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RESEARCH ARTICLE

Solving Time-fractional Order Radon Diffusion Equation in Water by Finite Difference Method

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Abstract

Objective: The aim of this research is to gain a comprehensive understanding of radon diffusion equation in water. **Methods:** A time fractional radon diffusion equation with Caputo sense is employed to find diffusion dynamics of radon in water medium. The fractional order explicit finite difference technique is used to find its numerical solution. A Python software is used to find numerical solution. **Findings:** The effect of fractional-order parameters on the distribution and concentration profiles of radon in water has been investigated. Furthermore, we study stability and convergence of the explicit finite difference method. **Novelty:** The fractional order explicit finite difference method can be used to estimate approximate solution of such fractional order differential equations.

Keywords: Radon Diffusion Equation; Finite Difference Method; Caputo; Fractional Derivative; Python

1 Introduction

Fractional calculus extends traditional calculus by allowing for non-integer or fractional orders of differentiation and integration[(1)]. Fractional calculus has a wide range of applications across various fields due to its ability to model complex systems and phenomena that involve non-integer order derivatives and integrals[(2)]. Some of the prominent applications of fractional calculus include, Physics, Engineering, Signal Processing, Biology and Medicine, Chemical Engineering, Astronomy and Astrophysics etc. The analytical solutions of fractional partial differential equations can be challenging to obtain, especially for non-trivial cases, as they often involve fractional derivatives. To handle this, numerical methods are used to obtain solution of fractional differential equations[(3)]. Finite difference method is one of the widely used numerical method for solving differential equations in both one and multiple dimensions[(4–7)]. They are particularly valuable when analytical solutions are not readily available or practical. Radon, a radioactive gas, is both colorless and odorless. It occurs naturally through the radioactive decay of elements like uranium, which are distributed in varying quantities within soil and rock all over the world. This gas, originating in the ground, can migrate into the atmosphere, as well as infiltrate both underground and surface water. Radon is pervasive in both outdoor and indoor environments. Given its hazardous nature, numerous researchers investigate the movement of radon through substances like soil, air, concrete, and activated charcoal[8]. In 2022, Shrimangale G.W and Raut S.R developed a crank-nicolson method to solve radon diffusion equation in soil medium[8]. In 2021, Rao T.D. and Chakraverty S. have used forward and inverse techniques for fuzzy fractional systems applied to radon transport in soil chambers[9]. Furthermore, A. Rybalkin study Radon difussion equation, and they obtained its exact solution[10]. In this cotext, we develop a fractional order Radon diffusion equation, but it is very difficult to obtain its exact solution. Therefore, we develop fractional order explicit finite difference method[11–13] to investigate the diffusion of radon in water medium. Furthermore, we create Python code to visually represent the approximate solution graphically.

2 Methodology

2.1 Some Preliminaries

Definition 1. The Caputo time-fractional derivative of order $\gamma, (0<\gamma\leq 1)$ is defined by[3],

$$\frac{\partial^{\gamma}V(\zeta,\tau)}{\partial\tau^{\gamma}}=\frac{1}{\Gamma(1-\gamma)}\int_{0}^{\tau}\frac{\partial V(\zeta,\tau)}{\partial\eta}\,\frac{d\eta}{(\tau-\eta)^{\gamma}}\tag{1}$$

2.2 Mathematical Modeling

The Fick's second law elaborates the change in concentration with respect to both time and position occurs. This law can be expressed as follows

$$\frac{\partial V(\zeta,\tau)}{\partial\tau}=D\frac{\partial^{2}V(\zeta,\tau)}{\partial\zeta^{2}}-\mu V(\zeta,\tau)\tag{2}$$

where μ is the radon decay constant.

In this paper, we focus on the time-fractional order radon diffusion equation given below[10],

$$\frac{\partial^{\gamma}V(\zeta,\tau)}{\partial\tau^{\gamma}}=D\frac{\partial^{2}V(\zeta,\tau)}{\partial\zeta^{2}}-\mu V\left(\zeta,\tau\right);\,0<\zeta<L,0<\gamma\leq 1,\tau\geq 0\tag{3}$$

initial condition:

$$V\left(\zeta,0\right)=0;\;\;0<\zeta<L\tag{4}$$

boundary conditions:

$$V\left(0,\tau\right)=V_{0}\;and\;\frac{\partial V\left(0,\tau\right)}{\partial\tau}=0;\;\tau\geq 0\tag{5}$$

Here, $V=V\left(\zeta,\tau\right)$ is the concentration of radon at space ζ and time τ, D is the diffusion coefficient, μ is the radon decay constant and $\frac{\partial^{\gamma}V(\zeta,\tau)}{\partial\tau^{\gamma}}$ is considered in Caputo sense.

2.3 Finite Difference Scheme

Let $(\zeta_i,\tau_k); i=0,1,2,...,M$ and $k=0,1,2,...,N$ be the exact solution of time fractional radon diffusion Equations (3), (4) and (5) at the mesh point (ζ_i,τ_k), where $\tau_k=k\tau',\;\;k=0,1,2,\ldots,N$ and $\zeta_i=ih,\;\;i=0,1,2,\ldots,M$, where $\tau'=\frac{T}{N}$ and $h=\frac{L}{M}$. Let V_i^k be the numerical approximation of the point $V(ih,k\tau')$. We approximate time-fractional derivative in the Caputo sense as follows[3]

$$\frac{\partial^{\gamma}V(\zeta_i,\tau_k)}{\partial\tau^{\gamma}}\approx\frac{1}{\Gamma(1-\gamma)}\int_{0}^{\tau_{k+1}}\frac{1}{(\tau_{k+1}-\eta)}\,\frac{\partial V(\zeta_i,\tau)}{\partial\eta}\,d\eta$$

$= \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)}\left[V_i^{k+1}-V_i^k\right]+\frac{\tau^{-\gamma}}{\Gamma(2-\gamma)}\sum_{j=1}^{k} b_j\left[V_i^{k-j+1}-V_i^{k-j}\right]+O(\tau)$
where $b_j=(j+1)^{1-\gamma}-j^{1-\gamma}, j=0,1,2,\cdots,N$.
Also, we approximate the second order space derivative as follows,
$\frac{\partial^2 V(\zeta,\tau)}{\partial\zeta^2}=\frac{V(\zeta_{i-1},\tau_k)-2V(\zeta_i,\tau_k)+V(\zeta_{i+1},\tau_k)}{h^2}$
$\frac{\partial^2 V(\zeta_i,\tau_k)}{\partial\zeta^2}=\frac{V_{i-1}^k-2V_i^k+V_{i+1}^k}{h^2}$
Therefore, substituting in Equation (4), we get
$\frac{\tau^{-\gamma}}{\Gamma(2-\gamma)}\left[V_i^{k+1}-V_i^k\right]+\frac{\tau^{-\gamma}}{\Gamma(2-\gamma)}\sum_{j=1}^{k} b_j\left[V_i^{k-j+1}-V_i^{k-j}\right]=D\left[\frac{V_{i-1}^k-2V_i^k+V_{i+1}^k}{h^2}\right]-\mu V\left(\zeta_i,t_k\right)$
$\left[V_i^{k+1}-V_i^k\right]+\sum_{j=1}^{k} b_j\left[V_i^{k-j+1}-V_i^{k-j}\right]$
$=\frac{D\Gamma(2-\gamma)\tau^{\gamma}}{h^2}\left[V_{i-1}^k-2V_i^k+V_{i+1}^k\right]-\mu\Gamma(2-\gamma)\tau^{\gamma}V(\zeta_i,t_k)$
Put $r=\frac{D\Gamma(2-\gamma)\tau^{\gamma}}{h^2}$ and $\mu'=\mu\Gamma(2-\gamma)\tau^{\gamma}$ we have,

$$\left[V_i^{k+1}-V_i^k\right]+\sum_{j=1}^{k} b_j\left[V_i^{k-j+1}-V_i^{k-j}\right]=r\left[V_{i-1}^k-2V_i^k+V_{i+1}^k\right]-\mu' V_i^k \tag{6}$$

After simplification,
$V_i^{k+1}=rV_{i-1}^k+(1-2r-\mu')V_i^k+rV_{i+1}^k-\sum_{j=1}^{k} b_j\left(V_i^{k-j+1}-V_i^{k-j}\right)$
$=rV_{i-1}^k+(1-2r-\mu')V_i^k+rV_{i+1}^k+b_1\left[V_i^{k-1}-V_i^k\right]+b_2\left[V_i^{k-2}-V_i^{k-1}\right]$
$+b_3\left[V_i^{k-3}-V_i^{k-2}\right]+\cdots+b_{k-1}\left[V_i^1-V_i^2\right]+b_k\left[V_i^0-V_i^1\right]$

$$V_i^{k+1}=rV_{i-1}^k+(1-2r-\mu'-b_1)V_i^k+rV_{i+1}^k+\sum_{j=1}^{k-1}(b_j-b_{j+1})V_i^{k-j}+b_kV_i^0 \tag{7}$$

We approximate initial condition as follows

$$V_i^0,\ i=0,1,2,\cdots,M.$$

Also, $V_0^k=0$ and $\frac{\partial V(L,\tau)}{\partial\zeta}=0$, which gives $V_{M+1}^k=V_{M-1}^k$.
Now, for $k=0$, we obtain
$V_i^1=rV_{i-1}^0+(1-2r-\mu)V_i^0+rV_{i+1}^0$
Therefore, the complete fractional approximated initial boundary value problem is,

$$V_i^1=rV_{i-1}^0+(1-2r-\mu)V_i^0+rV_{i+1}^0 \quad for\ k=0 \tag{8}$$

$$V_i^{k+1}=rV_{i-1}^k+(1-2r-\mu'-b_1)V_i^k+rV_{i+1}^k+\sum_{j=1}^{k-1}(b_j-b_{j+1})V_i^{k-j}+b_kV_i^0; for\ k\geq 1 \tag{9}$$

with initial condition:

$$V_i^0=0, i=0,1,2,\cdots,M \tag{10}$$

boundary conditions:

$$V_0^k=0\ \ and\ \frac{\partial V(L,\tau)}{\partial x}=0,\ V_{M+1}^k=V_{M-1}^k;\ k=0,1,2,\cdots \tag{11}$$

The finite difference scheme Equations (8), (9), (10) and (11) can be expressed in matrix form as follows,

$$V^1=AV^0+V';\quad for\ k=0 \tag{12}$$

$$V^{k+1}=BV^k+\sum_{j=1}^{k-1}(b_j-b_{j+1})V^{k-j}+b_kV^0+V';\quad for\ k\geq 1 \tag{13}$$

where, $V^k = \left[V_1^k, V_2^k, V_3^k, \cdots, V_M^k\right]$, $V' = 0 = \left[rV_0^0, 0, \cdots, 0\right]^T$,

$$A = [a_{ij}] = \begin{cases} 1-2r-\mu' & if\ i=j \\ r & if\ i=j+1 \\ r & if\ i=j-1 \\ 0 & otherwise \end{cases}$$

and,

$$B = [b_{ij}] = \begin{cases} 1-2r-\mu'-b_1 & if\ i=j \\ r & if\ i=j+1 \\ r & if\ i=j-1 \\ 0 & otherwise \end{cases}$$

The above system of equation is solved by using Python software.

2.4 Stability

Theorem 1. The solution of the finite difference scheme Equations (8), (9), (10) and (11) for time fractional radon diffusion Equations (3), (4) and (5), is stable, when $r \le min\left\{\frac{2-\mu'}{3}, \frac{2-\mu'}{4} | 0 \le \mu' \le 1\right\}$.

Proof: Let p_i be an eigenvalue of matrix A, so we have, $AX_i = p_i X_i$ for some nonzero vector X_i. Now, we choose i such that, $|X_i| = max\{|x_i| | j = 1, 2, \ldots, k\}$.

Then we have

$\sum_{j=1}^{k-1} a_{ij} x_j = p_i x_i$

and therefore,

$$p_i = a_{ii} + \sum_{j=1, i\neq 1}^{k} a_{ij} \frac{x_j}{x_i} \tag{14}$$

Now for $i = 1$, we get

$p_1 = a_{11} + \sum_{j=2, j\neq 1}^{k} a_{1j} \frac{x_j}{x_1} = 1 - 2r - \tau^{-\gamma} + a_{12}\frac{x_2}{x_1} = 1 - 2r - \mu' + r\frac{x_2}{x_1}$
$\le 1 - 2r - \mu' + r \le 1 \le 1 - r - \mu' < 1$

Theretofore, $p_1 < 1$.

Also, $p_1 v = 1 - 2r - v + r\frac{x_2}{x_1} \ge 1 - 2r - v - r\frac{x_2}{x_1} \ge 1 - 2r - \mu' - r \ge -1$

When, $1 - 2r - \mu' - r \ge -1$ implies that,

$1 - 3r - \mu' \ge -1 \Rightarrow 3r \le 2 - \mu' \Rightarrow r \le \frac{2-\mu'}{3}$

Now, for $2 \le i \le M-1$,

$$p_i = 1 - 2r - \mu' + a_{12}\frac{x_2}{x_1} + a_{23}\frac{x_3}{x_2} \le 1 - 2r - \mu' + r + r \le 1 - \mu' < 1$$

$$p_i = 1 - 2r - \mu' + a_{12}\frac{x_2}{x_1} + a_{23}\frac{x_3}{x_2} \ge 1 - 2r - \mu' - r - r \ge 1 - 4r - \mu' \ge -1$$

When, $1 - 4r - \mu' \ge -1$ implies that,

$4r \le 2 - \mu' \Rightarrow 4 \le \frac{2-\mu'}{4} \Rightarrow 0 \le v \le 1$

$-1 \le \mu' \le 1$ for $2 \le i \le M-1$ when $r \le \frac{2-\mu'}{4}$.

Finally, when $i = M$

$p_M = a_{MM} + \sum_{j=1, i\neq M}^{k-1} a_{ij}\frac{x_j}{p_i} \le 1 - 2r - \mu' + 0 + \cdots + 2r \le 1 - \mu' \le 1$

$\Rightarrow p_M \ge 1 - 2r - \mu' - 2r \ge -1$

Also,

$1 - 4r - \mu' \ge -1 \Rightarrow 4r \le 2 - \mu' \Rightarrow r \le \frac{2-\mu'}{4}$,

Therefore, for $1 \le i \le M$ we get, $-1 \le p_M \le 1$ when $r \le min\left\{\frac{2-v}{3}, \frac{2-\mu'}{4}\right\}$.

Implies, $|p_M| \le 1$ when $r \le min\left\{\frac{2-v}{3}, \frac{2-\mu'}{4}\right\}$

Therefore, $\| A \|_2 \le 1$ when $r \le min\left\{\frac{2-v}{3}, \frac{2-\mu'}{4}\right\}$, which gives

$\| V^1 \|_2 = \| AV^0 \|_2 \le \| A \|_2 \| V^0 \|_2$

Now, we have $\| V^n \|_2 \leq \| V^{n-1} \|_2 \leq \cdots \leq \| V^1 \|_2 \leq \| V^0 \|_2$.
Also, eigen values of B are given by,
When $i=1, p_1 = b_{11} + \sum_{j=2, j\neq 1}^{k} b_{ij}\frac{x_j}{x_i} = 1-2r-\mu' b_1 + b_{12}\frac{x_2}{x_1} \leq 1-2r-\mu'-b_1+r$
$\leq 1-r-\mu'-b_1 \leq 1$
Also $p_i \geq 1-2r-\mu'-b_1-b_{12}\frac{x_2}{x_1} \geq 1-2r-\mu'-b_1-r \geq 1$
Therefore, $\mu' \geq -1$ when $1-3r-\mu'-b_1 \geq -1$ i.e. $r \leq \frac{2-\mu'-b_1}{3}$
When, $2 \leq i \leq M-1$,
$p_i = b_{ii} + \sum_{j=1, j\neq 2}^{k-1} b_{ij}\frac{x_j}{x_i} = 1-2r-\mu' b_1 + b_{12}\frac{x_1}{x_2} + b_{23}\frac{x_3}{x_2}$
$\leq 1-2r-\mu'-b_1+r+r$
$\leq 1-\mu' \leq 1$
Also, $\mu' \geq 1-2r-\mu'-b_1-r-r \geq 1-4r-\mu'-b_1-r \geq -1$
Therefore, $\mu' \geq -1$ when $1-4r-\mu'-b_1 \geq -1$ i.e. $r \leq \frac{2-\mu'-b_1}{4}$.
$|\mu'| \leq 1$ for $2 \leq i \leq M$ when $r \leq \frac{2-\mu'-b_1}{4}$.
When, $i=M$, $\quad p_M = b_{MM} + \sum_{j=1, j\neq M}^{k} b_{ij}\frac{x_j}{x_i} = 1-2r-\mu' b_1 + \cdots + 2r\frac{x_M}{x_{M-1}}$
$\mu' \leq 1-2r-\mu'-b_1+2r \leq 1-\mu'-b_1 < 1$
Also, $\mu' \geq 1-2r-\mu'-b_1-2r \geq 1-4r-\mu'-b_1 \geq -1$
Therefore, $\mu' \geq -1$ when $1-4r-\mu'-b_1 \geq -1$ i.e. $r \leq \frac{2-\mu'-b_1}{4}$
Hence, $\| B \|_2 = \max_{1\leq i\leq M-1} |p_i| \leq 1$ that is $\| B \|_2 \leq 1$.
Hence,
$\| V^{k+1} \|_2 = \| BV^k + \sum_{j=1}^{k-1}(b_j - b_{j+1})\, V^{k-j} + b_k V^0 \|_2$
$\leq zB \|_2 \| V^k \|_2 + (b_1-b_2+b_2-b_3+\cdots+b_{k-1}) \| V^{k-j} \|_2 + b_k \| V^0 \|_2$
$\leq \| V^0 \|_2$
i.e.result is true for *k=n+1*. Hence, by induction $\| V^k \|_2 \leq V^0 \|_2$.
This shows that the scheme is stable when, $r \leq min\left\{\frac{2-\mu'-b_1}{3}, \frac{2-\mu'-b_1}{4} \,|\, 0 \leq \mu' \leq 1\right\}$.

2.5 Convergence

Theorem 2. Let $\overline{V}^k$ be the exact solution of the time fractional radon diffusion Equations (3), (4) and (5) and V^k be the approximate solution, then V^k converges to $\overline{V}^k$ as $(h, \tau') \rightarrow (0,0)$ when
$r \leq min\left\{\frac{2-\mu'-b_1}{3}, \frac{2-\mu'-b_1}{4} |0 \leq \mu' \leq 1\right\}$.

Proof: Let, $V^k = [V_1, V_2, \ldots, V_N]^T$, $\overline{V}^k = [\overline{V}_1, \overline{V}_2, \ldots, \overline{V}_N]$ and $E^k = \overline{V}^k - V^k$.
Let us assume that,
$|e_l^k| = \max_{1\leq i\leq N-1} |e_i^k| = ||E^k||_\infty;\ for\ l = 1, 2, \ldots$
and
$T_l^k = \max_{1\leq i\leq N-1} |T_i^k| = h^2 O(\tau' + h^2)$
For, $k=1$, we have,
$|e_l^1| = |re_{i-1}^0 + (1-2r-\mu')e_i^0 + re_{i+1}^0|$
$\leq |re_{i-1}^0| + (1-2r-\mu')|e_i^0| + |re_{i+1}^0| + |T_i^k|$
when $r \leq min\left\{\frac{2-\mu'-b_1}{3}, \frac{2-\mu'-b_1}{4} |0 \leq \mu' \leq 1\right\}$
$|e_l^1| \leq r|e_{i-1}^0| + (1-2r-\mu')|e_i^0| + r|e_{i+1}^0| + |T_i^k|$
$\leq (r+1-2r-\mu'+r)|e_i^0| + |T_i^k| \leq |e_i^0| + |T_i^k|$
This implies,
$\| E^1 \|_\infty \leq \| E^0 \|_\infty + r|T_i^k| = \| E^0 \|_\infty + rh^2(\tau' + h^2)$
$\leq \| E^\infty \|_\infty + \tau'^2 \Gamma(2-\gamma) 0(\tau' + h^2)$
That is the result hold for $n=1$.
For $n=k$, we assume,
$\| E^k \|_\infty \leq \| E^0 \|_\infty + kh^2 0(\tau' + h^2)$
For $n=k+1$, we have,

$\| E_l^{k+1} \| = |re_{i-1}^{k+1} + (1-2r-\mu')e_i^k + re_{i+1}^k| + |T_i^{k+1}| + \sum_{j=1}^{k-1} (b_j - b_{j+1})e_i^{k-j} + b_k e_i^0| + |T_i^k|$

when $r \leq min\left\{\frac{2-\mu'-b_1}{3}, \frac{2-\mu'-b_1}{4}|0 \leq \mu' \leq 1\right\}$.

$|E_l^{k+1}| \leq r|e_i^k| + (1-2r-\mu'-b_1)|e_l^k| + r|e_i^k|$
$+(b_1-b_2+b_2-b_3+\cdots+b_{k-1})|e_l^k| + r|T_l^k|$
$\leq (r+1-2r-\mu'-b_1+r+b_1-b_k+b_k)|e_l^k|$
$\leq |e_l^k| + r|T_l^k| \leq \| E^k \|_\infty + r|T_l^k|$
$\leq \| E^0 \|_\infty + r|T_l^k| + krh^2 0(\tau' + h^2)$
$\leq \| E^0 \|_\infty + k\tau'^2\Gamma(2-\gamma)O(\tau' + h^2) + \epsilon^{TM2}\Gamma(2-\gamma)$
$|E_i^{k+1}| \leq \| E^0 \|_\infty + (k+1)\,\tau'^2\Gamma(2-\gamma)O(\tau' + h^2)$
Therefore, we conclude that if we assume
$r \leq min\left\{\frac{2-\mu'-b_1}{3}, \frac{2-\mu'-b_1}{4}|0 \leq \mu' \leq 1\right\}$
then $\| E^k \|_\infty \to 0$ as $\tau' \to 0,\ h \to 0$ which results in the convergence of $\overline{V}_i^k$ to $V(\zeta_i, t_k)$.
Hence, the proof is complete.

3 Result and Discussion

In this section, we aim to derive approximate solutions for the time fractional Radon diffusion equation while considering both initial and boundary conditions in water medium. We adopt a finite difference scheme to obtain numerical solutions for the time fractional Radon diffusion equation. The equation under investigation is the following time-fractional Radon diffusion equation:

$$\frac{\partial^\gamma V(\zeta,\tau)}{\partial \tau^\gamma} = D\frac{\partial^2 V(\zeta,\tau)}{\partial \zeta^2} - \mu V(\zeta,\tau),\ 0<\gamma\leq 1,\ 0\leq \zeta \leq L\ \tau \geq 0, \tag{15}$$

Initial condition:

$$V(\zeta,0) = 0,\ 0<\zeta<L \tag{16}$$

Boundary conditions:

$$V(0,\tau) = dcV_0\ and\ \frac{\partial V(L,\tau)}{\partial \tau} = 0,\ \tau \geq 0 \tag{17}$$

Exact solution of Equations (15), (16) and (17) for $\gamma = 1$ is as follows [10],

$$V(\zeta,\tau) = dcV_0\left(\frac{cosh\sqrt{\frac{\mu}{D}}(L-\zeta)}{cosh\sqrt{\frac{\mu}{D}}L} - \frac{D\pi}{L^2}\sum_{n=0}^{\infty}\frac{(2n+1)e^{-\left(\frac{(2n+1)^2\pi^2 D}{4L^2}+\mu\right)\tau}}{\left(\frac{(2n+1)^2\pi^2 D}{4L^2}+\mu\right)} sin\frac{(2n+1)\pi\zeta}{2L}\right) \tag{18}$$

To estimate approximate solutions of the time –fractional radon diffusion Equations (15), (16) and (17), we consider the parameters given in the Table 1.

Table 1. Parameters and their values

Parameters	Value
Radon diffusivity coefficient in water D	$1\times 10^{-9}\ Bq/m^3$
Spatial length L	$1.7278\ cm$
Radon decay constant μ	2.1×10^{-6}
Adsorption coefficient c	$4\ m^2/kg$
Material density d	$0.5\ g/cm^3$
Radon concentration in air V_0	$200\ Bq/m^3$

The Python program is used to find the approximate solutions of Equations (15), (16) and (17) by developed finite difference method given in Equations (8), (9), (10) and (11). In Table 2, the approximate solution of time-fractional radon diffusion Equations (15), (16) and (17) derived from the fractional finite difference scheme Equations (8), (9), (10) and (11) is compared with the exact solution for $\gamma = 1$, which demonstrating the method's effectiveness.

Table 2. Absolute error

	Absolute Error				
$\zeta \rightarrow t \downarrow$	**0.2**	**0.4**	**0.6**	**0.8**	**1.0**
0.0	1.03×10^{-4}	5.41×10^{-5}	3.93×10^{-5}	3.34×10^{-5}	3.18×10^{-5}
0.2	8.45×10^{-5}	4.44×10^{-5}	3.22×10^{-5}	2.74×10^{-5}	2.61×10^{-5}
0.4	6.94×10^{-5}	3.64×10^{-5}	2.65×10^{-5}	2.25×10^{-5}	2.14×10^{-5}
0.6	5.69×10^{-5}	2.99×10^{-5}	2.17×10^{-5}	1.85×10^{-5}	1.76×10^{-5}
0.8	4.67×10^{-5}	2.45×10^{-5}	1.78×10^{-5}	1.51×10^{-5}	1.44×10^{-5}
1.0	3.83×10^{-5}	2.01×10^{-5}	1.46×10^{-5}	1.24×10^{-5}	1.18×10^{-5}

In Figures 1, 2 and 3, we plot the dynamics of radon concentration for the variation of control parameters with fixed time $\tau = 12\ hrs$. Figure 1 represents the behaviour of radon concentration in water for $\gamma = 1$. It is observed that, at $\zeta = 0$, that is beginning of the radon diffusion process the concentration is high in the water. As the distance is passed, the concentration is gradually decreased and moves to attain the resting concentration in the water.

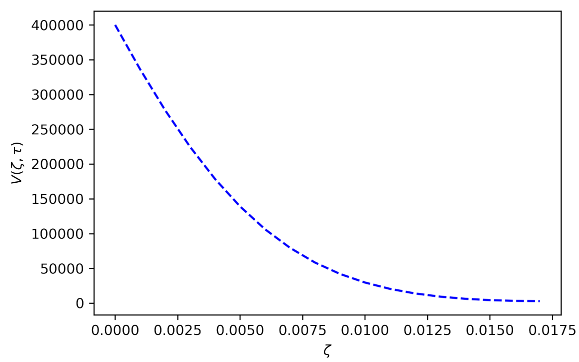

Fig 1. Radon concentration for $\tau = 12\ hrs,\ \ \gamma = 1,\ h = 0.001$

Furthermore, we study the effect of parameter γ on the radon concentration in water. Figure 2 shows the concentration of radon over space for different control parameter γ and $\tau = 12\ hrs$. The amount of radon concentration in Figure 2 is same as in Figure 1. It is observed that radon concentration decreases more rapidly as γ decreases.

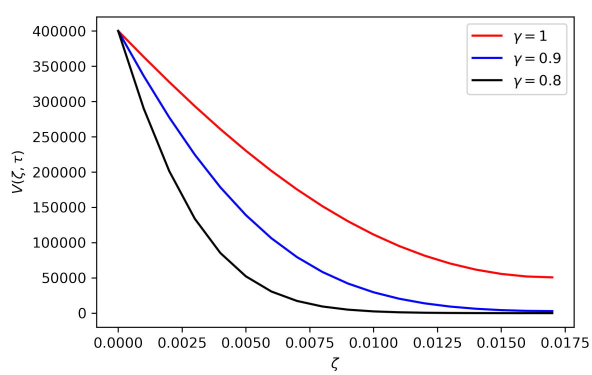

Fig 2. Radon concentration for $h = 0.001, \tau = 12\ hrs$

Figure 3 shows the simulation of radon concentration for different time slots and $\gamma = 1$. We observe that radon concentration gets increases as time passes.

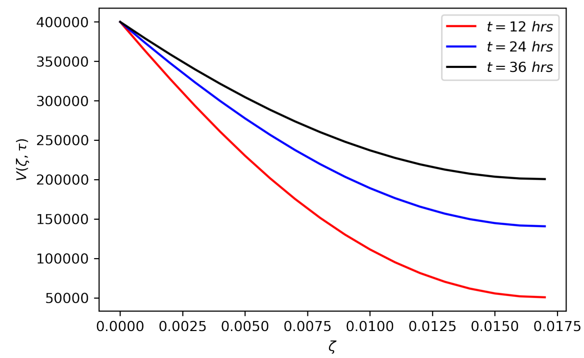

Fig 3. Radon concentration for $\gamma = 1,\ h = 0.001$

We observed that, the approximate solutions obtained by developed method is close to exact solution obtained by A. Rybalkin[10] for $\gamma = 1$. Therefore, our developed scheme is suitable for obtaining numerical solution of fractional order Radon diffusion equation in water medium.

4 Conclusion

We have successfully formulated an explicit finite difference scheme for fractional-order Radon diffusion equations. We have conducted a comprehensive analysis of the scheme's stability and convergence. As an application of this method, we have obtained numerical solutions for practical problems involving a water medium and have presented these solutions through graphical simulations. We study the effect of time-fractional order γ on radon concentration and observe that it decreases rapidly as γ decreases. Furthermore, we observe that radon concentration in water gets increases as time passes.

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