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Application of Fixed Point Theorems to Differential Equation in b-Multiplicative Metric Spaces

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Abstract

Objectives: In this manuscript, some fixed point theorems have been provided in b-multiplicative metric spaces. **Methods:** We proved the unique fixed point theorems using the Banach contraction principle and generalized Lipschitz contractive mappings. **Findings:** We established the P Property and the T-Stability of Picard's iteration in b-multiplicative metric spaces. We also provide an example to demonstrate the result. **Novelty:** Using our results, we obtained the existence and uniqueness of solutions for ordinary multiplicative differential equations with initial value problems.

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1 Introduction

Fixed point theory serves as a cornerstone in various mathematical disciplines, unveiling fundamental insights into stability. Research on stability results for fixed point iteration methods is widely used in many areas. The iteration scheme has been extended to various other spaces recently[1–3]. They also obtained some fixed point theorems in that space for mappings satisfying various types of contractive conditions. Picard's iteration, whose stability holds an essential place in several fields, is the most crucial iteration procedure among them. However, for some mappings, the P properties of fixed points have drawn the attention of several authors[4,5] because proving that a map holds the property P, where $F(T) = F(T^n)$, is essential for understanding the behavior of the map under iteration, it tells us that applying the map multiple times (n times) has the same effect as applying it once. This understanding can be crucial in various mathematical and scientific contexts, such as in dynamical systems theory or optimization algorithms. Bashirov et al. [6] established the concept of multiplicative metric spaces for transferring the functions of addition and subtraction to multiplication and division to ease the computation. In 2017[7], Muhammad Usman Ali introduced the idea of b-multiplicative metric spaces which provides the descriptions of distances increased versatility and enables the detection of some kinds of non-linearities that metric spaces could lack. Recently few authors have established some applications for integral equations[8],differential equations[9],and coupled fixed point problems[10].

In this paper, we need to leverage a different type of space known as b-multiplicative metric spaces. Here, we extend the mappings of b-metric spaces[3] to b-multiplicative metric spaces. A few fixed point theorems are constructed in this research for a class of contractive mappings in b-multiplicative metric spaces. We proved Lemma 3.1 for convergence in R^+ with the multiplicative sense. Lemma 3.2 generalizes the lemma of b-metric spaces to prove the sequence is a b-multiplicative Cauchy. The generalized Lipschitz contractive condition of rational type is used in Theorem 3.3 to determine the unique fixed point in the space called $b-MMS$. Definition 3.4 is introduced to state Picard's iteration is stable in $b-MMS$. Theorem 3.5 is proven to show T-stability for an iteration under the generalized Lipschitz Contractive Condition to ensure the stability of the solution over successive iterations because a stable iterative process ensures that it converges towards a solution. Without stability, the process may oscillate or diverge, failing to reach a solution. By using[5],Theorem 3.6 is proved to satisfy P Property in this space. Owing to our comprehension of the map's behaviour during iteration, we demonstrated Theorem 3.7 by the condition of theorem 3.3 to satisfy P – Property. The aforementioned theorems have never been proven in $b-MMS$ space before. In addition, we give an example that demonstrates the results. We also give an application for initial value problems in ordinary multiplicative differential equations in $b-MMS$ to be proved relatively easy and it confirms the solutions of the existence and uniqueness of these equations.

2 Preliminaries

Definition 2.1 Let (χ,σ) be a metric space and $T:\chi\to\chi$ be a mapping. The point $f\in\chi$ is called a fixed point of T if f is mapped onto itself. i.e., $T(f)=f$.

Definition 2.2[7]Let χ be a non-empty set and $\mu\geq 1$ be a given real number. A function $\sigma:\chi\times\chi\to[1,\infty]$ is called a $b-$multiplicative metric with coefficient μ if the following conditions hold:

i) $\sigma(f,g)>1$ for all $f,g\in\chi$ with $f\neq g$ and $\sigma(f,g)=1$ iff $f=g$

ii) $\sigma(f,g)=\sigma(g,f)$ for all $f,g\in\chi$

iii) $\sigma(f,h)\leq[\sigma(f,g).\sigma(g,h)]^{\mu}$ for all $f,g,h\in\chi$

The triplet (χ,σ,μ) is called b-multiplicative metric space.

Definition 2.3[7] Let (χ,σ,μ) be a $b-MMS$.

(i) A sequence $\{f_n\}$ is b-multiplicative convergent iff there exist $f\in\chi$ such that $\sigma(f_n,f)\to 1$ as $n\to\infty$.

(ii) A sequence $\{f_n\}$ is called b-multiplicative Cauchy iff $\sigma(f_m,f_n)\to 1$ as $n,m\to\infty$.

iii) A $b-MMS(\chi,\sigma,\mu)$ is said to be complete if every b-multiplicative Cauchy sequence in χ is b - convergent to a point in χ.

Definition 2.4[3] Let (χ,σ) be a metric space and T a self-map on χ. Let f_0 be a point of χ, and assume that $f_{n+1}=\Omega(T,f_n)$ is an iteration procedure, involving T, which yields a sequence $\{f_n\}$ of points from χ. Then the iteration procedure $f_{n+1}=\Omega(T,f_n)$is said to be T-stable with respect to T if $\{f_n\}$converges to a fixed point q of T and whenever $\{g_n\}$ is a sequence in χ with $\lim_{n\to\infty}\sigma(g_{n+1},\Omega(T,g_n))=0$, we have $\lim_{n\to\infty}g_n=q$. Particularly, if these conditions holds true for Picard's iteration procedure $f_{n+1}=T f_n$, then we will say that Picard's iteration is T-stable.

3 Result and discussion

In this section, we begin by providing an important lemma that generalizes the lemma of metric spaces. Following that, we established fixed point theorems in b-multiplicative metric spaces and verified the stability results. Finally, we give an example to demonstrate our results.

Lemma 3.1

Let $\{a_n\}$, $\{c_n\}$ be a non-negative sequences in $b-MMS$ satisfying $a_{n+1}\leq(a_n^h.c_n)$ for all $n\in N$,$0\leq h<1$, $\lim_{n\to\infty}c_n=1$. Then $\lim_{n\to\infty}a_n=1$.

Proof.

$a_{k+m}\leq a_{k+m-1}^h\cdot c_{k+m-1}\leq\left(a_{k+m-2}^h\cdot c_{k+m-2}\right)^h\cdot c_{k+m-1}$

$=\left(a_{k+m-2}\right)^{h^2}\cdot\left(c_{k+m-2}\right)^h\cdot c_{k+m-1}\leq\ldots\leq a_k^{h^m}\cdot c_k^{h^{m-1}}\cdot c_{k+1}^{h^{m-2}}\ldots c_{k+m-1}$

$\leq\ a_k^{h^m} . \left(c_{(k,k+m-1)}\right)^{h^{m-1}} . \left(c_{(k,k+m-1)}\right)^{h^{m-2}} \dots c_{(k,k+m-1)}$
$= a_k^{h^m} . \left(c_{(k.k+m-1)}\right)^{h^{m-1}+h^{m-2}+\dots+1}$
$\leq\ a_k^{h^m} . \left(c_{(k,k+m-1)}\right)^{\frac{1}{1-h}}$
where $c_{(k,k+m-1)} = max\{c_k, c_{k+1}, \dots, c_{k+m-1}\}$
Since $lim_{n\to\infty}\, c_n = 1$ for every $\varepsilon > 1$, there exists a natural number N_1, such that when $k\ \geq\ N_1$, we have

$$0\ \leq\ (c_n)^{\frac{1}{1-h}} < \varepsilon^{\frac{1}{2}} \tag{3.1}$$

suppose that $a_{N_1} = F$

$$0\ \leq\ F^{h^m} < \varepsilon^{\frac{1}{2}} \tag{3.2}$$

Since $0\ \leq h\ < 1$, there exists a natural number N_2, such that when $m\ >\ N_2$, we have
Suppose that $N =\ N_1 . N_2$, thus when $n\ > N$,

$$0\ \leq\ a_n\ =\ a_{m+N_1}\ \leq\ a_{N_1}{}^{h^m} . \left(c_{(N_1,N_1+m-1)}\right)^{\frac{1}{1-h}}$$

It follows from Equations (3.1) and (3.2) that $0\ \leq\ a_n < \varepsilon^{\frac{1}{2}} . \varepsilon^{\frac{1}{2}} = \varepsilon$. Therefore we have $lim_{n\to\infty}\, a_n = 1$. This completes the proof of the lemma.

Lemma 3. 2

Let (χ, σ, μ) be a $b-$multiplicative metric spaces with coefficient $\mu \geq 1$ and $T : \chi \to \chi$ be a mapping. Consider that $\{f_n\}$ is a sequence in χ induced by $f_{n+1} = Tf_n$ such that

$$\sigma(f_n, f_{n+1}) \leq \sigma(f_{n-1}, f_n)^{\lambda} \tag{3.3}$$

for all $n \in N$, where $\lambda \in [0,1]$ is constant. Then $\{f_n\}$ is a b-multiplicative Cauchy sequence.

Proof.

We can divide it into three separate cases. In each case, we have an $f_0\ \in\ \chi$ and define $f_{n+1} = Tf_n$ for all $n \in\ N$.
Case 1: $\lambda\ \in [0, \frac{1}{\mu}]\, (\mu > 1)$. Using Equation (3.3),
$\sigma\ \left(f_n, f_{n+1}\right)\ \leq\ \sigma\left(\ f_{n-1}, f_n\right)^{\lambda}$
$\leq\ \sigma\left(\ f_{n-2}, f_{n-1}\right)^{\lambda^2}$
$\vdots$
$\leq\ \sigma\left(\ f_0, f_1\right)^{\lambda^n}$
Thus, for any $n > m$ and $n, m \in N$, we have
$\sigma\ (f_m, f_n)\ \leq\ [\sigma\,(f_m, f_{m+1})\ .\sigma\ (f_{m+1}, f_n)]^{\mu}$
$\leq\ \sigma\,(f_m, f_{m+1})^{\mu}\ .[\sigma\,(f_{m+1}, f_{m+2})\ .\sigma\ (f_{m+2}, f_n)]^{\mu^2}$
$\leq\ \sigma\,(f_m, f_{m+1})^{\mu}\ .\sigma\,(f_{m+1}, f_{m+2})^{\mu^2} . [\sigma\,(f_{m+2}, f_{m+3})\ .\sigma\ (f_{m+3}, f_n)]^{\mu^3}$
$\leq\ \sigma\,(f_m, f_{m+1})^{\mu}\ .\sigma\,(f_{m+1}, f_{m+2})^{\mu^2} .\sigma\,(f_{m+2}, f_{m+3})^{\mu^3}\ .\sigma\ (f_{n-2}, f_{n-1})^{\mu^{n-m-1}}\ .\sigma\ (f_{n-1}, f_n)^{\mu^{n-m-1}}$
$\leq\ \sigma(f_0, f_1)^{\mu\lambda^m} . \sigma(f_0, f_1)^{\mu^2\lambda^{m+1}} . \sigma(f_0, f_1)^{\mu^3\lambda^{m+2}} . \sigma(f_0, f_1)^{\mu^{n-m-1}\lambda^{n-2}} . \sigma(f_0, f_1)^{\mu^{n-m-1}\lambda^{n-1}}$
$\leq\ \sigma(f_0, f_1)^{\mu\lambda^m[1+s\lambda+\mu^2\lambda 2+\dots+\mu^{n-m-2}\lambda^{n-m-2}+\mu^{n-m-1}\lambda^{n-m-1}]}$
$\leq \sigma\left(f_0, f_1\right)^{\mu\lambda^m\left[\Sigma_{i=0}^{\infty}(\mu\lambda)^i\right]}$
$\leq \sigma(f_0 . f_1)^{\frac{\mu\lambda^m}{1-\mu\lambda}} \quad \to 1 \quad (m \to \infty)$.
It indicates that $\{f_n\}$ is a b - multiplicative Cauchy sequence. Simply, $\{T^n\ f_0\}$ is a b-multiplicative Cauchy sequence.
Case 2: Let $\lambda\ \in \left[\frac{1}{\mu}, 1\right] (\mu > 1)$ In this case, $\lambda^n \to 1$ as $n \to \infty$, so there is $n_0\ \in N\ \ni\ \lambda^{n_0} < \frac{1}{\mu}$.
By case 1, we state that
$\{(T^{n_0})^n\ f_0\}_{n=0}^{\infty} = \left\{f_{n_0}, f_{n_0+1}, f_{n_0+2}, \dots, f_{n_0+n}, \dots\right\}$ is a b - Multiplicative Cauchy sequence in χ.
Case 3: Let $\mu = 1$. The process is same as case 1, and it verified the claim.

Theorem 3.3

Let (χ,σ,μ) be a $b-MMS$ with coefficient $\mu\geq 1$ and $T:\chi\to\chi$ be a mapping such that

$$\sigma(Tf,Tg)\leq\sigma(f,g)^{\psi_1}\cdot\left(\frac{\sigma(f,Tf)\sigma(g,Tg)}{1+\sigma(f,g)}\right)^{\psi_2}\cdot\left(\frac{\sigma(f,Tg)\sigma(g,Tf)}{1+\sigma(f,g)}\right)^{\psi_3}\cdot \left(\frac{\sigma(f,Tf)\sigma(f,Tg)}{1+\sigma(f,g)}\right)^{\psi_4}\cdot\left(\frac{\sigma(g,Tg)\sigma(g,Tf)}{1+\sigma(f,g)}\right)^{\psi_5} \tag{3.4}$$

Where $\psi_1,\psi_2,\psi_3,\psi_4,\psi_5$ are non-negative constants with $\psi_1+\psi_2+\psi_3+\mu\psi_4+\mu\psi_5<1$. Consequently, T has a unique fixed point in χ. Also, the iterative sequence $T^n f\,(n\in N)$ b-converges to the fixed point for any $f\in\chi$.

Proof.

Fix on $f_0\in\chi$ and construct an iterative Picard sequence $\{f_0\}$ using $f_{n+1}=Tf_n\ (n\in N)$. If there exists $n_0\in N\ni f_{n_0}=f_{n_0+1}$, then $f_{n_0}=f_{n_0+1}=Tf_{n_0}$. For the sake of simplicity, let $f_n\neq f_{n+1}\ \forall n\in N$. From Equation (3.4)

$$\sigma(f_n,f_{n+1})=(Tf_{n-1},Tf_n)$$

$$\leq\sigma(f_{n-1},f_n)^{\psi_1}\cdot\left(\frac{\sigma(f_{n-1},Tf_{n-1})\sigma(f_n,Tf_n)}{1+\sigma(f_{n-1},f_n)}\right)^{\psi_2}\cdot\left(\frac{\sigma(f_{n-1},Tf_n)\sigma(f_n,Tf_{n-1})}{1+\sigma(f_{n-1},f_n)}\right)^{\psi_3}\cdot$$

$$\left(\frac{\sigma(f_{n-1},Tf_{n-1})\sigma(f_{n-1},Tf_n)}{1+\sigma(f_{n-1},f_n)}\right)^{\psi_4}\cdot\left(\frac{\sigma(f_n,Tf_n)\sigma(f_n,Tf_{n-1})}{1+\sigma(f_{n-1},f_n)}\right)^{\psi_5}$$

$$=\sigma(f_{n-1},f_n)^{\psi_1}\cdot\left(\frac{\sigma(f_{n-1},Tf_n)\sigma(f_n,Tf_{n+1})}{1+\sigma(f_{n-1},f_n)}\right)^{\psi_2}\cdot\left(\frac{\sigma(f_{n-1},Tf_{n+1})\sigma(f_n,f_n)}{1+\sigma(f_{n-1},f_n)}\right)^{\psi_3}\cdot$$

$$\left(\frac{\sigma(f_{n-1},f_n)\sigma(f_{n-1},Tf_{n+1})}{1+\sigma(f_{n-1},f_n)}\right)^{\psi_4}\cdot\left(\frac{\sigma(f_n,Tf_{n+1})\sigma(f_n,f_n)}{1+\sigma(f_{n-1},f_n)}\right)^{\psi_5}$$

$$\leq\sigma(f_{n-1},f_n)^{\psi_1}\cdot\sigma(f_n,f_{n+1})^{\psi_2}\cdot(\sigma(f_{n-1},f_n).\sigma(f_n,f_{n+1}))^{\mu\psi_4}$$

It follows that

$$\sigma(f_n,f_{n+1})^{1-\psi_2-\mu\psi_4}\leq\sigma(f_{n-1},f_n)^{\psi_1+\mu\psi_4} \tag{3.5}$$

Again by Equation (3.4), we have

$$\sigma(f_n,f_{n+1})=\sigma(Tf_n,Tf_{n-1})$$

$$\leq\sigma(f_n,f_{n-1})^{\psi_1}\cdot\left(\frac{\sigma(f_n,Tf_n)\sigma(f_{n-1},Tf_{n-1})}{1+\sigma(f_n,f_{n-1})}\right)^{\psi_2}\cdot\left(\frac{\sigma(f_n,Tf_{n-1})\sigma(f_{n-1},Tf_n)}{1+\sigma(f_n,f_{n-1})}\right)^{\psi_3}\cdot$$

$$\left(\frac{\sigma(f_n,Tf_n)\sigma(f_n,Tf_{n-1})}{1+\sigma(f_n,f_{n-1})}\right)^{\psi_4}\cdot\left(\frac{\sigma(f_{n-1},Tf_{n-1})\sigma(f_{n-1},Tf_n)}{1+\sigma(f_n,f_{n-1})}\right)^{\psi_5}$$

$$=\sigma(f_n,f_{n-1})^{\psi_1}\cdot\left(\frac{\sigma(f_n,f_{n+1})\sigma(f_{n-1},f_n)}{1+\sigma(f_n,f_{n-1})}\right)^{\psi_2}\cdot\left(\frac{\sigma(f_n,f_n)\sigma(f_{n-1},f_{n+1})}{1+\sigma(f_n,f_{n-1})}\right)^{\psi_3}\cdot$$

$$\left(\frac{\sigma(f_n, f_{n+1})\sigma(f_n, f_n)}{1+\sigma(f_n, f_{n-1})}\right)^{\psi_4}\cdot\left(\frac{\sigma(f_{n-1}, f_n)\sigma(f_{n-1}, f_{n+1})}{1+\sigma(f_n, f_{n-1})}\right)^{\psi_5}$$

$$\le \sigma(f_n, f_{n-1})^{\psi_1}\cdot\sigma(f_n, f_{n+1})^{\psi_2}\cdot(\sigma(f_{n-1}, f_n).\sigma(f_n, f_{n+1}))^{\mu\psi_5}$$

$$\sigma(f_n, f_{n-1})^{1-\psi_2-\mu\psi_5} \le \sigma(f_{n-1}, f_n)^{\psi_1+\mu\psi_5} \tag{3.6}$$

Multiplying Equations (3.5) and (3.6)

$$\sigma(f_n, f_{n+1})^{2-2\psi_2-\mu\psi_4-\mu\psi_5} \le \sigma(f_{n-1}, f_n)^{2\psi_1+\mu\psi_4+\mu\psi_5}$$

$$\sigma(f_n, f_{n+1}) \le \sigma(f_{n-1}, f_n)^{\frac{2\psi_1+\mu\psi_4+\mu\psi_5}{2-2\psi_2-\mu\psi_4-\mu\psi_5}}$$

Put $h=\frac{2\psi_1+\mu\psi_4+\mu\psi_5}{2-2\psi_2-\mu\psi_4-\mu\psi_5}$. In view of $\psi_1+\psi_2+\psi_3+\mu\psi_4+\mu\psi_5<1$, then $0\le\psi<1$. According to lemma 3.2, $\{f_n\}$ is a b-Cauchy sequence in χ. As (χ,σ,μ) is b-complete, then $\exists$ some point $f^*\in\chi\ni f_n\to f^*$ as $n\to\infty$. It is simple to observe from Equation (3.4) that

$$\sigma(f_{n+1}, Tf^*) = \sigma(Tf_n, Tf^*) \tag{3.7}$$

$$\le \sigma(f_n, f^*)^{\psi_1}\cdot\left(\frac{\sigma(f_n, Tf_n)\sigma(f^*, Tf^*)}{1+\sigma(f_n, f^*)}\right)^{\psi_2}\cdot\left(\frac{\sigma(f_n, Tf^*)\sigma(f^*, Tf_n)}{1+\sigma(f_n, f^*)}\right)^{\psi_3}\cdot$$

$$\left(\frac{\sigma(f_n, Tf_n)\sigma(f_n, Tf^*)}{1+\sigma(f_n, f^*)}\right)^{\psi_4}\cdot\left(\frac{\sigma(f^*, Tf^*)\sigma(f^*, Tf_n)}{1+\sigma(f_n, f^*)}\right)^{\psi_5}$$

$$= \sigma(f_n, f^*)^{\psi_1}\cdot\left(\frac{\sigma(f_n, f_{n+1})\sigma(f^*, Tf^*)}{1+\sigma(f_n, f^*)}\right)^{\psi_2}\cdot\left(\frac{\sigma(f_n, Tf^*)\sigma(f^*, f_{n+1})}{1+\sigma(f_n, f^*)}\right)^{\psi_3}\cdot \left(\frac{\sigma(f_n, f_{n+1})\sigma(f_n, Tf^*)}{1+\sigma(f_n, f^*)}\right)^{\psi_4}\cdot\left(\frac{\sigma(f^*, Tf^*)\sigma(f^*, f_{n+1})}{1+\sigma(f_n, f^*)}\right)^{\psi_5} \tag{3.8}$$

Considering the limit $n\to\infty$ in Equation (3.8), we obtain $\lim_{n\to\infty}\sigma(f_{n+1}, Tf^*)=1$. That is, $f_n\to Tf^*$ as $n\to\infty$. Due to the uniqueness of the limit of the b-convergent sequence, it concludes that Tf^*=f^*. Alternatively, f^* is a fixed point of T. Eventually, to verify the uniqueness of the fixed point, use Equation (3.4), if $\exists$ a another fixed point g^*, then

$$\sigma(f^*, g^*) = \sigma(Tf^*, Tg^*)$$

$$\le \sigma(f^*, g^*)^{\psi_1}\cdot\left(\frac{\sigma(f^*, Tf^*)\sigma(g^*, Tg^*)}{1+\sigma(f^*, g^*)}\right)^{\psi_2}\cdot\left(\frac{\sigma(f^*, Tg^*)\sigma(g^*, Tf^*)}{1+\sigma(f^*, g^*)}\right)^{\psi_3}\cdot$$

$$\left(\frac{\sigma(f^*, Tf^*)\sigma(f^*, Tg^*)}{1+\sigma(f^*, g^*)}\right)^{\psi_4}\cdot\left(\frac{\sigma(g^*, Tg^*)\sigma(g^*, Tf^*)}{1+\sigma(f^*, g^*)}\right)^{\psi_5}$$

$$= \sigma\left(g^*, Tg^*\right)^{\psi_1} . \left(\frac{\sigma\left(f^*, g^*\right)\sigma\left(f^*, g^*\right)}{1+\sigma\left(f^*, g^*\right)}\right)^{\psi_3}$$

$$\leq \sigma\left(f^*, g^*\right)^{\psi_1+\psi_3} \tag{3.9}$$

Because $\psi_1+\psi_2+\psi_3+\mu\psi_4+\mu\psi_5<1$ implies $\psi_1+\psi_3<1$, we conclude that $\sigma\left(f^*, g^*\right)=1$ i.e. $f^*=g^*$.

Definition 2.4 is simplified here, and a concept of T -Stability of Picard iteration in b-multiplicative metric space is introduced.

Definition 3.4

Let (χ, σ, μ) be a $b-MMS$, $f_0 \in \chi$ and $T: \chi \rightarrow \chi$ be a mapping with $(T) \neq \emptyset$, where (T) denotes the set of all fixed points of T. Then Picard's iteration f_{n+1} = T f_n is said to be T-stable with respect to T if $\lim_{n\rightarrow\infty} f_n$ = q $\in$ (T) and whenever $\{g_n\}$ is a sequence in χ with $\lim_{n\rightarrow\infty} \sigma(g_{n+1}, Tg_n)$ =1, we have $\lim_{n\rightarrow\infty} g_n=q$.

Theorem 3.5

Under the generalized Lipschitz contractive mapping of previous theorem 3.3, if $2\mu\psi_1+2\psi_3+(\mu+\mu^2)(\psi_4+\psi_5)<2$, then Picard's iteration is T-Stable.

Proof.

To show that T has a unique fixed point f^* in χ , use Equation (3.4) and deduce that $\{g_n\}$ is a sequence in χ $\ni$ $\sigma\left(g_{n+1}, Tg_n\right) \rightarrow 1$ as n goes to ∞.

Using Equation (3.4),

$$\sigma\left(Tg_n, f^*\right)=\sigma\left(Tg_n, Tf^*\right)$$

$$\leq \sigma\left(g_n, f^*\right)^{\psi_1} . \left(\frac{\sigma\left(g_n, Tg_n\right)\sigma\left(f^*, Tf^*\right)}{1+\sigma\left(g_n, f^*\right)}\right)^{\psi_2} . \left(\frac{\sigma\left(g_n, Tf^*\right)\sigma\left(f^*, Tg_n\right)}{1+\sigma\left(g_n, f^*\right)}\right)^{\psi_3} .$$

$$\left(\frac{\sigma\left(g_n, Tg_n\right)\sigma\left(g_n, Tf^*\right)}{1+\sigma\left(g_n, f^*\right)}\right)^{\psi_4} . \left(\frac{\sigma\left(f_n, Tf_n\right)\sigma\left(f_n, Tf_{n-1}\right)}{1+\sigma\left(f_{n-1}, f_n\right)}\right)^{\psi_5}$$

$$\leq \sigma\left(g_n, f^*\right)^{\psi_1} . \sigma\left(f^*, Tg_n\right)^{\psi_3} . \sigma\left(g_n, Tg_n\right)^{\psi_4}$$

$$\leq \sigma\left(g_n, f^*\right)^{\psi_1} . \sigma\left(f^*, Tg_n\right)^{\psi_3} . \left(\sigma\left(g_n, f^*\right) . \sigma\left(f^*, Tg_n\right)\right)^{\mu\psi_4}$$

$$\leq \sigma\left(g_n, f^*\right)^{\psi_1+\mu\psi_4} . \sigma\left(f^*, Tg_n\right)^{\psi_3+\mu\psi_4}$$

which means that

$$\sigma(f^*, Tg_n)^{1-\psi_3-\mu\psi_4} \leq \sigma(g_n, f^*)^{\psi_1+\mu\psi 4} \tag{3.10}$$

On the other hand, we have

$\sigma\left(Tg_n, f^*\right)=\sigma\left(Tf^*, Tg_n\right)$

$\leq \sigma(f^*, g_n)^{\psi_1} . \sigma(f^*, Tg_n)^{\psi_3} . \sigma(g_n, Tg_n)^{\psi_5}$

$\leq \sigma(f^*, g_n)^{\psi_1} . \sigma(f^*, Tg_n)^{\psi_3}\left(\sigma\left(g_n, f^*\right) . \sigma\left(f^*, Tg_n\right)\right)^{\mu\psi_5}$

$\leq \sigma(f^*, g_n)^{\psi_1+\mu\psi_5} . \sigma(f^*, Tg_n)^{\psi_3+\mu\psi_5}$

which means that

$$\sigma(f^*, Tg_n)^{1-\psi_3-\mu\psi_5} \leq \sigma(g_n, f^*)^{\psi_1+\mu\psi_5} \tag{3.11}$$

Combining Equations (3.10) and (3.11)

$$\sigma(f^*,Tg_n)^{2-2\psi_3-\mu\psi_4-\mu\psi_5} \leq \sigma(g_n,f^*)^{2\psi_1+\mu\psi_4+\mu\psi_5} \tag{3.12}$$

As a result, we have

$\sigma(f^*,Tg_n)^{2-2\psi_3-\mu\psi_4-\mu\psi_5} \leq \sigma(g_n,f^*)^{\frac{2\psi_1+\mu\psi_4+\mu\psi_5}{2-2\psi_3-\mu\psi_4-\mu\psi_5}}$

Denote h = $\frac{\mu(2\psi_1+\mu\psi_4+\mu\psi_5)}{2-2\psi_3-\mu\psi_4-\mu\psi_5}$. It follows immediately from $2\mu\psi_1+2\psi_3+(\mu+\mu^2)(\psi_4+\psi_5)<2$ that $0\leq h\ <1$.

Let $a_n=\sigma(g_n,f^*)$, $c_n=\sigma(g_{n+1},Tg_n)^{\mu}$, by Equation (3.12), then

$$a_{n+1}=\sigma(g_{n+1},f^*)\leq(\sigma(g_{n+1},Tg_n)\ .\ \sigma(Tg_n,f^*))^{\mu}$$

$$\leq\ a_n^h.c_n$$

Thus, by lemma 3.1, it leads to $a_n=\sigma(g_n,f^*)\to 1$ as n goes to ∞, i.e. $g_n\to f^*$. Consequently, Picard's iteration is T-stable.

For each $n\in\ N,\ f^*$ is a fixed point of T^n if T is a map with a fixed point f^*. However, the converse is contradictory. A map T is said to hold the P property if $F(T)=F(T^n)$ for each $n\in N$ [4,5]. The implications of these findings in $b-MMS$ are generalized in the following results.

Theorem 3.6

Let (χ,σ,μ) be a $b-MMS$ with coefficient $\mu\geq 1$. Let $T:\chi\to\chi$ be a mapping such that $(T)\neq\emptyset$ and that

$$\sigma(Tf,\,T^2f)\leq\sigma(f,Tf)^{\psi} \tag{3.13}$$

for all $f\in\chi$, where $0\leq\psi<1$ is a constant. Then T has the P Property.

Proof.

As the statement for $n=1$ is trivial. Suppose $n>1$ and $\omega\in\ (T^n)$. It is obvious from the hypotheses that

$\sigma(\omega,T\omega)=\sigma(TT^{n-1}\omega,T^2T^{n-1}\omega)\leq\sigma(T^{n-1}\omega,T^n\omega)^{\omega}$

$=\sigma(TT^{n-2}\omega,T^2T^{n-2}\omega)^{\psi}\leq\sigma(T^{n-2}\omega,T^{n-1}\omega)^{\psi^2}$

$\leq\ldots\leq\ \sigma(\omega,T\omega)^{\psi^n}\to 1\,(n\to\infty)$

Hence, $\sigma(\omega,T\omega)=1$ that is $T\omega=\omega$.

Theorem 3.7

T holds the *P* property when the condition of theorem 3.3 is satisfied.

Proof.

The mapping T have to be satisfied Equation (3.13). In fact, for any $f\in\chi$, to start with, we have

$$\sigma(Tf,T^2f)=\sigma(Tf,TTf)$$

$$\leq\sigma(f,Tf)^{\psi_1}.\left(\frac{\sigma(f,\,Tf)\,\sigma(Tf,\,TTf)}{1+\sigma(f,Tf)}\right)^{\psi_2}.\left(\frac{\sigma(\,f,TTf)\,\sigma(Tf,\,Tf)}{1+\sigma(f,Tf)}\right)^{\psi_3}.$$

$$\left(\frac{\sigma(f,Tf)\,\sigma(f,TTf)}{1+\sigma(f,Tf)}\right)^{\psi_4}.\left(\frac{\sigma(Tf,TTf)\,\sigma(Tf,Tf)}{1+\sigma(f,Tf)}\right)^{\psi_5}$$

$$\leq\sigma(f,Tf)^{\psi_1}\,.\,\sigma(Tf,T^2f)^{\psi_2}\,.\,\sigma(f,T^2f)^{\psi_4}$$

$$\leq\sigma(f,Tf)^{\psi_1}\,.\,\sigma(Tf,T^2f)^{\psi_2}\,.\,(\sigma(f,Tf)\ .\sigma(Tf,T^2f))^{\mu\psi_4}$$

which implies that

$$\sigma\left(Tf,T^{2}f\right)^{1-\psi_{2}-\mu\psi_{4}} \leq \sigma\left(f,Tf\right)^{\psi_{1}+\mu\psi_{4}} \tag{3.14}$$

Moreover, we have

$$\sigma\left(Tf,T^{2}f\right)=\sigma(TTf,Tf)$$

$$\leq \sigma\left(Tf,f\right)^{\psi_{1}}.\left(\frac{\sigma\left(Tf,\ TTf\right)\sigma\left(f,\ Tf\right)}{1+\sigma\left(Tf,f\right)}\right)^{\psi_{2}}.\left(\frac{\sigma\left(Tf,Tf\right)\sigma\left(f,\ TTf\right)}{1+\sigma\left(Tf,f\right)}\right)^{\psi_{3}}.$$

$$\left(\frac{\sigma\left(Tf,TTf\right)\sigma\left(Tf,Tf\right)}{1+\sigma\left(Tf,f\right)}\right)^{\psi_{4}}.\left(\frac{\sigma\left(f,Tf\right)\sigma\left(f,TTf\right)}{1+\sigma\left(Tf,f\right)}\right)^{\psi_{5}}$$

$$\leq \sigma\left(Tf,f\right)^{\psi_{1}}.\sigma\left(Tf,T^{2}f\right)^{\psi_{2}}.\sigma\left(f,T^{2}f\right)^{\psi_{5}}$$

Which establishes that

$$\sigma\left(Tf,T^{2}f\right)^{1-\psi_{2}-\mu\psi_{5}} \leq \sigma\left(f,Tf\right)^{\psi_{1}+\mu\psi_{5}} \tag{3.15}$$

On multiplying up Equations (3.14) and (3.15), it follows that

$\sigma\left(Tf,T^{2}f\right)^{2-2\psi_{2}-\mu\psi_{4}-\mu\psi_{5}} \leq \sigma\left(f,Tf\right)^{2\psi_{1}+\mu\psi_{4}+\mu\psi_{5}}$.

This implies that

$\sigma\left(Tf,T^{2}f\right) \leq \sigma\left(f,Tf\right)^{\frac{2\psi_{1}+\mu\psi_{4}+\mu\psi_{5}}{2-2\psi_{2}-\mu\psi_{4}-\mu\psi_{5}}}$

Note that $\psi_1+\psi_2+\psi_3+\mu\psi_4+\mu\psi_5<1$, then $\psi<1$. Therefore, (3.13) is satisfied. As a result, following Theorem 3.3, T holds the P Property.

Example 3.8

Let $\chi=[0,\infty]$ and $\sigma:\chi\times\chi\to R^{+}$ by $\sigma\left(f,\ g\right)=4^{(f-g)^{2}}$. Consider that (χ,σ,μ) is a $b-MMS$ with coefficient $\mu=2$. Let $T:\chi\to\chi$ by $Tf=\frac{f+5}{2}$, $f\in\chi$. We have

$\sigma\left(Tf,Tg\right)=4^{\left(\frac{f+5}{2},\ \frac{g+5}{2}\right)^{2}}$

$=\ 4^{\left(\frac{f-g}{2}\right)^{2}}$

$=\ \sigma(f,g)^{\frac{1}{4}}$

$$\sigma\left(Tf,Tg\right)\leq\left(\sigma\left(f,\ g\right)\right)^{\psi_{1}}.\left(\frac{\sigma\left(f,\ Tf\right)\sigma\left(g,\ Tg\right)}{1+\sigma\left(f,g\right)}\right)^{\psi_{2}}.\left(\frac{\sigma\left(\ f,Tg\right)\sigma\left(g,\ Tf\right)}{1+\sigma\left(f,g\right)}\right)^{\psi_{3}}.$$

$$\left(\frac{\sigma\left(f,Tf\right)\sigma\left(f,Tg\right)}{1+\sigma\left(f,g\right)}\right)^{\psi_{4}}.\left(\frac{\sigma\left(g,Tg\right)\sigma\left(g,Tf\right)}{1+\sigma\left(f,g\right)}\right)^{\psi_{5}}$$

where $\psi_1=\frac{1}{4}$, $\psi_2=\psi_3=\psi_4=\psi_5=0$. Obviously $\psi_1+\psi_2+\psi_3+\mu\psi_4+\mu\psi_5<1$. Consequently, the condition outlined in theorem 3.3 has been fulfilled, establishing the uniqueness of a fixed point for T in χ.

Application: By applying theorem 3.3, the first-order multiplicative initial value problem is solved.

$$\begin{cases} f^{*}\left(\upsilon\right)=\Omega(\upsilon,\ f\left(\upsilon\right)) \\ f\left(\upsilon_{0}\right)=f_{0} \end{cases} \tag{3.16}$$

where $\Omega:\left[\upsilon_{0}-\left(\frac{1}{\eta}\right)^{\tau-1},\upsilon_{0}+\left(\frac{1}{\eta}\right)^{\tau-1}\right]\times[f_{0}-\frac{\eta}{2},\ f_{0}+\frac{\eta}{2}]\to[1,\infty]$ is continuous function and $\eta>1,\ \tau>2,\ f_0,\ \upsilon_0$ are real constants.

Theorem 3.9

Consider Equation (3.16) and assume that
(i) Ω satisfies the Lipschitz condition, i.e.,

$$\left|\frac{\Omega(\upsilon,\ f(\upsilon))}{\Omega(\upsilon,\ g(\upsilon))}\right| \leq \psi^{|f(\upsilon)/g(\upsilon)|} \tag{3.17}$$

for all $(\upsilon, f),\ (\upsilon, g) \in \mathbb{R}^{+}$, where $\mathbb{R}^{+}$=$\left\{(\upsilon, f) : |\upsilon - \upsilon_0| \leq \left(\frac{1}{\eta}\right)^{\tau-1}, |f - f_0| \leq \frac{\eta}{2}\right\}$;
(ii) Ω is bounded on $\mathbb{R}^{+}$, i.e.,

$$|\Omega(\upsilon, f)| \leq \frac{\eta^{\tau}}{2} \tag{3.18}$$

Then Equation (3.16) has a unique solution on the interval Γ = $\left[\upsilon_0 - \left(\frac{1}{\eta}\right)^{\tau-1},\ \upsilon_0 + \left(\frac{1}{\eta}\right)^{\tau-1}\right]$.

Proof.

$X(\Gamma)$ represents the collection of all continuous functions defined on Γ. Let $\chi = \left\{f \in X(\Gamma) : |f(\upsilon) - f_0| \leq \frac{\eta}{2}\right\}$. Define a mapping $\sigma : \chi \times \chi \to \mathbb{R}^{+}$ by

$$\sigma(f, g) = sup\left|\frac{f(\upsilon)}{g(\upsilon)}\right|^{2} \tag{3.19}$$

Evidently, $(X(\Gamma), \sigma)$ is a $b - MMS$ with $\mu = 2$. As χ constitutes a closed subspace within $X(\Gamma)$, it follows that (χ, σ) is a b – multiplicative metric space.

Integrate Equation (3.16),

$$f(\upsilon) = f_0 . \int_{\upsilon_0}^{*\upsilon} (\Omega(\phi, f(\phi)))^{d\phi} \tag{3.20}$$

Solving equation Equation (3.16) is similar to identifying the fixed point of the mapping $T : \chi \to \chi$ represented using

$$Tf(\upsilon) = f_0 . \int_{\upsilon_0}^{*\upsilon} (\Omega(\phi, f(\phi)))^{d\phi} \tag{3.21}$$

If $\phi \in \Gamma$, then $|\phi - \upsilon_0| \leq \left(\frac{1}{\eta}\right)^{\tau-1}$ and $f \in \chi$ means $|f(\phi) - f_0| \leq \frac{\eta}{2}$, so $(\phi, f(\phi)) \in \mathbb{R}^{+}$. Since Ω is continuous on the positive real numbers, we can establish the existence of the integral Equation (3.21) and the well-defined nature of T for all $f \in \chi$. Further, we can determine that T serves as a self-mapping on χ. In fact, using Equation (3.18) and Equation (3.21), it is clear that

$\left|\frac{Tf(\upsilon)}{f_0}\right| = \left|\int_{\upsilon_0}^{*\upsilon} \Omega(\phi, f(\phi))^{d\phi}\right|$
$\leq \int_{\upsilon_0}^{*\upsilon} |\Omega(\phi, f(\phi))|^{d\phi}$
$\leq \frac{\eta}{2}^{\tau|\upsilon - \upsilon_0|}$
$\leq \frac{\eta^{\tau}}{2}\left(\frac{1}{\eta}\right)^{\tau-1}$
$\leq \left(\frac{\eta^{\tau}}{2}\right)\eta^{1-\tau}$

Next by using Equations (3.17), (3.19) and (3.21) we get

$$\left|\frac{Tf(\upsilon)}{Tg(\upsilon)}\right|^{2} = \left[\int_{\upsilon_0}^{*\upsilon} \left|\frac{\Omega(\phi, f(\phi))}{\Omega(\phi, g(\phi))}\right|^{d\phi}\right]^{2}$$

$$\leq \left[\int_{\upsilon_0}^{*\upsilon} \left[\psi^{|f(\phi/g(\phi))|}\right]^{d\phi}\right]^{2}$$

$$\leq \int_{\upsilon_0}^{*\upsilon} \left[\psi^{sup|f(\phi/g(\phi))|^{2}}\right]^{d\phi}$$

$$\leq \int_{v_0}^{*v} \left(\psi^{\sigma(f,g)}\right)^{d\phi}$$

$$\leq \left(\int_{v_0}^{*v} 1^{d\phi}\right)^{\psi^{\sigma(f,g)}}$$

$$\leq |v - v_0|^{\psi^{\sigma(f,g)}}$$

$$\leq \left[\left(\frac{1}{\eta}\right)^{\tau-1}\right]^{\psi^{\sigma(f,g)}}$$

$$\leq \left(\eta^{\psi(1-\tau)}\right)^{\sigma(f,g)}$$

$$\leq \sigma(f,g)^{\eta^{\psi(1-\tau)}}$$

$$\sigma(Tf, Tg) \leq \sigma(f,g)^{\eta^{\psi(1-\tau)}}$$

$$\leq (\sigma(f,g))^{\psi_1} \cdot \left(\frac{\sigma(f,Tf)\sigma(g,Tg)}{1+\sigma(f,g)}\right)^{\psi_2} \cdot \left(\frac{\sigma(f,Tg)\sigma(y,Tx)}{1+\sigma(f,g)}\right)^{\psi_3} \cdot \left(\frac{\sigma(f,Tf)\sigma(f,Tg)}{1+\sigma(f,g)}\right)^{\psi_4} \cdot \left(\frac{\sigma(g,Tg)\sigma(g,Tf)}{1+\sigma(f,g)}\right)^{\psi_5}$$

where $\psi_1 = \eta^{\psi(1-\tau)}, \psi_2 = \psi_3 = \psi_4 = \psi_5 = 0$. Because $\eta > 1$ and $\tau > 2$, it means that $\psi_1 + \psi_2 + \psi_3 + \mu\psi_4 + \mu\psi_5 < 1$. Due to the aforementioned statement, conditions of theorem 3.3 are satisfied and there exists a unique fixed point to Equation (3.16).

4 Conclusion

In this paper, we have introduced some fixed point theorems for generalized Lipschitz contractive mappings in b-multiplicative metric spaces. Notably, Picard's iteration explores the stability of iteration procedures across various spaces. Here, we established stability results in b-multiplicative metric spaces. These mappings are verified by both the P property and T-Stability of the Picard's iteration. Our results provide an application to nonlinear differential equations with initial value problems, which is a tool to verify the existence and uniqueness of nonlinear problems.

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