\alpha,\alpha,...,\alpha^{'}) \le \frac{1}{(n-1)^3\vartheta^6}max\left\{\tilde{n}_b\left(\alpha,\alpha,...,\alpha^{'}\right),\tilde{n}_b\left(\beta,\beta,...,\beta^{'}\right)\right\}.$$

Similarly, we can show that

$$\tilde{n}_b(\beta,\beta,...,\beta^{'}) \le \frac{1}{(n-1)^3\vartheta^6}max\left\{\tilde{n}_b\left(\alpha,\alpha,...,\alpha^{'}\right),\tilde{n}_b\left(\beta,\beta,...,\beta^{'}\right)\right\}.$$

Thus

$$max\left\{\tilde{n}_b\left(\alpha,\alpha,...,\alpha^{'}\right),\tilde{n}_b\left(\beta,\beta,...,\beta^{'}\right)\right\} \le \frac{1}{(n-1)^3\vartheta^6}max\left\{\begin{matrix}\tilde{n}_b\left(\alpha,\alpha,...,\alpha^{'}\right),\\ \tilde{n}_b\left(\beta,\beta,...,\beta^{'}\right)\end{matrix}\right\}.$$

Therefore $\beta = \beta^{'}, \alpha = \alpha^{'}$. The Unique Common Coupled Fixed Point of F and f is hence (α,β)

Case (ii): If $f\text{æ}_z = f\text{æ}_{z+1}, f\text{œ}_z = f\text{œ}_{z+1}$for some z then

$f\text{æ}_z = F\left(\text{æ}_z,\text{œ}_z\right)$, $f\text{œ}_z = F\left(\text{œ}_z,\text{æ}_z\right)$so that $\left(\text{æ}_z,\text{œ}_z\right)$ is the coupled coincidence point of F & f. Moreover, following as in case (i) with$f\text{æ}_z = \alpha, f\text{œ}_z = \beta$, we can show that$(\alpha,\beta)$is the Unique Common Coupled Fixed Point of F & f

Corollary 3.6: Suppose $(\mathfrak{J},\tilde{n}_b)$ be the A_b-metric space and $F:\mathfrak{J}^2 \to \mathfrak{J}$ & $F:\mathfrak{J} \to \mathfrak{J}$ be two mappings which satisfy $(\varphi,\wp)$ -type contraction with

$a)$ $F\left(\mathfrak{J}^2\right) \subseteq f\left(\mathfrak{J}\right)\&f\left(I\right)$is complete,

$b)$ $(F,f)$is ω -compatible.

Following which, we have a Unique Common Coupled Fixed Point of F and f in $\mathfrak{J}$.

Corollary 3.7: Let $(\mathfrak{J},\tilde{n}_b)$ be a complete A_b-Metric Space. Suppose that the mapping $F:\mathfrak{J}^2 \to \mathfrak{J}$ satisfying $(\varphi,\wp)$-type suzuki contraction, for all æ, œ, ß, $d \in \mathfrak{J}$

$$\frac{1}{n\vartheta^2}\tilde{n}_b\left(\text{æ},\text{æ},...,F(\text{æ},\text{œ})\right) \le max\left\{\begin{matrix}\tilde{n}_b\left(\text{æ},\text{æ},...,\text{ß}\right),\tilde{n}_b\left(\text{œ},\text{œ},...,d\right),\\ \tilde{n}_b\left(\text{æ},\text{æ},...,F(\text{æ},\text{œ})\right),\tilde{n}_b\left(\text{œ},\text{œ},...,F(\text{œ},\text{æ})\right)\end{matrix}\right\}\ implies$$

$$\wp\left((n-1)^3\vartheta^6\tilde{n}_b\left(F\left(\text{æ},\text{œ}\right),F\left(\text{æ},\text{œ}\right),\cdots,F\left(\text{ß},d\right)\right)\right) \le \wp\left(M\left(\text{æ},\text{œ},\text{ß},d\right)\right) - \varphi\left(M\left(\text{æ},\text{œ},\text{ß},d\right)\right)$$

$$where\ M(\text{æ},\text{œ},\text{ß},d) = max\left\{\begin{matrix}\tilde{n}_b\left(\text{æ},\text{æ},...,\text{ß}\right),\tilde{n}_b\left(\text{œ},\text{œ},...,d\right),\\ \tilde{n}_b\left(\text{æ},\text{æ},...,F(\text{æ},\text{œ})\right),\tilde{n}_b\left(\text{œ},\text{œ},...,F(\text{œ},\text{æ})\right),\\ \tilde{n}_b\left(\text{ß},\text{ß},...,F(\text{ß},d)\right),\tilde{n}_b\left(d,d,...,F(d,\text{ß})\right),\\ \tilde{n}_b\left(\text{ß},\text{ß},...,F(\text{æ},\text{œ})\right),\tilde{n}_b\left(d,d,...,F(\text{œ},\text{æ})\right)\end{matrix}\right\}$$

Then there is a Unique Fixed Point of F in $\mathfrak{J}$.

Example 3.8: Suppose $\mathfrak{J} = [0,+\infty)$ we define $\tilde{n}_b : \mathfrak{J}^n \to [0,+\infty)$ as $\tilde{n}_b(æ_1, æ_2, ..., æ_{n-1}, æ_n) = \sum_{i=1}^{n}\sum_{i<j}\left|æ_i - æ_j\right|^2 \forall\, æ_i \in \mathfrak{J},\ i = 1,2,...$

Following which $(\mathfrak{J}, \tilde{n}_b)$ is a complete A_b-Metric Space with $\vartheta = 2$

Let $F : \mathfrak{J}^2 \to \mathfrak{J}$ and $f : \mathfrak{J} \to \mathfrak{J}$ be given by $F(d,n) = sin(\frac{6d-3n+72n-3}{72n})$, $f(d) = \frac{2d+3n-2}{3n}$ and $\wp, \varphi : [0,\infty) \to [0,\infty)$ by $\wp(t) = t, \varphi(t) = \frac{t}{2}$, Then clearly, $F(\mathfrak{J}^2) \subseteq f(\mathfrak{J})$& (F,f) are

ω -compatible & for every$d$, n , Z,ß $\in \mathfrak{J}$

$\frac{1}{8}\tilde{n}_b(fd, fd, \cdots, F(d,n)) \le \tilde{n}_b(fd, fd, \cdots, F(d,n))$.

$\le max\begin{Bmatrix} \tilde{n}_b(fd, fd, \cdots, fß), \tilde{n}_b(fn, fn, \cdots, fZ), \\ \tilde{n}_b(fd, fd, \cdots, F(d,n)), \tilde{n}_b(fn, fn, \cdots, F(n,d)), \end{Bmatrix}$

Now we have

$$\wp((n-1)^3\vartheta^6\tilde{n}_b(F(d,n), F(d,n), ..., F(ß,Z)))$$

$$= (n-1)^4\vartheta^6|F(d,n) - F(ß,Z)|^2$$

$$= (n-1)^4\vartheta^6|sin(\frac{6d-3n+72n-3}{72n}) - sin(\frac{6ß-3Z+72n-3}{72n})|^2$$

$$= 4(n-1)^4\vartheta^6|cos(\frac{6d-3n+144n-6+6ß-3Z}{72n})sin(\frac{6d-3n-6ß+3Z}{72n})|^2$$

$$\le 4(n-1)^4\vartheta^6|\frac{6d-3n-6ß+3Z+144n-6}{72n}|^2$$

$$\le \frac{(n-1)}{4}\left(|\frac{2d+3n-2}{3n} - sin(\frac{6ß-3Z+72n-2}{72n})|^2 + |\frac{2n+3n-2}{3n} - sin(\frac{6Z-3ß+72n-2}{72n})|^2\right)$$

$$\le \frac{(n-1)}{4}\left(|fd - F(ß,Z)|^2 + |fn - F(Z,ß)|^2\right)$$

$$= \frac{1}{4}\left(\tilde{n}_b(fd, fd, ..., F(ß,Z)) + \tilde{n}_b(fn, fn, ..., F(Z,ß))\right)$$

$$\le \frac{1}{2}max\{\tilde{n}_b(fd, fd, ..., F(ß,Z)), \tilde{n}_b(fn, fn, ..., F(Z,ß))\}$$

$$\le \frac{1}{2}M(d,n,ß,Z) \le \wp(M(d,n,ß,Z)) - \varphi(M(d,n,ß,Z))$$

Plug in $n = 2$ and $\vartheta = 2$ this satisfies all the requirements of the **Theorem 3.5** and $(1,1)$ is the only unique coupled common fixed point of $F$and f.

4 Application to Integral Equations

As an application to **Corollary 3.7** we examine the existence of a unique solution to an initial value problem in this section

Theorem 4.1. Take into consideration the initial value problem

$$ß^1(t) = T(t,(ß,Z)(t)), t \in I = [0,1], (ß,Z)(0) = (ß_0, Z_0) \tag{4.1}$$

Here $T : I \times R \to R$ along with $\int_0^t T(s,(ß,Z)(s))ds = max\left\{\int_0^t T(s,(ß(s))ds, \int_0^t T(s,(Z(s))ds\right\}$

& $ß_0, Z_0 \in R$. Then there exists unique solution for the initial value problem (Equation (4.1)) in $C(I,R)$

Proof. Corresponding integral equation for the initial value problem (Equation (4.1)) is

$$ß(t) = ß_0 + \frac{\sqrt{(n-1)^3}}{2}\vartheta^5 \int_0^t T(s,(ß,Z)(s))ds.$$

Let $\mathfrak{J} = C(I,R)$and $\tilde{n}_b(æ_1, æ_2, ..., æ_{n-1}, æ_n) = \sum_{i=1}^n \sum_{i<j} \left|æ_i - æ_j\right|^2$ $to\, each\, æ_i \in \mathfrak{J}$, $i = 1,2,...$ define $F : \mathfrak{J}^2 \to \mathfrak{J}$by

$$F(ß,Z)(t) \quad = \quad \frac{2ß_{,0}}{\sqrt{(n-1)^3}\vartheta^5} + \int_0^t T(s,(ß,Z)(s))ds.$$

and $\wp, \varphi : [0,\infty) \to [0,\infty)$ as $\wp(t) = t$ & $\varphi(t) = \frac{3t}{4}$. Clearly, for all $ß, Z \in \mathfrak{J}$, we have

$$\frac{1}{n\vartheta^2}\tilde{n}_b(ß,ß,...,F(ß,Z)) \leq max\left\{\begin{matrix} \tilde{n}_b(ß,ß,...,d), \tilde{n}_b(Z,Z,...,n), \\ \tilde{n}_b(ß,ß,....,F(ß,Z)), \tilde{n}_b(Z,Z,...,F(Z,ß)) \end{matrix}\right\}.$$

Now

$$\wp((n-1)^3\vartheta^6\tilde{n}_b(F(ß,Z)(t), F(ß,Z)(t), \cdots, F(d,n)(t))) = (n-1)^4\vartheta^6|F(ß,Z)(t) - F(d,n)(t)|^2$$

$$= |\frac{2ß_0}{\sqrt{(n-1)^3\vartheta^5}} + \int_0^t T(s,(ß,Z)(s))ds - \frac{2ß_0}{\sqrt{(n-1)^3\vartheta^5}} + \int_0^t T(s,(d,n)(s))ds|^2$$

$$\leq \frac{4(n-1)}{\vartheta^4} max\{|ß(t) - d(t)|^2, |Z(t) - n(t)|^2\} = \frac{1}{4} max\left\{\begin{matrix} \tilde{n}_b(ß,ß,\cdots,d), \\ \tilde{n}_b(Z,Z,\cdots,n) \end{matrix}\right\}$$

$$\leq \frac{1}{4}M(ß,Z,d,n) \leq \wp(M(ß,Z,d,n) - \varphi(M(ß,Z,d,n))$$

Based on **Corollary 3.7**, we can infer that there is only one fixed point for F in $\mathfrak{J}$.

5 Conclusion

We proved the existence and uniqueness of a common coupled fixed-point for two mappings in the setup of complete A_b-metric space, via $(\varphi, \wp)$ -type Suzuki contraction with an example. And also illustrated the application to integral equations.

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