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Application of Polynomial Collocation Method Based on Successive Integration Technique for Solving Delay Differential Equation in Parkinson's Disease

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Abstract

Background/Objective: Parkinson's disease (PD) is a neurological disorder that is often prevalent in elderly people. This is induced by the reduction or loss of dopamine secretion. The main objective of this work is to apply the polynomial collocation method using successive integration technique for solving delay differential equations (DDEs) arising in PD models. **Methods:** The polynomial collocation method based on successive integration techniques is proposed to obtain approximate solutions of the PD models. In this study, the most widely used classical orthogonal polynomials, namely, the Bernoulli polynomials, the Chebyshev polynomials, the Hermite polynomials, and the Fibonacci polynomials are considered. **Findings:** Numerical examples of two PD models have been considered to demonstrate the efficiency of the proposed method. Numerical simulations of the proposed method are well comparable to the simulation by step method using Picard approximation. **Novelty:** The numerical simulation demonstrates the reliability and efficiency of the proposed polynomial collocation method. The proposed method is very effective, simple, and suitable for solving the nonlinear DDEs model of PD and similar real-world problems exist in different fields of science and engineering.

Keywords: Polynomial collocation method; Successive Integration Technique; Delay differential equation; Parkinson's disease; Simulation

1 Introduction

Delay Differential Equations (DDEs) are a type of differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. The terms involving previous times are called delay terms. DDEs play an important role in real-life problems such as in biology, chemistry, mechanical engineering, physics, control theory, fluid mechanics and many other physical processes are modelled in terms of DDEs. Some notable applications of DDEs are in electrochemical biosensor [1], cancer cells growth [2] and population model [3].

A familiar neurodegenerative disease is Parkinson's Disease (PD). The primary cause of PD is due to insufficient secretion of the chemical dopamine in the nerve cells of the brain. The elderly people of age above sixty are mostly affected by this disease. The patients may experience different symptoms such as trembling of the hands, stiffness and slow movement. They may also experience the neurological systems such as psychosis and depression. Medications are given to reduce the symptoms as well as to increase the production of dopamine. Mathematical models are made in understanding the dynamics of neurodegeneration of PD patients. Lainscsek et al. [4] presented a non-linear DDE model of finger tapping movements of PD patients. Elfouly and Sohaly [5] described the Van der Pol delay model in PD. Salma and Asma Badrah [6] built a mathematical model using DDEs to study the effect of immunotherapies in delaying the progression of PD.

It is too complicated to obtain the exact solution of a DDE that arises in real world applications. In such cases, numerical methods are essential. Many authors have developed various numerical, analytical methods for solving DDEs. Agiza, et al. [7] solved nonlinear DDE in PD by using step method. Jafari, et al. [8] applied numerical method involving Legendre pseudo spectral method to solve pantograph DDEs. Alshehri, et al. [9] derived Reproducing Hilbert space method for solving DDEs. Vinci Shaalini and Emimal Kanaga Pushpam [10] presented the analysis of fourth order Composite Runge Kutta method and a new one-step technique for solving stiff DDEs. Dhinesh Kumar and Emimal Kanaga Pushpam [11] proposed higher order derivative Runge Kutta method for solving DDEs.

Khan and Ali [12] applied Legendre polynomial-based collocation method for solving system of DDEs. Yuzbasi, et al. [13] presented collocation method based on the Pell-Lucas polynomials for solving DDEs. Kumbinarasaiah Srinivasa and Ravikiran Ashok Mundewadi [14] proposed Laguerre wavelet approach for solving nonlinear variable DDEs. Methi, et al. [15] applied a hybrid technique of the differential transform and the Bell polynomial for solving nonlinear DDEs. The above-mentioned collocation methods using different polynomials are based on operational matrices. In this study, a new approach on polynomial collocation methods which is based on successive integration technique has been proposed to solve DDEs in PD models.

This paper is organized as follows: In Section 2, the basic definitions of different polynomials are given. In Section 3, the description of the proposed polynomial collocation method for solving DDEs is provided. In Section 4, numerical simulations of DDE model in Parkinson's disease are provided.

2 Basic Definition of Polynomial

In this study, the most widely used classical orthogonal polynomials are considered. Namely, the Bernoulli polynomials, the Chebyshev polynomials, the Hermite polynomials and the Fibonacci polynomials.

2.1 Bernoulli Polynomial

The Bernoulli polynomial is named after Jacob Bernoulli which combines the Bernoulli numbers and binomial coefficients. The generating function for the Bernoulli polynomial of order n is defined by

$$\sum_{n=0}^{\infty} B_n(t)\frac{x^n}{n!} = \frac{xe^{xt}}{e^x - 1} \tag{1}$$

The explicit formula for Bernoulli polynomial is:

$$B_n(t) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}\left(t^k\right) \tag{2}$$

for n $\geq$ 0, where B_k are the Bernoulli numbers.

$B_0(t)$ can be obtained from Equation (1) and the remaining terms are determined by using the recursion relation. Thus, few terms of the Bernoulli polynomials as:

$$B_0(t) = 1$$

$$B_1(t) = t - \frac{1}{2}$$

$$B_2(t) = t^2 - t + \frac{1}{6}$$

$$B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t$$

$$B_4(t) = t^4 - 2t^3 + t^2 - \frac{1}{30}$$

2.2 Chebyshev Polynomial

The Chebyshev polynomial related to cosine functions on the interval $[-1,\ 1]$ of order n is defined as

$$T_n(cost) = cos(nt) \tag{3}$$

The recursion relation of Chebyshev polynomial is:

$$T_{n+1}(t) = 2t\ T_n(t) - T_{n-1}(t) \tag{4}$$

$T_0(t)$ and $T_1(t)$ can be obtained from Equation (3). Then the remaining terms are determined by from Equation (4). Thus, the following sequence of polynomials:

$$T_0(t) = 1$$

$$T_1(t) = t$$

$$T_2(t) = 2t^2 - 1$$

$$T_3(t) = 4t^3 - 3t$$

$$T_4(t) = 8t^4 - 8t^2 + 1$$

2.3 Hermite Polynomial

The Hermite polynomial $H_n(t)$ of order n is defined on the interval $(-\infty,\ \infty)$. There are different ways to define for Hermite polynomial, one of them is the so-called Rodrigues' formula

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2} \tag{5}$$

From Equation (5), the recurrence relation for the polynomials can be derived as

$$H_n(t) = 2t\ H_{n-1}(t) - H'_{n-1}(t) \tag{6}$$

$H_0(t)$ can be obtained from Equation (5) and the remaining terms are determined by using the recursion relation Equation (6). Thus, the following sequence of polynomials:

$$H_0(t) = 1$$

$$H_1(t) = 2t$$

$$H_2(t) = 4t^2 - 2$$

$$H_3(t) = 8t^3 - 12t$$

$$H_4(t) = 16t^4 - 48t^2 + 12$$

and so on. The n^{th} order Hermite polynomial $H_n(t)$ has a leading coefficient 2^n.

2.4 Fibonacci Polynomial

In Mathematics, the Fibonacci polynomials are a polynomial sequence which can be considered of Fibonacci numbers. The Fibonacci polynomials are defined by a recurrence relation

$$F_n(t) = \begin{cases} 0, & if\ n = 0 \\ 1, & if\ n = 1 \\ tF_{n-1}(t) + F_{n-2}(t), & if\ n \geq 2. \end{cases}$$

The first few Fibonacci polynomials are:

$$F_0(t) = 0$$

$$F_1(t) = 1$$

$$F_2(t) = t$$

$$F_3(t) = t^2 + 1$$

$$F_4(t) = t^3 + 2t$$

3 Description of the Proposed Method

Consider the n^{th} order DDE of the form

$$y^{(n)}(t) = f\left(t, y(t), y(t-\tau), y'(t), y'(t-\tau), \right), \left(\ldots, y^{(n-1)}(t), y^{(n-1)}(t-\tau)\right), \quad t > t_0 \tag{7}$$

with initial conditions

$$y^{(i)}(t_0) = \varnothing(t), \ i = 1, 2, 3, \ldots \ t \leq t_0 \tag{8}$$

Here $\varnothing(t)$ is the initial function and τ is the delay term.

Let P(t) represent any orthogonal polynomials. For the proposed method, assume that

$$y^{(n)}(t) \approx B^T P(t)^T = \sum_{j=0}^{N} c_j P_j(t) \tag{9}$$

where N being any positive integer,

$$B^T = (c_0, c_1, \ldots c_N)$$

$$P(t) = (P_0(t), P_1(t) \ldots P_N(t))$$

To determine the polynomial coefficients $c_j^{'}s$, integrate the Equation (9) with respect to t from t_0 to t,

$$\left.\begin{array}{c} y^{(n-1)}(t) = y(t_0) + \int_{t_0}^{t} B^T P_j(t)\,dt \\ y^{(n-2)}(t) = y(t_0) + y'(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} B^T P_j(t)\,dt \\ y'(t) = \sum_{i=0}^{N-1} y^{(i)}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t}\cdots\int_{t_0}^{t} B^T P_j(t)\,dt \\ y(t) = \sum_{i=0}^{N} y^{(i)}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t}\cdots\int_{t_0}^{t} B^T P_j(t)\,dt \end{array}\right\} \tag{10}$$

Now, for delay terms

$$\left.\begin{array}{l} y^{(n-1)}(t-\tau) = y(t_0) + \int_{t_0}^{t} B^T P_j(t-\tau)dt \\ y^{(n-2)}(t-\tau) = y(t_0) + y'(t_0) + \int_{t_0}^{t}\int_{t_0}^{t} B^T P_j(t-\tau)dt \\ y'(t-\tau) = \sum_{i=0}^{N-1} y^{(i)}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t}\cdots\int_{t_0}^{t} B^T P_j(t-\tau)dt \\ y(t-\tau) = \sum_{i=0}^{N} y^{(i)}(t_0) + \int_{t_0}^{t}\int_{t_0}^{t}\cdots\int_{t_0}^{t} B^T P_j(t-\tau)dt \end{array}\right\} \tag{11}$$

Then substitute Equations (10) and (11) in Equation (7) and use the collocating points $t_i = \frac{i}{N}$, where i = 0, 1…N. This yields a system of linear or nonlinear equations subject to the linear and nonlinear terms in Equation (7). On solving this system of equations, the respective Polynomial coefficients c_j's are obtained from which the solution of the DDE Equation (7) can be obtained.

4 Numerical Simulations

In this section, two DDE models in Parkinson's disease have solved by using Hermite Collocation Method (HCM), Chebyshev Collocation Method (CCM), Bernoulli Collocation Method (BCM) and Fibonacci Collocation Method (FCM) based on successive integration technique. The numerical results are compared with the results available in the literature.

Example 1 – PD Model 1 [7]

$y^{'}(t) = 0.1y(t-2) + 0.2y(t-5) + 0.3y(t-2)y(t-3)$

with history function as

$y(t) = 0.5, t > 0$ and $-3 < t < 0$.

For this example, the numerical results are obtained by using the proposed method based on the above mentioned four polynomials. The numerical simulations by the proposed methods are compared with the simulation by step method using Picard approximation [7] These have been shown in Figures 1 and 2.

Example 2 - PD Model 2 [7]

$y^{'}(t) = 0.1y(t-2) + 0.2y(t-3) + 0.3y(t-2)\,y(t-3) - (y(t))^2$

with history function as

$y(t) = 0.5, t > 0$ and $-3 < t < 0$.

For this example, the numerical results are obtained by using the proposed method based on Hermite polynomial. The numerical simulation by the proposed HCM is compared with the simulation by step method using Picard approximation. [7] These have been shown in Figures 3 and 4.

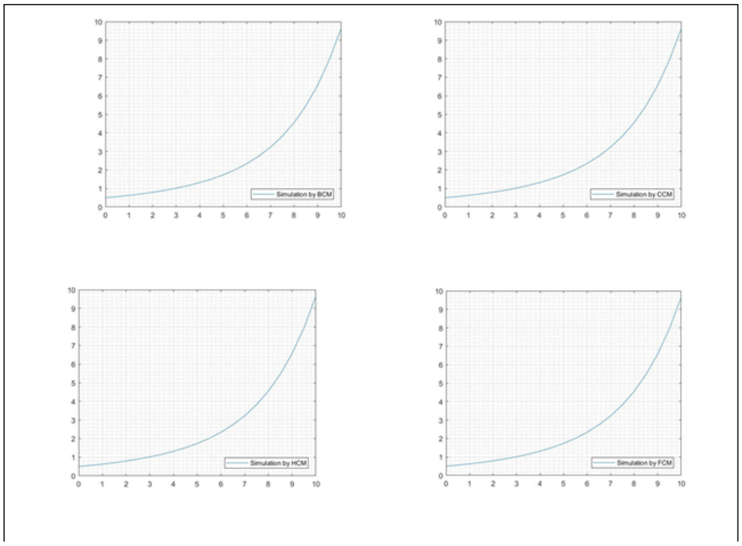

Fig 1. Numerical Simulations by Proposed Method

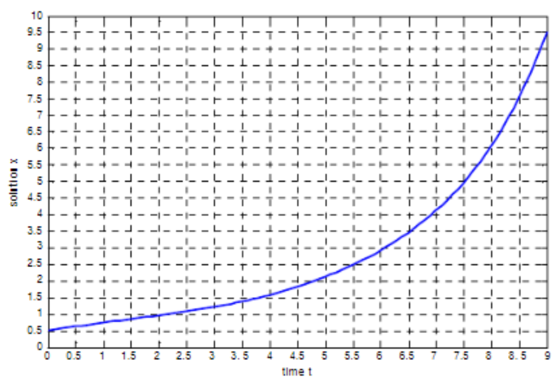

Fig 2. Numerical Simulations by Step-Method [7]

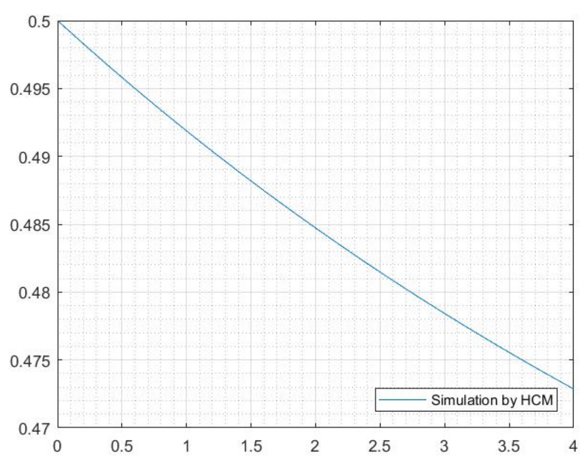

Fig 3. Numerical Simulation of Example 2 by HCM

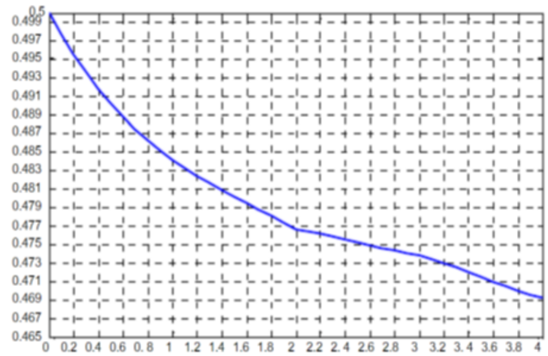

Fig 4. Numerical Simulation of Example 2 by Step-Method [7]

5 Conclusion

In this study, polynomial collocation method based on successive integration technique is applied for solving nonlinear DDEs model of Parkinson's Disease. Numerical examples of two PD models [7] have been considered to demonstrate the efficiency of the proposed method. Numerical simulations of the proposed method have been compared with the numerical simulation of step-method using Picard approximation [7].

The simulations with different polynomials are similar and well comparable to the simulation by step-method. Hence, it is evident that the proposed polynomial collocation method based on successive integration technique with different polynomials is very effective, simple and suitable for solving nonlinear DDEs model of PD and similar real-world problems exist in different fields of science and engineering.

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