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Type (K) Compatible Mappings and Common Fixed Points in Complete Cone S-metric Spaces

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Abstract

Objective: This research work is an attempt to explore common fixed points for four self-maps that are pairwise type (K) compatible in a complete cone S-metric space. **Method:** We have applied a contractive condition together with a weaker condition, compatible of type (*K*) to establish fixed points that are common for four self-maps. **Findings:** Some already existing results in the literature have been generalized, expanded upon, and new fixed point results have been obtained in cone *S*-metric spaces. To back up our conclusions, a suitable example is provided. **Novelty:** By adopting an appropriate contractive condition, we have proven the existence of a unique common fixed point for four maps in a cone S-metric space, despite the weak compatible condition.
MSC 2020: 47H10, 54H25.

Keywords: Normal Cone; Partial Ordering; Cone S-metric Space; Compatible Mappings of Type (K); Common Fixed Point

1 Introduction

Stefan Banach proposed the first theorem on fixed points, also known as the Banach Contraction Principle. One of the fixed point theorems most frequently applied in analysis is the Banach Contraction Principle. It serves several purposes in nonlinear analysis. It was used in several directions as a result of this generalization. Cone metric space was proposed by Huang and Zhang[(1)] in 2007, where the set of real numbers was replaced with a Banach space E that has a partial order with respect to a cone. A new kind of metric space, that is, an S - metric space, was proposed, and some topological properties were proven by Sedghi et al. in 2012[(2)]. Later, by fusing S- metric space with cone metric space, Dhamodaran and Krishnakumar[(3)] developed cone S - metric space.

In 1982, by proposing the idea of weak commutativity, Sessa[(4)] established a fixed point theorem. After that, the concept of compatible maps was put forth by Jungck[(5)] in 1986. Later, weaker classes of mappings known as weakly compatible maps were proposed by Jungck and Rhoades[(6)]. In common fixed point theory, several existence theorems have been proven using this notion. The idea of type (K) compatibility was just recently suggested by Jha et al.[(7)]

The theory of fixed points is drawing the attention of academic researchers due to its significance. The results regarding fixed points that are common to self-maps in cone S-metric space and that meet various contractive conditions were explored and determined by several researchers (see, e.g., [3,8–11]). Recently, several kinds of metric spaces, such as vector and digital metric spaces have been developed, and some fixed point results have been established (see, e.g., [12–17]).

Inspired by the work of several authors, we establish the presence of fixed points that are common for four self-maps that meet compatibility of type (K) in a cone S-metric space. At the end, a befitting example is provided to back up our result. Several similar findings in the literature are expanded upon by our findings.

2 Methodology

We start with the definitions and lemmas given below that we can use in our main work.

Definition 2.1. [1] Let E be a real Banach space and $P \subseteq E$. Let θ denote the zero element of E and int P or P^0 denote the interior of P.

P is said to be a cone if the following conditions hold:

1. P is a non-empty closed set and $P \neq \{\theta\}$
2. $au+bv \in P$ for every $a,b \in R, a,b \geq 0$ and $u,v \in P$.
3. $u \in P$ and $-u \in P \Rightarrow u=\theta$.

We define a partial ordering $\leq$ in E w.r.t. P by $u \leq v$ iff $v-u \in P$.

We shall write $u < v$ whenever $u \leq v$ and $u \neq v$, while $u < v$ will stand for $v-u \in intP$. If there is a number $K>0$ such that $\| u \| \leq K \| v \|$ for all $u,v \in P$ with $0 \leq u \leq v$, then P is said to be a normal cone. The smallest positive number K that satisfies above mentioned inequality is known as the normal constant of P. A non-normal cone is one that is not normal.

Example 2.2. [1] Let $E=R^2$ and $P=\{(u,v) \in R^2, u \geq 0, v \geq 0\}$. Then P is a normal cone in E, with normal constant $K=1$.

Definition 2.3. [1] Let E be a real Banach space equipped with a partial ordering $\leq$ w.r.t. a cone P, with int $P \neq \phi$. Suppose that X is a non-empty set. Then a map $d: X^2 \rightarrow E$ is said to be a cone metric on X, if it satisfies the condition

(CM1) $d(u,v) \geq 0$ for all $u,v \in X$ and $d(u,v)=0$ iff $u=v$;

(CM2) $d(u,v)=d(v,u)$ for all $u,v \in X$;

(CM3) $d(u,v) \leq d(u,w)+d(w,v)$ for all $u,v,w \in X$.

In this case, the pair (X,d) is said to be a cone metric space.

Example 2.4. [1] Let $E=R^2, P=\{(u,v) \in R^2 : u \geq 0, v \geq 0\}, X=R$ and $k \geq 0$ is a constant. Then a map $d: X^2 \rightarrow E$ given by $d(u,v)=(|u-v|, k|u-v|)$ is a cone metric on R.

Definition 2.5. [2] A function $S: X^3 \rightarrow R$ where X is a non-empty set, is said to be an $S-$ metric if for each $u,v,w \in X$,

(SM1) $S(u,v,w) \geq 0$ and $S(u,v,w)=0$ iff $u=v=w$;

(SM2) $S(u,v,w) \leq S(u,u,a)+S(v,v,a)+S(w,w,a)$ for every $a \in X$.

The pair (X,S) is called an S - metric space.

Example 2.6. [18] The function $S: R^3 \rightarrow \infty$, defined by $S(u,v,w)=|u-w|+|v-w|$ is an $S-$ metric on R.

Definition 2.7. [3] Let E be a real Banach space equipped with a partial ordering $\leq$ w.r.t. a cone P, with int $P \neq \phi$. Suppose that X is a non-empty set. Then the function $S: X^3 \rightarrow E$ is said to be a cone $S-$ metric if for each $u,v,w \in X$,

(CSM1) $S(u,v,w) \geq 0$ and $S(u,v,w)=0$ iff $u=v=w$;

(CSM2) $S(u,v,w) \leq S(u,u,a)+S(v,v,a)+S(w,w,a)$ for every $a \in X$.

In this case, (X,S) is said to be a cone $S-$ metric space.

Example 2.8. [3] Let $E=R^2, P=\{(u,v) \in R^2, u \geq 0, v \geq 0\}$. Then the function $S: R^3 \rightarrow E$ given by $S(u,v,w)=(|u-v|+|u-w|, k[|u-v|+|u-w|])$, where $k \geq 0$ is a cone $S-$ metric on X.

Lemma 2.9. [3] In a cone S-metric space $X, S(u,u,v)=S(v,v,u)$ for all $u,v \in X$.

Lemma 2.10. In a cone S - metric space $X, S(u,u,v) \leq 2S(u,u,w)+S(w,w,v)$ for all $u,v,w \in X$.

Proof. For all $u,v,w \in X, S(u,u,v) \leq S(u,u,w)+S(u,u,w)+S(v,v,w)$

$=2S(u,u,w)+S(w,w,v)$

Definition 2.11. [3] Let (X,S) be a cone S-metric space.

(i) A sequence $\{u_n\}$ in X is said to converge to some u in X if for each $0 < c \in E$ there exists $n_0 \in N$ such that $S(u_n,u_n,u) < c$ whenever $n \geq n_0$, i.e., $S(u_n,u_n,u) \rightarrow 0$ as $n \rightarrow \infty$. In this case, we write $lim_{n \rightarrow \infty} u_n = u$.

(ii) A sequence $\{u_n\}$ in X is called a Cauchy sequence, if for each $0 < c \in E$, there exists $n_0 \in N$ such that $S(u_n,u_n,u_m) < c$ whenever $n,m \geq n_0$, i.e., $S(u_n,u_n,u_m) \rightarrow 0$ as $m,n \rightarrow \infty$.

(iii) If every Cauchy sequence in X is convergent in X, then X is said to be complete.

Definition 2.12.[7] Self-maps F,G of a cone S - metric space X are said to be compatible of type (*K*), if $GGu_n \to Fu, FFu_n \to Gu$ for every sequence $\{u_n\}$ in X such that $Fu_n \to u$ and $Gu_n \to u$ for some $u \in X$

3 Results and Discussion

We now prove a theorem regarding fixed points common to four self-maps that are type (K) compatible.

Theorem 3.1. Let E be a real Banach space and $P \subseteq E$ be a solid cone. Let (X,S) be a complete cone S-metric space with cone P, and $F,\ G,\ I,\ J$ be self-maps of X satisfying the conditions

(i) $F(X) \subseteq J(X)$ and $G(X) \subseteq I(X)$;

(ii) $S(Fu,\ Fu,\ Gv) \le p_1 S(Iu,\ Iu,\ Jv) + p_2 S(Iu,\ Iu,\ Fu) + p_3 S(Jv,\ Jv,\ Gv)$
$+ p_4 S(Iu,\ Iu,\ Gv) + p_5 S(Jv,\ Jv,\ Fu)$

for all $u,v \in X$, where $p_i > 0\,(i \in Z^+, 1 \le i \le 5)$ satisfying

$$p_1 + p_2 + p_3 + 3max\{p_4, p_5\} < 1; \tag{3.1}$$

(iii) *J* And *I* are continuous and

(iv) The pairs (*G, J*) and (*F, I*) of mappings are compatible of type (*K*) on *X*.

Then $F,\ G,\ I,\ J$ have a unique common fixed point in X.

Proof. Let u_0 be an arbitrary point in X. Since $F(X) \subseteq J(X)$ and $G(X) \subseteq I(X)$, there exist $u_1, u_2 \in X$, such that $Fu_0 = Ju_1$ and $Gu_1 = Iu_2$.

Continuing this process, we get,

$$l_{2n} = Fu_{2n} = Ju_{2n+1}, l_{2n+1} = Gu_{2n+1} = Iu_{2n+2}\, for\, all\, n \tag{3.2}$$

We now prove that the sequence $(l_n\}$ is a Cauchy sequence.

We know that

$$\begin{aligned}
S(l_{2n}, l_{2n}, l_{2n+1}) = S(Fu_{2n}, Fu_{2n}, Gu_{2n+1}) \\
&\le p_1 S(Iu_{2n}, Iu_{2n}, Ju_{2n+1}) + p_2 S(Iu_{2n}, Iu_{2n}, Fu_{2n}) + p_3 S(Ju_{2n+1}, Ju_{2n+1}, Gu_{2n+1}) \\
&+ p_4 S(Iu_{2n}, Iu_{2n}, Gu_{2n+1}) + p_5 S(Ju_{2n+1}, Ju_{2n+1}, Fu_{2n}) \\
&\le p_1 S(l_{2n-1}, l_{2n-1}, l_{2n}) + p_2 S(l_{2n-1}, l_{2n-1}, l_{2n}) + p_3 S(l_{2n}, l_{2n}, l_{2n+1}) \\
&+ p_4 S(l_{2n-1}, l_{2n-1}, l_{2n+1}) + p_5 S(l_{2n}, l_{2n}, l_{2n}) \\
&\le p_1 S(l_{2n-1}, l_{2n-1}, l_{2n}) + p_2 S(l_{2n-1}, l_{2n-1}, l_{2n}) + p_3 S(l_{2n}, l_{2n}, l_{2n+1}) \\
&+ p_4 [2S(l_{2n-1}, l_{2n-1}, l_{2n}) + S(l_{2n+1}, l_{2n+1}, l_{2n})]
\end{aligned}$$

which implies that

$$S(l_{2n}, l_{2n}, l_{2n+1}) \le \frac{p_1 + p_2 + 2p_4}{1 - p_3 - p_4} S(l_{2n-1}, l_{2n-1}, l_{2n})$$

Thus, $S(l_{2n+1}, l_{2n+1}, l_{2n}) \le \mu_1 S(l_{2n}, l_{2n}, l_{2n-1})$, where

$$\mu_1 = \frac{p_1 + p_2 + 2p_4}{1 - p_3 - p_4}. \tag{3.3}$$

Also,

$$\begin{aligned}
S(l_{2n+2}, l_{2n+2}, l_{2n+1}) = S(Fu_{2n+2}, Fu_{2n+2}, Gu_{2n+1}) \\
&\le p_1 S(Iu_{2n+2}, Iu_{2n+2}, Ju_{2n+1}) + p_2 S(Iu_{2n+2}, Iu_{2n+2}, Fu_{2n+2}) \\
&+ p_3 S(Ju_{2n+1}, Ju_{2n+1}, Gu_{2n+1}) + p_4 S(Iu_{2n+2}, Iu_{2n+2}, Gu_{2n+1}) \\
&+ p_5 S(Ju_{2n+1}, Ju_{2n+1}, Fu_{2n+2}) \\
&\le p_1 S(l_{2n+1}, l_{2n+1}, l_{2n}) + p_2 S(l_{2n+1}, l_{2n+1}, l_{2n+2}) \\
&+ p_3 S(l_{2n}, l_{2n}, l_{2n+1}) + p_4 S(l_{2n+1}, l_{2n+1}, l_{2n+1}) \\
&+ p_5 S(l_{2n}, l_{2n}, l_{2n+2}) \\
&\le p_1 S(l_{2n+1}, l_{2n+1}, l_{2n}) + p_2 S(l_{2n+1}, l_{2n+1}, l_{2n+2}) + p_3 S(l_{2n}, l_{2n}, l_{2n+1}) \\
&+ p_5 [2S(l_{2n}, l_{2n}, l_{2n+1}) + S(l_{2n+1}, l_{2n+1}, l_{2n+2})]
\end{aligned}$$

which implies that

$$S(l_{2n+2},l_{2n+2},l_{2n+1}) \leq \frac{p_1+p_3+2p_5}{1-p_2-p_5} S(l_{2n+1},l_{2n+1},l_{2n})$$

Therefore, $S(l_{2n+2},l_{2n+2},l_{2n+1}) \leq \mu_2 S(l_{2n+1},l_{2n+1},l_{2n})$, where

$$\mu_2 = \frac{p_1+p_3+2p_5}{1-p_2-p_5}. \tag{3.4}$$

From Equations (3.3) and (3.4),

$$\begin{aligned} S(l_{2n+2},l_{2n+2},l_{2n+1}) &\leq \mu_2 S(l_{2n+1},l_{2n+1},l_{2n}) \\ &\leq \mu_2\mu_1 S(l_{2n},l_{2n},l_{2n-1}) \\ &\leq \mu_1\mu_2.\mu_2 S(l_{2n-1},l_{2n-1},l_{2n-2}) \end{aligned}$$

$$\leq \ldots \leq \mu_2(\mu_1\mu_2)^n S(l_1,l_1,l_0) \tag{3.5}$$

Also,

$$\begin{aligned} S(l_{2n+3},l_{2n+3},l_{2n+2}) &\leq \mu_1 S(l_{2n+2},l_{2n+2},l_{2n+1}) \\ &\leq \mu_1\mu_2 S(l_{2n+1},l_{2n+1},l_{2n}) \end{aligned}$$

$$\leq \ldots \leq (\mu_1\mu_2)^{n+1} S(l_1,l_1,l_0) \tag{3.6}$$

Since $p_1+p_2+p_3+3max\{p_4,p_5\} < 1, 0 < \mu_1 = \frac{p_1+p_2+2p_4}{1-p_3-p_4} < 1$ and $0 < \mu_2 = \frac{p_1+p_3+2p_5}{1-p_2-p_5} < 1$.

Therefore, $0 < \mu_1\mu_2 < 1$.

Let $n > m > 1; n,m \in Z^+$.

If both n and m are odd, then $n = 2i+1$ and $m = 2j+1$ for some $i,j \in Z^+$.

Therefore, from Equations (3.5) and (3.6), using lemma 2.10,

$$\begin{aligned} S(l_{2i+1},l_{2i+1},l_{2j+1}) &\leq 2S(l_{2i+1},l_{2i+1},l_{2i}) + S(l_{2j+1},l_{2j+1},l_{2i}) \\ &= 2S(l_{2i+1},l_{2i+1},l_{2i}) + S(l_{2i},l_{2i},l_{2j+1}) \\ &\leq 2S(l_{2i+1},l_{2i+1},l_{2i}) + 2S(l_{2i},l_{2i},l_{2i-1}) + S(l_{2j+1},l_{2j+1},l_{2i-1}) \\ &= 2S(l_{2i+1},l_{2i+1},l_{2i}) + 2S(l_{2i},l_{2i},l_{2i-1}) + S(l_{2i-1},l_{2i-1},l_{2j+1}) \\ &\leq 2S(l_{2i+1},l_{2i+1},l_{2i}) + 2S(l_{2i},l_{2i},l_{2i-1}) \\ &+2S(l_{2i-1},l_{2i-1},l_{2i-2}) + 2S(l_{2i-2},l_{2i-2},l_{2i-3}) \\ &+\cdots+2S(l_{2j+3},l_{2j+3},l_{2j+2}) + 2S(l_{2j+2},l_{2j+2},l_{2j+1}) \\ &\leq 2(\mu_1\mu_2)^i S(l_1,l_1,l_0) + 2\mu_2(\mu_1\mu_2)^{i-1} S(l_1,l_1,l_0) \\ &+2(\mu_1\mu_2)^{i-1} S(l_1,l_1,l_0) + 2\mu_2(\mu_1\mu_2)^{i-2} S(l_1,l_1,l_0) + \ldots \\ &\cdots + 2(\mu_1\mu_2)^{j+1} S(l_1,l_1,l_0) + 2\mu_2(\mu_1\mu_2)^j S(l_1,l_1,l_0) \\ &\leq 2\left((\mu_1\mu_2)^{j+1} + (\mu_1\mu_2)^{j+2} + \ldots\right\} S(l_1,l_1,l_0) \\ &+2\mu_2\left((\mu_1\mu_2)^j + (\mu_1\mu_2)^{j+1} + \ldots\right\} S(l_1,l_1,l_0) \\ &= \left\{\frac{2(\mu_1\mu_2)^{j+1}}{1-\mu_1\mu_2} + \frac{2\mu_2(\mu_1\mu_2)^j}{1-\mu_1\mu_2}\right\} S(l_1,l_1,l_0) \end{aligned}$$

$$= \frac{2(1+\mu_1)\mu_2}{1-\mu_1\mu_2}(\mu_1\mu_2)^j S(l_1,l_1,l_0) \to 0, \ as \ n > m = 2j+1 \to \infty \tag{3.7}$$

If n is odd and m is even, then $n = 2i+1$ and $m = 2j$ for some $i,j \in Z^+$.

From Equations (3.6) and (3.7),

$$\begin{aligned} S(l_{2i+1},l_{2i+1},l_{2j}) &\leq 2S(l_{2i+1},l_{2i+1},l_{2j+1}) + S(l_{2j},l_{2j},l_{2j+1}) \\ &\leq 2S(l_{2i+1},l_{2i+1},l_{2j+1}) + S(l_{2j+1},l_{2j+1},l_{2j}) \\ &\leq \frac{4(1+\mu_1)\mu_2}{1-\mu_1\mu_2}(\mu_1\mu_2)^j S(l_1,l_1,l_0) + (\mu_1\mu_2)^j S(l_1,l_1,l_0) \\ &= \left\{\frac{4(1+\mu_1)\mu_2}{1-\mu_1\mu_2} + 1\right\}(\mu_1\mu_2)^j S(l_1,l_1,l_0) \to 0, \ as \ n > m = 2j \to \infty \end{aligned}$$

If n is even and m is odd, then $n = 2i$ and $m = 2j+1$ for some $i, j \in Z^+$.

Again, from Equations (3.5) and (3.6), using lemma 2.10,

$$\begin{array}{l} S(l_{2i}, l_{2i}, l_{2j+1}) \leq 2S(l_{2i}, l_{2i}, l_{2i-1}) + 2S(l_{2i-1}, l_{2i-1}, l_{2i-2}) \\ +2S(l_{2i-2}, l_{2i-2}, l_{2i-3}) + 2S(l_{2i-3}, l_{2i-3}, l_{2i-4}) + \dots \\ \dots + 2S(l_{2j+3}, l_{2j+3}, l_{2j+2}) + S(l_{2j+2}, l_{2j+2}, l_{2j+1}) \\ \leq 2S(l_{2i}, l_{2i}, l_{2i-1}) + 2S(l_{2i-1}, l_{2i-1}, l_{2i-2}) \\ +2S(l_{2i-2}, l_{2i-2}, l_{2i-3}) + 2S(l_{2i-3}, l_{2i-3}, l_{2i-4}) + \dots \\ \dots + 2S(l_{2j+3}, l_{2j+3}, l_{2j+2}) + S(l_{2j+2}, l_{2j+2}, l_{2j+1}) \\ \leq 2\mu_2(\mu_1\mu_2)^{i-1} S(l_1, l_1, l_0) + 2(\mu_1\mu_2)^{i-1} S(l_1, l_1, l_0) \\ +2\mu_2(\mu_1\mu_2)^{i-2} S(l_1, l_1, l_0) + 2(\mu_1\mu_2)^{i-2} S(l_1, l_1, l_0) + \dots \\ \dots + 2(\mu_1\mu_2)^{j+1} S(l_1, l_1, l_0) + 2\mu_2(\mu_1\mu_2)^{j} S(l_1, l_1, l_0) \\ \leq 2\left\{(\mu_1\mu_2)^{j+1} + (\mu_1\mu_2)^{j+2} + \dots\right\} S(l_1, l_1, l_0) \\ +2\mu_2\left\{(\mu_1\mu_2)^{j} + (\mu_1\mu_2)^{j+1} + \dots\right\} S(l_1, l_1, l_0) \\ = \left\{\frac{2(\mu_1\mu_2)^{j+1}}{1-\mu_1\mu_2} + \frac{2\mu_2(\mu_1\mu_2)^{j}}{1-\mu_1\mu_2}\right\} S(l_1, l_1, l_0) \end{array}$$

$$= \frac{2(1+\mu_1)\mu_2}{1-\mu_1\mu_2}(\mu_1\mu_2)^j S(l_1, l_1, l_0) \to 0,\ as\ n > m = 2j+1 \to \infty. \tag{3.8}$$

If both n and m are even, then $n = 2i$ and $m = 2j$ for some $i, j \in Z^+$.

From Equations (3.6) and (3.8),

$$\begin{array}{l} S(l_{2i}, l_{2i}, l_{2j}) \leq 2S(l_{2i}, l_{2i}, l_{2j+1}) + S(l_{2j+1}, l_{2j+1}, l_{2j}) \\ \leq \frac{4(1+\mu_1)\mu_2}{1-\mu_1\mu_2}(\mu_1\mu_2)^j S(l_1, l_1, l_0) + (\mu_1\mu_2)^j S(l_1, l_1, l_0) \end{array}$$

$$= \left\{\frac{4(1+\mu_1)\mu_2}{1-\mu_1\mu_2} + 1\right\}(\mu_1\mu_2)^j S(l_1, l_1, l_0) \to 0,\ as\ n > m = 2j \to \infty. \tag{3.9}$$

Thus, in all cases, $S(l_n, l_n, l_m) \to 0$ as $m, n \to \infty$, showing that the sequence $\{l_n\}$ is Cauchy.

From the completeness of X, there exists $t \in X$ such that $lim_{n\to\infty} l_n = t$.

Thus,

$$lim_{n\to\infty} Fu_{2n} = lim_{n\to\infty} Ju_{2n+1} = lim_{n\to\infty} Gu_{2n+1} = lim_{n\to\infty} Iu_{2n+2} = t. \tag{3.10}$$

Since the pair (G, J) is type (K) compatible,

$$lim_{n\to\infty} GGu_{2n+1} = Jt\, and\, lim_{n\to\infty} JJu_{2n+1} = Gt \tag{3.11}$$

Since J is continuous,

$$lim_{n\to\infty} JJu_{2n+1} = Jt = lim_{n\to\infty} JGu_{2n+1} \tag{3.12}$$

Since the pair (F, I) is type (K) compatible,

$$lim_{n\to\infty} FFu_{2n} = It\, and\, lim_{n\to\infty} IIu_{2n} = Ft \tag{3.13}$$

Since I is continuous,

$$lim_{n\to\infty} IIu_{2n} = lim_{n\to\infty} IIu_{2n+2} = It = lim_{n\to\infty} IFu_{2n}. \tag{3.14}$$

Now, we show that $Jt = t$.

Putting $u = u_{2n}$ and $v = Gu_{2n+1}$ in Equation (3.1),

$$\begin{array}{l} S(Fu_{2n}, Fu_{2n}, GGu_{2n+1}) \leq p_1 S(Iu_{2n}, Iu_{2n}, JGu_{2n+1}) + p_2 S(Iu_{2n}, Iu_{2n}, Fu_{2n}) \\ +p_3 S(JGu_{2n+1}, JGu_{2n+1}, GGu_{2n+1}) + p_4 S(Iu_{2n}, Iu_{2n}, GGu_{2n+1}) \\ +p_5 S(JGu_{2n+1}, JGu_{2n+1}, Fu_{2n}) \end{array}$$

Letting $n \to \infty$,

$$S(t,t,Jt) \le p_1 S(t,t,Jt) + p_2 S(t,t,t) + p_3 S(Jt,Jt,Jt) + p_4 S(t,t,Jt) + p_5 S(Jt,Jt,t)$$

This implies, $(1 - p_1 - p_4 - p_5)\, S(t,t,Jt) \le 0$
and hence $Jt = t$ as $p_1 + p_4 + p_5 \le p_1 + p_2 + p_3 + 3max\{p_4, p_5\} < 1$.
Therefore, from Equations (3.11) and (3.12),

$$Gt = Jt = t. \tag{3.15}$$

Now we prove that $It = t$.
Putting $u = Fu_{2n}$ and $v = u_{2n+1}$ in Equation (3.1),

$$\begin{aligned} S(FFu_{2n}, FFu_{2n}, Gu_{2n+1}) \le\; & p_1 S(IFu_{2n}, IFu_{2n}, Ju_{2n+1}) + p_2 S(IFu_{2n}, IFu_{2n}, FFu_{2n}) \\ & + p_3 S(Ju_{2n+1}, Ju_{2n+1}, Gu_{2n+1}) + p_4 S(IFu_{2n}, IFu_{2n}, Gu_{2n+1}) \\ & + p_5 S(Ju_{2n+1}, Ju_{2n+1}, FFu_{2n}) \end{aligned}$$

Letting $n \to \infty$

$$S(It,It,t) \le p_1 S(It,It,t) + p_2 S(It,It,It) + p_3 S(t,t,t) + p_4 S(It,It,t) + p_5 S(t,t,It).$$

This Equations (3.14) and (3.15) implies, $(1 - p_1 - p_4 - p_5)\, S(It,It,t) \le 0$
and hence $It = t$ as $p_1 + p_4 + p_5 \le p_1 + p_2 + p_3 + 3max\{p_4, p_5\} < 1$.
Therefore, from Equations (3.13) and (3.14),

$$Ft = It = t. \tag{3.16}$$

Therefore, from Equations (3.14) and (3.15), $Ft = It = Gt = Jt = t$, showing that t is a common fixed point of F, G, I, J.
Uniqueness: Let t, s be two fixed points common to F, G, I, J.
Then $t = Ft = Gt = It = Jt$ and $s = Fs = Gs = Is = Js$.
Putting $u = t$ and $v = s$ in Equation (3.1),
We get,

$$\begin{aligned} S(Ft,Ft,Gs) \le\; & p_1 S(It,It,Js) + p_2 S(It,It,Ft) + p_3 S(Js,Js,Gs) \\ & + p_4 S(It,It,Gs) + p_5 S(Js,Js,Ft) \end{aligned}$$

Thus,

$$S(t,t,s) \le p_1 S(t,t,s) + p_2 S(t,t,t) + p_3 S(s,s,s) + p_4 S(t,t,s) + p_5 S(s,s,t),$$

which implies, $(1 - p_1 - p_4 - p_5)\, S(t,t,s) \le 0$.
Hence $s = t$, showing that t is unique.

Corollary 3.2. Let E be a real Banach space and $P \subseteq E$ be a solid cone. Let (X, S) be a complete cone S - metric space with cone P, and F, G, I be self-maps of X satisfying the conditions
(i) $F(X) \cup G(X) \subseteq I(X)$;
(ii) $S(Fu,Fu,Gv) \le p_1 S(Iu,Iu,Iv) + p_2 S(Iu,Iu,Fu) + p_3 S(Iv,Iv,Gv) + p_4 S(Iu,Iu,Gv) + p_5 S(Iv,Iv,Fu)$
for all $u, v \in X$, where $p_i > 0\, (i \in Z^+, 1 \le i \le 5)$ satisfying

$$p_1 + p_2 + p_3 + 3max\{p_4, p_5\} < 1; \tag{3.17}$$

(iii) I is continuous and
(iv) The pairs (G, I) and (F, I) of mappings are compatible of type (K) on X.
Then F, G, I have a unique common fixed point in X.
Proof. By assuming $J = I$ in Theorem 3.1, the proof follows.

Corollary 3.3. Let E be a real Banach space and $P \subseteq E$ be a solid cone. Let (X, S) be a complete cone S-metric space with cone P, and G, I be self-maps of X satisfying the conditions

(i) $G(X) \subseteq I(X)$;

(ii) $$S(Gu,Gu,Gv) \leq p_1 S(Iu,Iu,Iv) + p_2 S(Iu,Iu,Gu) + p_3 S(Iv,Iv,Gv) + p_4 S(Iu,Iu,Gv) + p_5 S(Iv,Iv,Gu)$$

for all $u,v \in X$, where $p_i > 0\,(i \in Z^+, 1 \leq i \leq 5)$ satisfying

$$p_1 + p_2 + p_3 + 3max\{p_4, p_5\} < 1; \tag{3.18}$$

(iii) I Is continuous and

(iv) The pair (G, I) of mappings is compatible of type (K) on X.

Then G and I have a unique common fixed point in X.

Proof. By assuming $F = G$ and $J = I$ in Theorem 3.1, the proof follows.

In support of Theorem 3.1, we give the following example.

Example 3.4. Let $E = R^2$ and $P = \{(u,v) \in R^2 : u \geq 0, v \geq 0\}$.

Let the function $S : R^3 \to E$ be given by $S(u,v,w) = (|u-w| + |v-w|, \frac{1}{4}[|u-w| + |v-w|])$.

Then S is a cone S-metric on X.

In this case, $S(u,u,v) = (2|u-v|, \frac{1}{2}|u-v|)$.

Let $X = [0,1]$ and define the maps F, G, I and J on X by

$F(u) = \frac{u}{10}, G(u) = 0, I(u) = u, J(u) = \frac{u}{2}$ for all $u \in [0,1]$.

Then $S(Fu,Fu,Gv) = (2|Fu - Gv|, \frac{1}{2}|Fu - Gv|) = (\frac{u}{5}, \frac{u}{20})$,

$S(Iu,Iu,Gv) = (2|Iu - Gv|, \frac{1}{2}|Iu - Gv|) = (2u, \frac{u}{2})$.

Now, $\frac{1}{8}S(Iu,Iu,Gv) - S(Fu,Fu,Gv) = (\frac{u}{20}, \frac{u}{80}) \in P$ since $u \geq 0$.

Therefore, $S(Fu,Fu,Gv) \leq \frac{1}{8}S(Iu,Iu,Gv)$.

If we choose p_1, p_2, p_3, p_4, p_5 such that $p_1 + p_2 + p_3 < \frac{5}{8}, p_4 = \frac{1}{8}$ and $p_5 < \frac{1}{8}$,

then $p_1 + p_2 + p_3 + 3max\{p_4, p_5\} < 1$.

Hence, inequality (3.1.1) holds for $p_4 = \frac{1}{8}, p_5 < \frac{1}{8}$ and for any p_1, p_2, p_3 such that $p_1 + p_2 + p_3 < \frac{5}{8}$.

Also, $F(X) = [0, \frac{1}{10}] \subseteq [0, \frac{1}{2}] = J(X)$ and $G(X) = \{0\} \subseteq [0,1] = I(X)$.

Clearly J and I are continuous.

For the sequence, $u_n = \frac{1}{n^2}, n = 1, 2, \ldots$

$S(Gu_n, Gu_n, 0) = S(0,0,0) = (0,0)$,

$S(Ju_n, Ju_n, 0) = S\left(\frac{1}{2n^2}, \frac{1}{2n^2}, 0\right) = (\frac{1}{n^2}, \frac{1}{4n^2}) \to (0,0)$ as $n \to \infty$,

$S(Fu_n, Fu_n, 0) = S\left(\frac{1}{10n^2}, \frac{1}{10n^2}, 0\right) = (\frac{1}{5n^2}, \frac{1}{20n^2}) \to (0,0)$ as $n \to \infty$,

and $S(Iu_n, Iu_n, 0) = S\left(\frac{1}{n^2}, \frac{1}{n^2}, 0\right) = (\frac{2}{n^2}, \frac{1}{2n^2}) \to (0,0)$ as $n \to \infty$.

Then $Gu_n, Ju_n, Fu_n, Iu_n \to 0$ as $n \to \infty$.

$S(GGu_n, GGu_n, J(0)) = S(0,0,0) = (0,0)$

and $S(JJu_n, JJu_n, G(0)) = S\left(\frac{1}{4n^2}, \frac{1}{4n^2},, 0\right) = (\frac{1}{2n^2}, \frac{1}{8n^2}) \to (0,0)$ as $n \to \infty$.

Hence, $GGu_n \to J(0)$ and $JJu_n \to G(0)$.

Also $S(FFu_n, FFu_n, I(0)) = S\left(\frac{1}{100n^2}, \frac{1}{100n^2}, 0\right) = (\frac{1}{50n^2}, \frac{1}{200n^2}) \to (0,0)$ as $n \to \infty$

and $S(IIu_n, IIu_n, F(0)) = S\left(\frac{1}{n^2}, \frac{1}{n^2}, 0\right) = (\frac{2}{n^2}, \frac{1}{2n^2}) \to (0,0)$ as $n \to \infty$.

Hence, $FFu_n \to I(0)$ and $IIu_n \to F(0)$.

It shows that the pairs (G,J) and (F,I) are type (K) compatible.

Also, by the definition itself, it is clear that I and J are continuous.

All the requirements of Theorem 3.1 are met.

Also, the maps F, G, I and J possess a common fixed point $0 \in X$, and it is unique as asserted by the theorem.

4 Conclusion

In this research work, we prove a theorem regarding the existence of fixed points in a cone S-metric space. In Theorem 3.1, we establish a fixed point common to four type (K) compatible self-maps defined on a complete cone S-metric space that satisfy a generalised contraction and prove its uniqueness. We also provide two corollaries wherein our result is validated in the case of three or two maps. Finally, a befitting example is provided in support of our result. Thus, we generalise previous results in the literature by replacing stronger conditions with a weaker condition, type (K) compatibility.

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