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RESEARCH ARTICLE

Difference Ordered $\Gamma-$ Semirings

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Abstract

Objectives: In this study, we generalize some of the difference ordered and weak uniquely difference ordered $\Gamma-$semirings results. **Methods:** To establish the results in $\Gamma-$semirings, we use conditions like commutativity, simple, multiplicative cancellative, additively idempotent on difference-ordered, and weak uniquely difference-ordered $\Gamma-$semirings. **Findings:** First, we give some examples of difference ordered $\Gamma-$semiring, and weak difference ordered $\Gamma-$semirings. Then generalize some of the results of semirings to $\Gamma-$semirings and discuss some of the properties of additive idempotent $\Gamma-$semifield. **Novelty:** We find that if R is a non-zeroic difference ordered $\Gamma-$semiring then $Z(R)$ is a strong ideal of R. Let R be a positive difference ordered Gel'fand $\Gamma-$semiring then every maximal non-unit of R is prime. Further, we find that if R is a simple difference ordered additively idempotent $\Gamma-$semiring and $x \in R$ which is not a unit then x is prime if and only if there exists a character $f_x : R \to B = \{0, 1\}$ satisfying $ker(f_x) = \{r \in R | r \leq x\}$. Moreover, if x and y are distinct prime elements of R Then f_x and f_y are also distinct. Finally, we consider some properties of additive idempotent $\Gamma-$semifield and then introduce the concept of weak uniquely difference-ordered $\Gamma-$semirings.

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1 Introduction

Semiring is a universal algebra with two binary operations called addition and multiplication, one of which is distributive over the other. The theory of rings and semigroup considerably impacted the development of the theory of semirings. In structure, semirings lie between semigroups and rings. A natural example of a semiring, which is not a ring, is the set of all non-negative integers under usual addition and multiplication. There are numerous different examples of a semiring; for instance, for a given integer n, the set $(a_{ij})_{n\times n}$ over semiring structures a semiring with the usual addition and multiplication of matrices. Rao[(1)], as a generalization of semirings and $\Gamma-$rings presented the concept of $\Gamma-$semirings in 1995. In 2021, Sharma and Ranote[(2)]

studied some properties on $\Gamma-$ semiring. In 2024, Sharma and Kumar [3,4] studied the concept of lattices and positive partially ordered $\Gamma-$ semiring and derived many exciting results in this field. Further, Koam Ali NA, Haider A., Ansari MA [5] introduced the concept of ordered Quasi (BI) $-\Gamma-$Ideals in Ordered $\Gamma-$Semirings and obtain the results regarding ordered quasi $\Gamma-$ideal, ordered prime quasi $\Gamma-$ideal, ordered semiprime quasi $\Gamma-$ideal, ordered maximal quasi $\Gamma-$ideal, ordered irreducible, and ordered strongly irreducible quasi $\Gamma-$ideal of an ordered $\Gamma-$semiring

As a continuation of the paper "Positive Partially Ordered $\Gamma-$ semiring" [4] we here, investigate some of the results of difference ordered $\Gamma-s$emiring in terms of uniquely difference ordered $\Gamma-$ semiring by using additively idempotent, Zeroid, Gel'fand $\Gamma-s$emiring, simple, strong ideals, maximal ideals, prime ideals, centreless, multiplicatively cancellative commutative $\Gamma-$ semiring and weak uniquely difference ordered $\Gamma-$ semirings.

2 Preliminaries

For the definitions of a $\Gamma-$semiring and their identity elements 0 and 1, strong identity, simple, set of units ($U(\Gamma R)$), multiplicative cancellative, $\Gamma-$semifield, centreless and additive idempotent, one can refer to [2,6,7].

Now we include some necessary preliminaries for the sake of completeness. A non-empty subset I of R is said to be the left (right) ideal of R if I is sub semigroup of $(R, +)$ and $x\alpha y \in I(y\alpha x \in I)$ for all $y \in I,\ x \in R$ and $\alpha \in \Gamma$. An ideal I of a $\Gamma-$ semiring R is called a strong ideal if for $x,\ y \in R,\ \ x + y \in I,$ implies that $x \in I$ and $y \in I$. A $\Gamma-$ semiring R is said to be semi-subtractive if for every $x,\ y \in R$ there exists $r \in R$ such that either $r + x = y$ or $r + y = x$. A $\Gamma-$ semiring R is said to satisfy cancellation law if for all $x,y,z \in R$ and $\alpha \in \Gamma$ we have that $x \neq 0, x\alpha y = x\alpha z$ and $y\alpha x = z\alpha x$ implies $y = z$. Let R be a $\Gamma-$ semiring and define $G_F(R) = \{r \in R|\, 1 + r \in U(\Gamma R)\}$. Then $G_F(R)$ is a Gel'fand $\Gamma-$ semiring if and only if $R = G_F(R)$.

Let R be a $\Gamma-$ semiring. Then R is called partially ordered $\Gamma-$ semiring if and only if there exists a partial order relation $\leq$ on R satisfying the following conditions if $x \leq y$ and $z \geq 0$ then $(i)\ x + z \leq y + z\ (ii)\ x\alpha z \leq y\alpha z\ (iii)\ z\alpha x \leq z\alpha y$, for all $x,\ y,\ z \in R$ and $\alpha \in \Gamma$. A partially ordered $\Gamma-$ semiring R is difference ordered if and only if $x \leq y$ in R, there exists an element z of R such that $x + z = y$. Difference ordered $\Gamma-$ semirings are positive and hence centreless. A partially ordered $\Gamma-$ semiring R is uniquely difference ordered if and only if $x \leq y$ in R, there exists a unique element z of R such that $x + z = y$.

We have the following theorems from [2,4,6].

Theorem 2.1. [2] Let R be a $\Gamma-$semiring. If R is multiplicatively cancellative additively idempotent commutative $\Gamma-$semiring. Then $[((x+y)\alpha)^{n-1}(x+y)] = (x\alpha)^{n-1}x + (y\alpha)^{n-1}y$ for all $x,y \in R$, and all positive integers n.

Theorem 2.2. [4] Let R be an additively idempotent $\Gamma-$ semiring with a strong identity. Then R is partially ordered by the relation $x \leq y$ if and only if $x + y = y$. Moreover, R is positive and a join semilattice with $x \vee y = x + y$. Further, if $x,\ y \in U(\Gamma R)$ then $x \geq y$ if and only if $w \leq t$, where w and t are inverses of x and y respectively.

Theorem 2.3. [6] Let R be a $\Gamma-$semiring. Then

(i) R is simple if and only if $x = x + x\alpha y$, for all $x,\ y \in R,\ \alpha \in \Gamma$

(ii) R is simple if and only if $x = x + y\alpha x$, for all $x,\ y \in R,\ \alpha \in \Gamma$

(iii) R is simple if and only if xαy = xαy+(xβz)αy , for all $x,\ y,\ z \in R,\ \alpha,\ \beta \in \Gamma$.

3 Methodology

Following [8], we will generalize some results for difference ordered $\Gamma-$semiring with conditions of commutativity, simple, additively idempotent, etc. Further, we define the middle function and prove some conditions. Finally, we discuss the concept of weak uniquely difference ordered $\Gamma-$ semiring.

4 Results and Discussion

Main R esults

We start this section by giving some examples and then generalize some of the results of regarding difference ordered and weak uniquely difference ordered $\Gamma-$ semiring.

Example 4.1 . The $\Gamma-$ semiring $\mathbb{N}$, of non-negative integers is uniquely difference ordered in its usual ordering.

Example 4.2 . The order on an additively idempotent $\Gamma-$ semiring defined in Theorem 2.2 is the difference order. Further, if $x + y = y$ then in the difference order $x \leq y$. Conversely, let $x \leq y$ in a difference order. Then there exists an element z of R such that $x + z = y$. Now, $y = x + z + z = y + z$. This implies that $x + y = x + y + z = y + y = y$.

Example 4.3 . Since every uniquely difference –ordered $\Gamma-$ semiring is a weak uniquely difference –ordered $\Gamma-$ semiring the set $B = \{0,1\}$ is totally ordered, weak uniquely difference ordered $\Gamma-$ semiring but not unique difference ordered $\Gamma-$ semiring.

Now we have the following results, which are straightforward.

Theorem 4.4. Let R be a $\Gamma-$ semiring. Then every difference ordered $\Gamma-$ semiring is positive. Further, if R is both simple and difference ordered then $x\alpha y \leq y$ and $y\alpha x \leq y$ for all $x, y \in R, \alpha \in \Gamma$.

Theorem 4.5. Let R be a difference ordered $\Gamma-$ semiring. Then R is uniquely difference ordered if and only if it is cancellative.

Theorem 4.6. The only simple difference ordered division $\Gamma-$ semiring is $B = \{0,1\}$ if it has a strong identity.

Definition 4.7. A difference ordered $\Gamma-$ semiring is called totally ordered if it is semi-subtractive.

Theorem 4.8. Let R be an additively idempotent partially ordered $\Gamma-$semiring satisfying $0 < 1$ then $S = \{r \in R \,|\, 0 \leq r \leq 1\}$ is a sub-Γ-semiring of R.

Theorem 4.9 . Let R be a $\Gamma-$semiring. Then R is the difference ordered if and only if $x, y, z \in R$ satisfying $x = x + y + z$ then $x = x + y$.

Proof. Let R be the difference ordered and $y, z \in R$. If $x = x + y + z$ then $x \leq x + y \leq x$. Thus, $x = x + y$. Conversely, define the relation $\leq$ on R by $r \leq r'$ if and only if there exists an element $r'' \in R$ satisfying $r + r'' = r'$. Then clearly $x \leq x$ for all $x \in R$ and $x \leq y, y \leq z$ implies that $x \leq z$ for all $x, y, z \in R$. If $x, y \in R$ such that there exist $z, t \in R$ satisfying $x + z = y$ and $y + t = x$ then $x + z + t = x$. So, $x + z = x$. This implies that $y = x + z = x$, Thus, $\leq$ is a partial order on R. This implies that R is a partially ordered $\Gamma-$semiring. Hence, R is difference ordered.

The following definition is analogous to the definition in [8].

Definition 4.10. The Zeroid of a $\Gamma-$semiring R is $Z(R) = \{r \in R \,|\, r+a = a \text{ for some } a \in R\}$. Thus, if $Z(R) = R$ then R is zeroic, otherwise non-zeroic. If R has an infinite element then it is surely zeroic.

Now one can easily verify the following result by using theorem 4.9.

Theorem 4.11. Let R be a non-zeroic difference ordered $\Gamma-$semiring. Then $Z(R)$ is a strong ideal of R.

Theorem 4.12 . Let R be a difference ordered $\Gamma-$semiring. Then an ideal I of R is strong if and only if $x \leq y$ and $y \in I$ implies that $x \in I$.

Proof. Let I be the strong ideal of a $\Gamma-$semiring R. If $x \leq y$ and $y \in I$ then there exists an element z such that $x + z \in I$. So, by assumption $x \in I$. Conversely, assume that given conditions hold. If $x, y \in R$ satisfying $x + y \in I$ then $x \leq x + y \in I$ and so $x \in I$. Similarly, $y \in I$.

The following results are straightforward.

Theorem 4.13 . Let R be a difference ordered $\Gamma-$semiring and S is a subset of R then $(0 : S)$ is a strong ideal of R.

Theorem 4.14. Let R be a $\Gamma-$semiring with a strong identity. Then R is a Gel'fand $\Gamma-$semiring if and only if $r + x \in U(\Gamma R)$ for all $r \in R, x \in U(\Gamma R)$.

Theorem 4.15. Let R be a difference ordered Gel'fand $\Gamma-$semiring with strong identity and $t \geq z \in U(\Gamma R)$ in R. Then $t \in U(\Gamma R)$.

Definition 4.16. An element x of a partially ordered $\Gamma-$semiring R is prime if and only if x is not a unit and $y\alpha z \leq x$ in $R\alpha \in \Gamma$, implies that either $y \leq x$ or $z \leq x$.

Definition 4.17. An element x of a partially ordered $\Gamma-semiring$ R is semiprime if and only if x is not a unit and $y\alpha y \leq x$ in R implies that either $y \leq x$. If R is multiplicatively $\Gamma-$idempotent then clearly non-unit of R is semiprime.

Definition 4.18. A maximal non-unit of a partially ordered $\Gamma-$ semiring R is an element x of $R \setminus U(\Gamma R)$ satisfying the condition that $\{r \in R \,|\, r > x\}$ is a non-empty subset of $U(\Gamma R)$.

Theorem 4.19. Let R be a positive difference ordered Gel'fand $\Gamma-$semiring then every maximal non-unit of R is prime.

Proof. Let x be a maximal non-unit of R and $y, z \in R, \alpha \in \Gamma$ satisfying $y \not\leq x, z \not\leq x$ and $y\alpha z \leq x$. Then $x \leq x + y, \ y \leq x + y$. So $x < x + y$. Thus, $x + y \in U(\Gamma R)$. Similarly, $x + z \in U(\Gamma R)$ and so $t = (x+y)\alpha(x+z) \in U(\Gamma R)$. But $t = (x+y)\alpha x + x\alpha z + y\alpha z \leq (x+y)\alpha x + x\alpha z + x\alpha 1 = (x+y)\alpha x + x\alpha(z+1) \leq (x + y + 1)\alpha z\alpha(z + 1)$, where $x + y + 1$ and $z + 1$ are units of R since R is a Gel' fand $\Gamma-$ semiring. Therefore, as R is positive, so $u\alpha(t\alpha w) \leq x$, where u is the inverse of $(x + y + 1)$ and v is inverse of $(z + 1)$. Thus, by theorem 4.15, $x \in U(\Gamma R)$. This is a contradiction. Hence, x must be prime.

Theorem 4.20. Let R be a simple difference ordered $\Gamma-$semiring and A be a nonempty subset of R satisfying the condition that if $x, x' \in A$ then there exists an element $x'' \in A, \alpha \in \Gamma$ with $x'' \leq x\alpha x'$. If $S = \{r \in R \,|\, x \not\leq r \text{ for all } x \in A\}$. Then every additively idempotent maximal element of S is prime.

Proof. Let y be an additively idempotent maximal element of S. Let $r, r' \in R$ be such that $r, r' \not\leq y$. Then $r + y, r' + y > y$ So, by the choice of y, clearly $r + y, r' + y \notin S$. So there exist elements x, x' and x'' of A satisfying $x \leq r + y, x' \leq r' + y$

and for $\alpha \in \Gamma$, $x'' \leq x\alpha x'$. Thus, $x'' \leq (r+y)\alpha(r'+y) = r\alpha r' + y\alpha r' + r\alpha y + y\alpha y$. Therefore, by theorem 2.3, $y\alpha r' + r\alpha y + y\alpha y \leq y + y + y = y$ and so $x'' \leq r\alpha r' + y$. If $r\alpha r' \leq y$ then $x'' \leq y + y = y$, contradicting the fact that $y \in S$. Hence, $r\alpha r' \not\leq y$, proving that y is prime.

Theorem 4.21. Let R be a simple difference ordered additively idempotent $\Gamma-$semiring and $x \in R$ which is not a unit. Then x is prime if and only if there exists a character f_x on R satisfying $ker(f_x) = \{r \in R \,|\, r \leq x\}$. Moreover, if x and y are distinct prime elements of R Then f_x and f_y are also distinct.

Proof. Let x be prime. Define $f_x : R \to B = \{0,1\}$ by $f_x(r) = 0$ if and only if $r \leq x$. Then $f_x(0) = 0$, since R is difference ordered and $f_x(1) = 1$, since $1 \not\leq x$. Moreover, if $r,\ r' \in R$ then $f_x(r + r') = 0$ if and only if $r + r' \leq x$ if and only if $r \leq x$ and $r' \leq x$ if and only if $f_x(r) = 0 = f_x(r')$. Therefore, $f_x(r) + f_x(r') = f_x(r + r')$. Similarly, for $\alpha \in R$ and by primness $f_x(r\alpha r') = 0$ if and only if $r\alpha r' \leq x$ if and only if either $r \leq x$ or $r' \leq x$ if and only if $f_x(r)\alpha\, f_x(r') = 0$. Thus, $f_x(r\alpha r') = f_x(r)\alpha\, f_x(r')$. Conversely, assume that there exists a morphism $f_x : R \to B$ satisfying $ker\ f_x) = \{r \in R \,|\, r \leq x\}$. If $r\alpha r' \in R$ satisfy $r\alpha r' \leq x$ then for some $\alpha \in \Gamma$, $f_x(r)\alpha f_x(r') = f_x(r\alpha r') = 0$. Therefore, either $f_x(r) = 0$ or $f_x(r') = 0$. Hence, either $r \leq x$ or $r' \leq x$, proving that x is prime. Further, if x and y are prime elements in R satisfying $f_x = f_y$ then $f_y(x) = 0$. This implies that $x \leq y$. Similarly, $y \leq x$. Hence, $x = y$.

Definition 4.22.[3] Let R be a partially ordered $\Gamma-$semiring. Then a function $h : R \to R$ is called a middle function if and only if the following conditions are satisfied. (i) If $x \leq y$ in R then $h(x) \leq h(y)$ (ii) If $x \in R$ then $h(h(x)) = h(x) \geq x$ (iii) If $x,\ y \in R$ then $h(x\alpha y) \geq h(x)\alpha h(y)$, $\alpha \in \Gamma$.

Note that if h is a middle function then $x \leq h(y)$ if and only if $h(x) \leq h(y)$.

Theorem 4.23. Let R be a partially ordered $\Gamma-$ semiring and $h : R \to R$ be a middle function on R. If R is positive then $h(x\alpha y) = h(h(x)\alpha y) = h(h(x)\alpha h(y)) = h(x\alpha h(y))$ for all $x,\ y \in R$, $\alpha \in \Gamma$.

Proof. Let R be positive and $x,\ y \in R$ then $h(x) \geq x$ and $h(y) \geq y$. So, for any $\alpha \in \Gamma$, $h(x)\alpha h(y) \geq h(x)\alpha y \geq x\alpha y$. Therefore, $h(h(x)\alpha y) \geq h(x\alpha y) = h(h(x\alpha y)) \geq h(h(x)\alpha h(y) \geq h(h(x)\alpha y)$, Hence, $h(x\alpha y) = h(h(x)\alpha y) = h(h(x)\alpha h(y))$. Similarly, $h(x\alpha y) = h(x\alpha h(y)) = h(h(x)\alpha h(y))$.

Theorem 4.24. Let R be a partially ordered $\Gamma-$ semiring and $h : R \to R$ be a middle function on R. If R is additively idempotent then $h(h(x) + h(y)) = h(x + y)$ for all $x,\ y \in R$.

Theorem 4.25 . Let R be commutative additively idempotent difference ordered $\Gamma-$ semifield then every pair of elements of R has infimum in R.

Proof. Let $x,\ y \in R$. If $x = 0$ or $y = 0$ then 0 is the infimum of $\{x,\ y\}$, since R is difference ordered. So, we can assume that $x,\ y \in R\setminus\{0\}$. Since R is additively idempotent and hence centreless, therefore, $x + y \neq 0$. Set $z = (x\alpha y)\beta u, \alpha,\ \beta \in \Gamma,\ u$ is a strong inverse of $(x + y)$. That is, $u\alpha(x + y) = (x + y)\alpha u = 1$, for all $\alpha \in \Gamma$. Now, $(z+x)\alpha(x+y) = [(x\alpha y)\beta u+x]\alpha(x+y) = (x\alpha y)\beta u\alpha(x+y)+x\alpha(x+y) = (x\alpha y)\beta 1 + x\alpha x + x\alpha y = x\alpha y + x\alpha x + x\alpha y = x\alpha y + x\alpha x = x\alpha(x + y)$. This implies that $z + x = x$. Thus, $z \leq x$. Similarly, $z \leq y$. Further, suppose that $t \leq x,\ t \leq y$ then $t + x = x$ and $t + y = y$. So, $t\alpha(x + y) + x\alpha y = t\alpha x + t\alpha y + (t + x)\alpha(t + y) = t\alpha x + t\alpha y + t\alpha t + t\alpha x + t\alpha y + x\alpha y = t\alpha t + t\alpha x + t\alpha y + x\alpha y = (t+x)\alpha(t+y) = x\alpha y$ and hence, $t\alpha(x + y) \leq x\alpha y$ implies that $t\alpha(x + y)\beta u \leq (x\alpha y)\beta u$ implies that $t\alpha 1 \leq (x\alpha y)\beta u$. Thus, $t \leq z$. Hence, z is the infimum of $\{x,\ y\}$ in R.

We now, consider some of the properties of additive idempotent $\Gamma-$ semifield.

Theorem 4.26. A sufficient condition for a $\Gamma-$ semiring R to be centreless is that there exists an element $z \in R$ satisfying $z + 1 = z$.

Theorem 4.27. An additively idempotent $\Gamma-$ semiring R is centreless.

Theorem 4.28. Let $n > 1$ be a positive integer. Let R be an additively idempotent $\Gamma-$semifield and $y_1, y_2, y_3, \ldots y_n$ be elements of R with $y_n \neq 0$ and $\alpha_i \in \Gamma, 1 \leq i \leq n$. Let $f : R \to R$ be the function defined by $f(x) = \sum_{i=1}^{n} y_i\alpha_i[(x\alpha)^{i-1}x]$. Then a sufficient condition for $z \geq t$ in R is that $f(z) \geq f(t)$.

Proof. Let $f(z) \geq f(t)$. If $z = 0$ or $t = 0$ then the result is obvious. So let $z \neq 0$ and $t \neq 0$. Therefore $z + t \neq 0$. Since R is Γ-semifield so for $\alpha \in \Gamma$, $1 \leq i \leq n((z + t)\alpha)^{i-1}\ (z + t) \neq 0$, and thus $f(z + t) \neq 0$. Let $g : R \to R$ be the function defined by $g(x) = \sum_{i=1}^{n} y_i\alpha_i[(x\alpha)^{i-2}x]$ then $f(x) = \sum_{i=1}^{n} y_i\alpha_i[(x\alpha)^{i-1}x] = x\alpha\sum_{i=1}^{n} y_i\alpha_i[(x\alpha)^{i-2}x] = x\alpha g(x)$ for all $x \in R$, $\alpha \in \Gamma$. Now, by theorem 2.1, if $x,\ y \in R$ then

$$g(x + y) = \sum_{i=1}^{n} y_i\alpha_i[((x+y)\alpha)^{i-2}\ (x+y)]$$

$$= \sum_{i=1}^{n} y_i\alpha_i[(x\alpha)^{i-2}x + [(y\alpha)^{i-2}y]$$

$$= \sum_{i=1}^{n} y_i \alpha_i [(x\alpha)^{i-2} x] + \sum_{i=1}^{n} y_i \alpha_i [(y\alpha)^{i-2} y]$$

$$= g(x) + g(y).$$

Further, if $x,\ y\ \in\ R$, and j is any positive integer, then

$$[(x+y)\,\alpha)^{j-1}]\ (x+y) = \sum_{i=0}^{j} [(x\alpha)^{i-1} x]\alpha[(y\alpha)^{j-i-1} y]$$

$$=\ [(y\alpha)^{j-1} y] + x\alpha \sum_{i=0}^{j-i} [(x\alpha)^{i-1} x]\alpha[(y\alpha)^{j-i-1} y]$$

$$= [(y\alpha)^{j-1} y] + x\alpha\,((x+y)\,\alpha)^{j-2}\,(x+y)].$$

Therefore,

$$f\,(x\ +\ y) = (x+y)\,\alpha g\,(x\ +\ y)$$

$$= x\alpha g\,(x\ +\ y) + y\alpha g\,(x\ +\ y)$$

$$= x\alpha g\,(x\ +\ y) + y\alpha g(x) + y\alpha g(y)$$

$$= x\alpha g\,(x\ +\ y) + y\alpha g(y)$$

$$= x\alpha g\,(x\ +\ y) + f(y)$$

This implies that $f\,(z\ +\ t)=\ z\alpha g\,(z\ +t) + f\,(t),\ z,\ t\ \in R$. So by assumption, $z\alpha g(z\ +\ t)\ =\ z\alpha g(z)\ +\ z\alpha g(t)\ =\ f(z)\ +\ z\alpha g(t)\ \geq\ f(z)\ \geq\ f(t)$. This implies that $z\alpha g(z\ +\ t)\ +\ f(t)\ =\ z\alpha g(z\ +\ t)$, since $x \leq y$ implies that $x+y=y$. Thus, $(z\ +\ t)\,\alpha g\,(z\ +\ t) =\ f\,(z\ +\ t) =\ z\alpha g\,(z\ +\ t) +\ f(t)\ =\ z\alpha g(z\ +\ t)$. Since $g(z\ +\ t) \neq\ 0$ and R is a $\Gamma-$ semifield, so $z\ +\ t\ =\ z$ and so $z\ \geq$ t.

Theorem 4.29. Let R be an additively idempotent multiplicatively cancellative commutative $\Gamma-$ semiring. Let n be a positive integer and $x\ \in\ R$. Then there is at most one element $y\ \in\ R$ such that $(y\alpha)^{n-1}\ y\ =\ x$.

Proof. Put $n\ =\ 1$, the result is obvious. Let $n\ >\ 1$. If $(z\alpha)^{n-1} z = 0$ for $z \neq\ 0$ then we have $(z\alpha)^{n-1}\ z\ =\ 0(z\alpha)^{n-2}\ z$. So, by multiplicative cancellation, $z\ =\ 0$which is a contradiction and so the result is true for $x\ =\ 0$. Thus, we can assume that $x \neq\ 0$. Let $y,\ z\ \in\ R$ satisfying $(y\alpha)^{n-1}\ y\ =\ (z\alpha)^{n-1}\ z\ =\ x$. Then $x \neq\ 0 \neq\ z$. Moreover, $(y\ +\ z)\alpha)^{\ n-1}\ (x\ +\ y) = \sum_{i=0}^{j} [(y\alpha)^{i-1} y]\alpha[(z\alpha)^{n-i-1} z]$ is obvious. So, we have

$$(y\alpha)^{n-1}\, y\ =\ x$$

$$=\ x\ +\ x$$

$$= (y\alpha)^{n-1}\, y + (z\alpha)^{n-1}\, z$$

$$= ((y + z)\alpha))^{n-1}\, (y + z)$$

$$\geq y\alpha(z\alpha)^{\,n-1}\,)z.$$

This implies that $(y\alpha)^{n-1}\, y = (y\alpha)^{n-1}\, y + y\alpha(z\alpha)^{\,n-1}\,)z$.

Thus,

$$((y + z)\alpha))^{n-1}\, (y + z) = (y\alpha)^{\,n-1}\, y$$

$$= (y\alpha)^{n-1}\, y + y\alpha\,(z\alpha)^{n-2}\,)z$$

$$= y\alpha\left((y\alpha)^{n-2}\, y + (z\alpha)^{n-2}\, z\right)$$

$$= y\alpha((y+z)\alpha)^{n-2}\, (y+z).$$

Therefore, $((y+z)\,\alpha((y+z)\alpha))^{n-2}\, (y+z) = y\alpha((y + z)\alpha))^{n-2}\, (y + z)$, thus, multiplication cancellation implies that $y + z = y$. Similarly $y + z = z$ and hence $y = z$.

In 1998, W.U. Fuming[9], introduced the concept of Weak uniquely difference-ordered semirings. He suggested the notion of Weak uniquely difference ordered (WUDO) semiring, namely a difference ordered semiring satisfying the condition that the set $\{z \in R \,|\, x + z = y\}$ is either empty or a singleton set whenever $x \neq y$.

Now we define the concept of weak uniquely difference ordered $\Gamma-$ semiring, which is analogous to the concept of weak uniquely difference-ordered semirings.

Definition 4.30. A difference ordered $\Gamma-$ semiring R is called a weak uniquely difference ordered $\Gamma-$ semiring if and only if $x \leq y$ implies that there is a unique element z of R satisfying $x + z = y$. We denote this unique element z by $y \ominus x$. We also set $x \ominus x = 0$, for all $x \in R$.

OR

If $x \leq y$ are elements of a weak uniquely difference ordered $\Gamma-$ semiring then

$$y \ominus x = \begin{cases} 1 & if\; y = x \\ the\; unique\; element\; z such\; that\; x + z = y & if\; otherwise. \end{cases}$$

Theorem 4.31. Let x be an element of a weak uniquely difference ordered $\Gamma-$ semiring R and let $y \neq z$ be elements of R satisfying $x + y = x + z$ then $x = x + y$.

Theorem 4.32. Let R be a weak uniquely difference ordered $\Gamma-$ semiring. Then

(i) If $x \ominus 0 = x$ for all $x \in R$.

(ii) If $y = (y \ominus x) + x$ for all $x \leq y$ in R.

(iii) $z\alpha(y \ominus x) = z\alpha y \ominus z\alpha x$ and $(y \ominus x)\alpha z = y\alpha z \ominus x\alpha z$ for $x \leq y$ and $z \in R$, $\alpha \in \Gamma$.

(iv) $(y + z) \ominus (x + z) = y \ominus x$ for $x \leq y$ and $z \in R$.

(v) $x \leq y \leq z\; in\; R$ then $z \ominus y \leq z \ominus x$.

(vi) If $x \leq z$ and $y \leq z$ then $z \ominus x \geq y$ and $z \ominus y \geq x$ implies that $(z \ominus y) \ominus x = (z \ominus x) \ominus y$.

(vii) $x + y = x + z > x$ then $y = z$.

(viii) If $y \geq x$ and $z > 0$ then $(y + z) \ominus x = (y \ominus x) + z$ if and only if $y + z > x$.

Proof. (i) and (ii) follows from the definition.

(iii) Since R is difference ordered so $z \geq 0$. If $x = y$ then $y \ominus x = 0$ and so $z\alpha\,(y \ominus x) = 0 = z\alpha y \ominus z\alpha x$ for some $\alpha \in \Gamma$. Otherwise, $z\alpha x + z\alpha(y \ominus x) = z\alpha(x + (y \ominus x)) = z\alpha y$ and so by uniqueness $z\alpha(y \ominus x) = z\alpha y \ominus z\alpha x$. Similarly, $(y \ominus x)\alpha z = y\alpha z \ominus x\alpha z$.

Results from (iv) to (viii) are analogous to the results in[8].

5 Conclusion

This study is centered on the concept of difference-ordered $\Gamma-$ semiring by using some of the conditions like commutativity, simple, multiplicative cancellative, additively idempotent, etc. We find that in positive difference ordered Gel'fand semiring, every maximal non-unit of R is prime. Further, we define a middle function and give its characteristics. Finally, we find some properties of additive idempotent $\Gamma-$ semifield, and then the concept of weak uniquely difference ordered $\Gamma-$ semiring is discussed. So, the concept of the structure of difference ordered and weak uniquely difference ordered $\Gamma-$ semiring is helpful to explore the ideas in this field by different researchers.

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