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Modified Detour Index of Hamiltonian Connected (Laceable) Graphs

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Abstract

Objectives: To explore the bounds for the modified detour index of certain Hamiltonian connected and laceable graphs. **Methods:** The Wiener index $W(G) = \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} d(u,v)$, detour index $\omega(G) = \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} l(u,v)$ and the modified detour index $\omega^c(G) = \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} l(u,v)d(u,v)$ are used. **Findings:** Here we introduce the modified detour index and its least upper bounds for Hamiltonian connected and laceable graphs, by formulating the constraints. **Novelty:** Based on the modified detour index, the bounds for some special graphs such as: Hamiltonian connected graphs of two families of convex polytopes (R_n and S_n) and Hamiltonian laceable graphs of spider graph ($S_n(m)$) and image graph of prism graph ($I_m(Y_n)$) are encountered here.

Keywords: Hamiltonian graph; Hamiltonian connected; Hamiltonian laceable; Wiener index; detour index

1 Introduction

The detour index[1,2] of a connected graph G is defined as the sum of the detour distances (lengths of the longest paths) between unordered pairs of vertices of the graph. i.e

$$\omega(G) = \frac{1}{2}\sum_{\{u,v\}\subseteq V(G)} l(u,v).$$

where The detour distance between vertices u and v in the graph G is the length of a longest path between them, denoted by $l(u,v)$.

The detour index has important applications in chemistry. Applications of this parameter in quantitative structure activity and property relationship models were put forward in[2,3]. In[4] a comparative analysis with the Wiener index in terms of applicability in correlating structure boiling point of organic compounds. Moreover, its further applications in predicting the normal boiling points of cyclic and acyclic alkanes were studied. The Wiener index of a graph G is defined as:

$$W(G) = \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} d(u,v).$$

A graph G is called Hamiltonian if there exist a Hamiltonian cycle in it. Since, determining Hamiltonian cycle in a given graph is NP-complete problem [5] and necessary sufficient conditions for a Hamiltonian graphs is still an open problem in graph theory. Therefore, it is hard to identify a graph which is Hamiltonian. A graph which contains a Hamiltonian path between every two of its vertices is called Hamilton-connected [6]. However, any bipartite graph is not Hamiltonian connected. Since any Hamiltonian path in a bipartite graph consists of the same number of vertices of the two partite sets. There exist no Hamiltonian path between two vertices belonging to the same partite sets. In that case, a bipartite graph is called Hamilton-laceable [7–9] if there exist Hamiltonian paths between vertices of different partite sets.

Since, the Winer index is the sum of arbitrary pair of vertices at a shortest distance and the detour index is the sum of arbitrary pair of vertices at a longest distance. The combination of these two indices leads to some interesting results. Particularly, for Hamiltonian connected and laceable graphs this study may be helpful to other researchers for finding the necessary and sufficient conditions for a Hamiltonian graphs.

Therefore, we define the modified detour index as follows:

$$\omega^c(G) = \frac{1}{2} \sum_{\{u,v\} \subseteq V(G)} l(u,v)d(u,v).$$

where $l(u,v)$ and $d(u,v)$ denotes the length of the longest path and distance between the vertices u and v respectively.

Since, finding the detour index of a given graph is computationally NP-complete. Therefore, we assume that the problem of finding the modified detour index of a given graph is also computationally NP-complete.

2 Methodology

In this section two families of convex polytopes (R_n and S_n), spider graph ($S_n(m)$) and image graph of prism graph ($I_m(Y_n)$) are considered for the study.

2.1 Convex poytopes

For $n \geq 4$ by R_n we denote the graph of the convex polytope [10] which is obtained as a combination of the graph of a prism and the graph of an antiprism, where $V(R_n) = \{x_j, y_j, z_j : 1 \leq j \leq n\}$ and the edge set of R_n is given by:

$E(R_n) = \{(x_j,x_{j+1}),\ (x_j,y_j),\ (x_{j+1},y_j),\ (y_j,y_{j+1}),\ (y_j,z_j),\ (z_j,z_{j+1});\ 1 \leq j \leq n\}$

We make the convention that $u_{n+l} = u_l$, $v_{n+l} = v_l$ and $w_{n+l} = w_l$ to simplify the notation. See Figure 1 to view the n-dimensional convex polytope R_n.

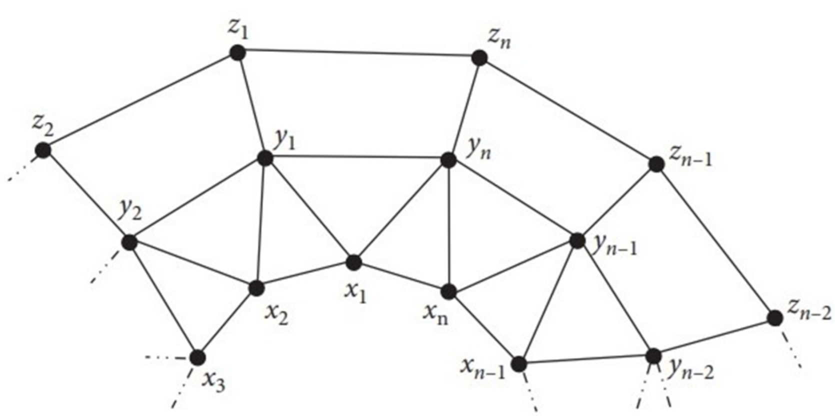

Fig 1. The n-dimensional convex polytope R_n

The graph of convex polytope S_n [11] consists of 3-sided faces, 2n-4-sided faces and a pair of n-sided faces, and is obtained by the combination of the graph of convex polytope R_n and the graph of a prism D_n. We have $V(S_n) = \{x_j, w_j, v_j, u_j : 1 \leq j \leq n\}$ and the edge set of R_n is given by:

$E(S_n) = \{(x_j, x_{j+1}),\ (w_j, w_j+1),\ (v_j, v_{j+1}),\ (u_j, u_{j+1})\} \cup \{(x_{j+1}, w_j),\ (x_j, w_j),\ (w_j, v_j),\ (v_j, u_j);\ 1 \leq j \leq n\}$

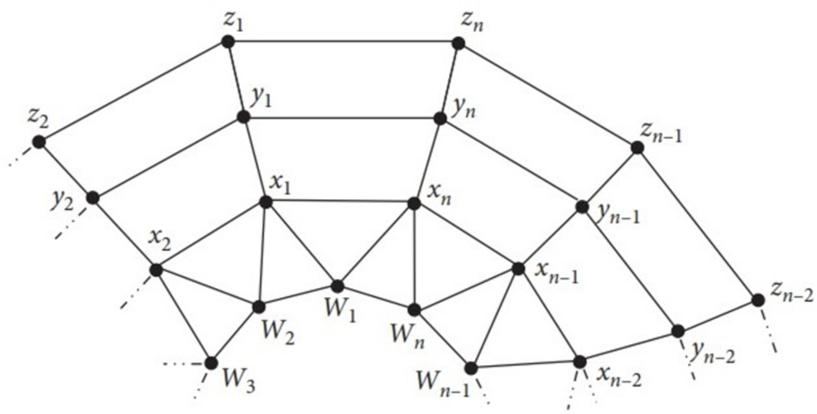

Fig 2. The n-dimensional convex polytope

2.2 Spider web networks

A spider web network[12] $SW(r,s)$ is the graph with vertex set $\{(x,y)|0 \leq x < r, 0 \leq y < s\}$ such that (x,y) and (u,v) are adjacent if they satisfy one of the following conditions:

- $x = u$ and $y = v \pm 1$;
- $y = v$ and $u = [x+1]_m$ if $x+y$ is odd or $y = n-1$; and
- $y = v$ and $u = [x-1]_m$ if $x+y$ is even or $y = 0$.

For example, two different layouts of spider web network graph $SW(8,6)$ are shown in Figure 3.

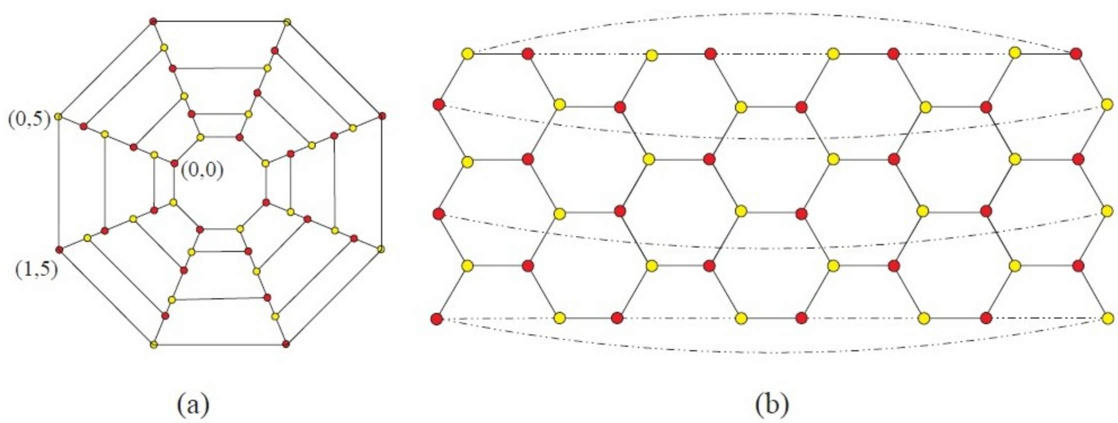

Fig 3. Spider web network graph $SW(8,6)$

2.3 Image graph of a prism graph Y_n

The vertex set of Y_n, the prism graph[13] having $2n$ vertices and $3n$ edges is defined as

$V(Y_n) = \{u_1, u_2, u_3, \cdots, u_n\} \cup \{v_1, v_2, v_3, \cdots, v_n\}$

$E(Y_n) = \{(v_i, v_{i+1}),\ (v_i, u_i),\ (u_i, u_{i+1}):\ 1 \leq i \leq n\}$.

The image graph of a connected graph G, denoted by $I_m(G)$, is the graph obtained by joining the vertices of the original graph G to the corresponding vertices of a copy of G.

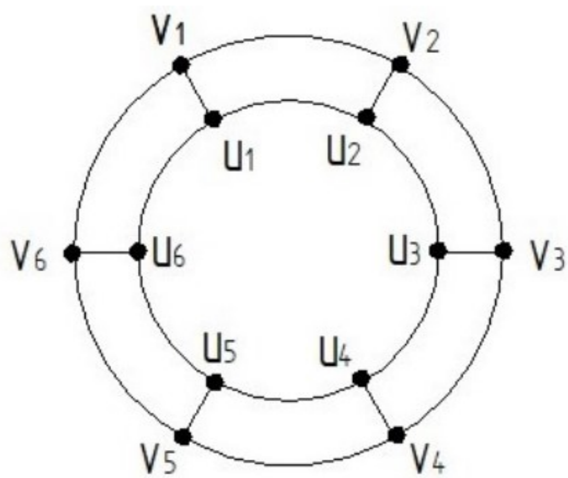

Fig 4. Prism graph Y_6

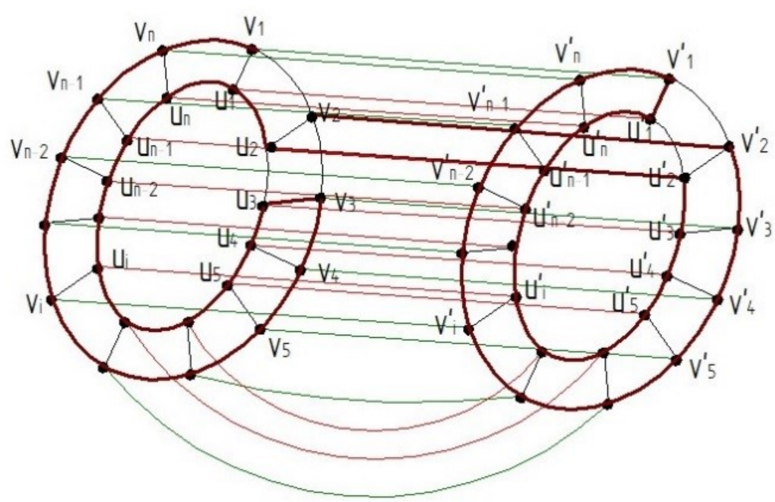

Fig 5. Image graph of Prism graph Y_n

3 Results and Discussion

Lemma A. Let $G = P_n$ be a path graph with $n \geq 3$ vertices. Then

$$W(P_n) = \frac{1}{6}n(n^2-1)$$

Lemma B. Let $G = C_n$ be a cycle graph with $n \geq 3$ vertices. Then

$$W(C_n) = \frac{n}{8}\left(n^2-1\right) \quad if\ n\ is\ odd$$

$$W(C_n) = \frac{n^3}{8} \quad if\ \ n\ is\ even$$

Theorem 4.1. The modified detour index of n-dimensional convex polytope R_n is given by $\omega^c(R_n) \leq \frac{3}{8}[n(3n-1)^2][5n^3+(n-2)^3+32]$.

Proof. Let $G = R_n$ be n-dimensional convex polytope. By definition $V(R_n) = \{u_j, v_j, w_j : 1 \leq j \leq n\}$ and the edge set of R_n is given by:

$$E(R_n) = \{(x_j, x_{j+1}),\ (x_j, y_j),\ (x_{j+1}, y_j),\ (y_j, y_{j+1}),\ (y_j, z_j),\ (z_j, z_{j+1});\ 1 \leq j \leq n\}$$

Also, $deg(z_i) = 3$, $deg(y_i) = 5$ and $deg(x_i) = 4$ for $1 \leq i \leq n$. Since, R_n contains a cycle of length three as a subgraph. Therefore, R_n is non-bipartite graph. Further, in (5), it is proved that R_n is Hamiltonian connected. That is, R_n is attainable for every arbitrary pair of vertices. Also, we can see that R_n has $3n(3n-1)^2$ Hamiltonian paths. Hence,

$$\omega^c(R_n) = \frac{1}{2} \sum_{\{u,v\} \subseteq V(R_n)} l(u,v) d(u,v)$$

$$= \frac{1}{2} \sum_{\{u,v\} \subseteq V(R_n)} 3n(3n-1)^2 d(u,v)$$

$$= (3n(3n-1)^2) . \frac{1}{2} \sum_{\{u,v\} \subseteq V(R_n)} d(u,v)$$

$$= 3n(3n-1)^2 W(R_n) \tag{1}$$

To find the Wiener index of R_n we need to calculate the distance between every pair of vertices. i.e.,

$$W(R_n) = \frac{1}{2} \sum_{\{u,v\} \subseteq V(R_n)} d(u,v)$$

$$= \frac{1}{2} \sum_{\{z_i,z_j\} \subseteq Z} d(z_i,z_j) + \frac{1}{2} \sum_{\{y_i,y_j\} \subseteq Y} d(y_i,y_j) + \frac{1}{2} \sum_{\{x_i,x_j\} \subseteq X} d(x_i,x_j)$$

$$+ \frac{1}{2} \sum_{\{z_i,y_j\} \subset V(R_n)} d(z_i,y_j) + \frac{1}{2} \sum_{\{z_i,x_j\} \subset V(R_n)} d(z_i,x_j) + \frac{1}{2} \sum_{\{y_i,x_j\} \subset V(R_n)} d(y_i,x_j)$$

$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \tag{2}$$

Where

$$I_1 = \frac{1}{2} \sum_{\{z_i,z_j\} \subseteq Z} d(z_i,z_j),\ I_2 = \frac{1}{2} \sum_{\{y_i,y_j\} \subseteq Y} d(y_i,y_j)$$

$$I_3 = \frac{1}{2} \sum_{\{x_i,x_j\} \subseteq X} d(x_i,x_j),\ I_4 = \frac{1}{2} \sum_{\{z_i,y_j\} \subset V(R_n)} d(z_i,y_j)$$

$$I_5 = \frac{1}{2} \sum_{\{z_i,x_j\} \subset V(R_n)} d(z_i,x_j),\ I_6 = \frac{1}{2} \sum_{\{y_i,x_j\} \subset V(R_n)} d(y_i,x_j)$$

Since, each layer of R_n is a cycle of even length. Therefore, by Lemma B, we have

$$I_1 = I_2 = I_3 = \frac{n^3}{8}. \tag{3}$$

It can be easily verified that $d(z_i, y_i) = 1$ and $d(z_i, y_j) = d(z_i, y_i) + d(y_i, y_j)$ for $2 \leq i \leq n$. Therefore,

$$I_4 = \frac{1}{2} \sum_{\{z_i, y_j\} \subset V(R_n)} d(z_i, y_j)$$

$$\leq 1 + \frac{1}{2} \sum_{\{y_i, y_j\} \subseteq Y} d(y_i, y_j)$$

$$= \frac{n^3 + 8}{8}. \tag{4}$$

Similar argument holds for I_6. Therefore,

$$I_6 \leq \frac{n^3 + 8}{8}. \tag{5}$$

Since, $d(z_i, x_i) = d(z_i, x_{i+1}) = 2$ and $d(z_i, x_j) = d(z_i, x_i) + d(x_i, x_j)$ or $d(z_i, x_j) = d(z_i, x_{i+1}) + d(x_{i+1}, x_j)$. Therefore,

$$I_5 = \frac{1}{2} \sum_{\{z_i, x_j\} \subset V(R_n)} d(z_i, x_j)$$

$$\leq 2 + \frac{1}{2} \sum_{\{x_i, x_j\} \subseteq X} d(x_i, x_j)$$

$$= 2 + \frac{1}{2} \sum_{\{x_i, x_j\} \subseteq X} d(x_{i+1}, x_j)$$

$$\leq \frac{(n-2)^3 + 16}{8} \tag{6}$$

Using Equations (3), (4) and (5) in Equation (2) and replacing $W(R_n)$ inEquation (1) we get the desired result.

Theorem 4. 2. The modified detour index of n-dimensional convex polytope S_n is given by
$\omega^c(S_n) \leq n(4n-1)^2[4n^3 + (n-2)^3 + 40]$.

Proof. Let $G = S_n$ be n-dimensional convex polytope. By definition $V(S_n) = \{u_j, v_j, w_j, x_j : 1 \leq j \leq n\}$ and the edge set of R_n is given by:

$$E(S_n) = \{(x_j, x_{j+1}),\ (w_j, w_j + 1),\ (v_j, v_{j+1}),\ (u_j, u_{j+1})\} \cup \{(x_{j+1}, w_j),\ (x_j, w_j),\ (w_j, v_j),\ (v_j, u_j);\ 1 \leq j \leq n\}$$

Also, $deg(z_i) = 3$, $deg(y_i) = 4 = deg(w_i)$ and $deg(x_i) = 5$ for $1 \leq i \leq n$. Since, S_n contains a cycle of length three as a subgraph. Therefore, R_n is non-bipartite graph. Further, in (10), it is proved that S_n is Hamiltonian connected. That is, S_n is attainable for every arbitrary pair of vertices. Also, we can see that S_n has $4n(4n-1)^2$ Hamiltonian paths. Hence,

$$\omega^c(S_n) = \frac{1}{2} \sum_{\{u,v\} \subseteq V(S_n)} l(u,v) d(u,v)$$

$$= \frac{1}{2} \sum_{\{u,v\} \subseteq V(S_n)} 4n(4n-1)^2 d(u,v)$$

$$= (4n(4n-1)^2) . \frac{1}{2} \sum_{\{u,v\} \subseteq V(S_n)} d(u,v)$$

$$= 4n(4n-1)^2 W(S_n) \tag{7}$$

To find the Wiener index of S_n we need to calculate the distance between every pari of vertices. i.e

$$W(S_n) = \frac{1}{2} \sum_{\{u,v\} \subseteq V(S_n)} d(u,v)$$

$$= \frac{1}{2} \sum_{\{z_i,z_j\} \subseteq Z} d(z_i,z_j) + \frac{1}{2} \sum_{\{y_i,y_j\} \subseteq Y} d(y_i,y_j) + \frac{1}{2} \sum_{\{x_i,x_j\} \subseteq X} d(x_i,x_j) + \frac{1}{2} \sum_{\{w_i,w_j\} \subseteq W} d(w_i,w_j)$$

$$+ \frac{1}{2} \sum_{\{z_i,y_j\} \subset V(S_n)} d(z_i,y_j) + \frac{1}{2} \sum_{\{z_i,x_j\} \subset V(S_n)} d(z_i,x_j) + \frac{1}{2} \sum_{\{z_i,w_j\} \subset V(S_n)} d(z_i,w_j + \frac{1}{2} \sum_{\{y_i,x_j\} \subset V(S_n)} d(y_i,x_j)$$

$$+ \frac{1}{2} \sum_{\{y_i,w_j\} \subset V(S_n)} d(y_i,w_j) + \frac{1}{2} \sum_{\{x_i,w_j\} \subset V(S_n)} d(x_i,w_j)$$

$$= \textstyle\sum_{i=1}^{10} I_i. \tag{8}$$

Where

$$I_1 = \frac{1}{2} \sum_{\{z_i,z_j\} \subseteq Z} d(z_i,z_j),\ I_2 = \frac{1}{2} \sum_{\{y_i,y_j\} \subseteq Y} d(y_i,y_j)$$

$$I_3 = \frac{1}{2} \sum_{\{x_i,x_j\} \subseteq X} d(x_i,x_j),\ I_4 = \frac{1}{2} \sum_{\{w_i,w_j\} \subseteq W} d(w_i,w_j)$$

$$I_5 = \frac{1}{2} \sum_{\{z_i,y_j\} \subset V(S_n)} d(z_i,y_j),\ I_6 = \frac{1}{2} \sum_{\{z_i,x_j\} \subset V(S_n)} d(z_i,x_j)$$

$$I_7 = \frac{1}{2} \sum_{\{z_i,w_j\} \subset V(S_n)} d(z_i,w_j),\ I_8 = \frac{1}{2} \sum_{\{y_i,x_j\} \subset V(S_n)} d(y_i,x_j)$$

$$I_9 = \frac{1}{2} \sum_{\{y_i,w_j\} \subset V(S_n)} d(y_i,w_j),\ I_{10} = \frac{1}{2} \sum_{\{x_i,w_j\} \subset V(S_n)} d(x_i,w_j)$$

Since, each layer of S_n is a cycle of even length. Therefore, by Lemma B, we have

$$I_1 = I_2 = I_3 = I_4 = \frac{n^3}{8}. \tag{9}$$

It can be easily verified that $d(z_i,y_i) = 1$ and $d(z_i,y_j) = d(z_i,y_i) + d(y_i,y_j)$ for $2 \leq i \leq n$. Therefore,

$$I_4 = \frac{1}{2} \sum_{\{z_i,y_j\} \subset V(R_n)} d(z_i,y_j)$$

$$\leq 1+\frac{1}{2}\sum_{\{y_i,y_j\}\subseteq Y} d(y_i,y_j)$$

$$=\frac{n^3+8}{8}. \tag{10}$$

Similar argument holds for I_8 and I_{10}. Therefore,

$$I_8 \leq \frac{n^3+8}{8}. \tag{11}$$

$$I_{10} \leq \frac{n^3+8}{8}. \tag{12}$$

Since, $d(z_i,x_i)=2$ and $d(z_i,x_j)=d(z_i,x_i)+d(x_i,x_j)$. Therefore,

$$I_6=\frac{1}{2}\sum_{\{z_i,x_j\}\subset V(R_n)} d(z_i,x_j)$$

$$\leq 2+\frac{1}{2}\sum_{\{x_i,x_j\}\subseteq X} d(x_i,x_j)$$

$$\leq \frac{n^3+16}{8}. \tag{13}$$

Since, $d(z_i,w_i)=d(z_i,w_{i+1})=3$ and $d(z_i,w_j)=d(z_i,w_i)+d(w_i,w_j)$ or $d(z_i,w_j)=d(z_i,w_{i+1})+d(w_{i+1},w_j)$. Therefore,

$$I_7=\frac{1}{2}\sum_{\{z_i,w_j\}\subset V(R_n)} d(z_i,w_j)$$

$$\leq 3+\frac{1}{2}\sum_{\{w_i,w_j\}\subseteq W} d(w_i,w_j)$$

$$=3+\frac{1}{2}\sum_{\{w_i,w_j\}\subseteq W} d(w_{i+1},w_j)$$

$$\leq \frac{(n-2)^3+24}{8}. \tag{14}$$

Since, $d(y_i,w_i)=d(y_i,w_{i+1})=2$ and $d(y_i,w_j)=d(y_i,w_i)+d(w_i,w_j)$ or $d(y_i,w_j)=d(y_i,w_{i+1})+d(w_{i+1},w_j)$. Therefore,

$$I_9=\frac{1}{2}\sum_{\{y_i,w_j\}\subset V(R_n)} d(y_i,w_j)$$

$$\leq 2+\frac{1}{2}\sum_{\{w_i,w_j\}\subseteq W} d(w_i,w_j)$$

$$= 2 + \frac{1}{2} \sum_{\{w_i, w_j\} \subseteq W} d(w_{i+1}, w_j)$$

$$\leq \frac{(n-2)^3 + 16}{8}. \tag{15}$$

Using Equations (9), (10), (11), (12), (13), (14) and (15) in Equation (8) and replacing $W(S_n)$ in Equation (7) we get the desired result.

Theorem 4.3. The modified detour index of spider web network $SW(r,s)$ is given by

$$\omega^c(SW(r,s)) \leq (\tfrac{1}{4}[r^2 + 2(r+s) - 2] + \tfrac{1}{3}[(s+1)(s^2+2s) + \tfrac{1}{3}(s-1)(s^2-2s)] + \tfrac{1}{3}[s(s^2-1)]) \times (\tfrac{1}{4}[rs(rs-2)(rs-1)] + \tfrac{1}{16}[(rs)^2(rs-2)]).$$

Proof. By definition of spider web network $SW(r,s)$, it has $|V(SW(r,s))| = rs$. Since $SW(r,s)$ is a connected 3-regular bipartite graph. Therefore, $|E(SW(r,s))| = \frac{3rs}{2}$. The structure of $SW(r,s)$ is described as follows: It has an inner cycle

$C_1 = \langle (0,0), (1,0), (2,0), \cdots, (m-1,0), (0,0) \rangle$

and an outer cycle

$C_2 = \langle (0,n-1), (1,n-1), (2,n-1), \cdots, (m-1,n-1), (0,n-1) \rangle$

It has the following pats of length $s-1$:

$P_{0,0} = \{(0,0), (0,1), (0,2), \cdots, (0,s-1)\}$

$P_{1,0} = \{(1,0), (1,1), (1,1), \cdots, (1,s-1)\}$

$P_{2,0} = \{(2,0), (2,1), (2,2), \cdots, (2,s-1)\}$

$\vdots$

$P_{r-1,s-1} = \{(r-1,0), (r-1,1), (r-1,2), \cdots, (r-1,s-1)\}$

With this structural information, consider the modified detour index of $SW(r,s)$:

$\omega^c(SW(r,s)) = \frac{1}{2} \sum_{\{u,v\} \subseteq V(SW(r,s))} l(u,v) d(u,v)$

$$\leq (\frac{1}{2} \sum_{\{u,v\} \subseteq V(SW(r,s))} d(u,v))(\frac{1}{2} \sum_{\{u,v\} \subseteq V(SW(r,s))} l(u,v)). \tag{16}$$

Now we can solve RHS of Equation (16) separately.

$$W(SW(r,s)) = \frac{1}{2} \sum_{\{u,v\} \subseteq V(SW(r,s))} d(u,v) \tag{17}$$

Case 1. Let $v_i, v_j \in C_1$, Then

$$\frac{1}{2} \sum_{\{v_i, v_j\} \subseteq C_1} d(v_i, v_j) = W(C_r) = \frac{r^3}{8}$$

Case 2. Let $v_i, v_j \in C_2$, Then

$$\frac{1}{2} \sum_{\{v_i, v_j\} \subseteq C_2} d(v_i, v_j) = W(C_r) = \frac{r^3}{8}$$

Case 3. Let $v_i \in C_1$ and $v_j \in P_{0,0}$, Then

$$\frac{1}{2} \sum_{\{v_i, v_j\} \subset V(SW(r,s))} d(v_i, v_j) = W(P_s) = \frac{1}{6}[s(s^2-1)]$$

Case 4. Let $v_i \in C_1$ and $v_j \in P_{0,1}$, Then

$$\frac{1}{2}\sum_{\{v_i,v_j\}\subset V(SW(r,s))} d(v_i,v_j) = W(P_{s+1}) = \frac{1}{6}[(s+1)(s^2+2s)$$

Case 5. Let $v_i \in C_1$ and $v_j \in P_{r-1,s-1}$, Then

$$\frac{1}{2}\sum_{\{v_i,v_j\}\subset V(SW(r,s))} d(v_i,v_j) = W(P_{s+1}) = \frac{1}{6}[(s+1)(s^2+2s)]$$

Case 6. Let $v_i, v_k \in C_1$ and $v_j \in P_{x,y}$; $0 \le x \le r-1$ and $0 \le y \le r-1$ Then

$$\frac{1}{2}\sum_{\{v_i,v_j\}\subset V(SW(r,s))} d(v_i,v_k)+d(v_k,v_j) \le \frac{r}{4}+W(s) = \frac{r}{4}+\frac{1}{6}[s(s^2-1)]$$

Case 7. Let $v_i, v_j \in P_{x,y}$; $0 \le x \le r-1$ and $0 \le y \le r-1$ Then

$$\frac{1}{2}\sum_{\{v_i,v_j\}\subset V(SW(r,s))} d(v_i,v_j) \le W(P_{s-1}) = \frac{1}{6}[(s-1)(s^2-2s)]$$

Case 8. Let $v_i \in P_{x,y}$ and $v_j \in P_{x',y'}$; $0 \le x \le r-1$, $0 \le y \le r-1$, $0 \le x' \le r-1$ and $0 \le y' \le r-1$. Further, $v_k, v_l \in C_1$, $x \ne x'$ and $y \ne y'$. Then

$$\frac{1}{2}\sum_{\left(v_i,v_j\right\}\subset V(SW(r,s))} d\left(v_i,v_k\right)+d\left(v_k,v_l\right)+d\left(v_l,v_j\right) \le \frac{1}{2}\left[(s-1)+\frac{r}{2}\right]+W(P_{s-1})$$

$$= \frac{2(s-1)+r}{4}\frac{1}{6}[(s-1)(s^2-2s)].$$

Since $SW(r,s)$ is a 3-regular bipartite graph. Therefore, $V(SW(r,s)) = V_1 \cup V_2$. It has been proved that $SW(r,s)$ is Hamiltonian-laceable graph. Therefore, the number of Hamiltonian paths between the vertices v_i and v_j at odd distance is at most $\frac{1}{4}[rs(rs-2)(rs-1)]$. Also observe that $diam(SW(r,s)) \le \frac{rs}{2}$. Therefore, the longest paths between the vertices v_i and v_j at even distance is at most $\frac{1}{16}[(rs)^2(rs-2)]$. Therefore, we conclude that

$$\frac{1}{2}\sum_{\{u,v\}\subseteq V(SW(r,s))} l(u,v) \le \frac{1}{4}[rs(rs-2)(rs-1)]+\frac{1}{16}[(rs)^2(rs-2)]. \tag{18}$$

Thus, combining Cases 1 to 8 we get Equation (17) after simplifying and by putting Equation (18) in Equation (16) we get the desired result.

Theorem 4.4. The modified detour index of image graph of prism $I_m(Y_n)$ is given by
$\omega^c(I_m(Y_n)) \le (\frac{n}{4}(5n^2+16)) \times (2n(2n-1)(n-1)+n(2n-1)(n+3))$.

Proof. By definition of image graph of prism $I_m(Y_n)$, it has $|V(I_m(Y_n))| = 4n$. Since $I_m(Y_n)$ is a connected 4-regular bipartite graph. Therefore, $|E(I_m(Y_n))| = 8n$. The structure of $I_m(Y_n)$ is described as follows: It has the following cycles of length n:

$C_1 = \{v_1, v_2, v_3, \cdots, v_{n-1}, v_n = v_1\}$
$C_2 = \{u_1, u_2, u_3, \cdots, u_{n-1}, u_n = u_1\}$
$C_3 = \{v'_1, v'_2, v'_3, \cdots, v'_{n-1}, v'_n = v'_1\}$
$C_4 = \{u'_1, u'_2, u'_3, \cdots, u'_{n-1}, u'_n = u'_1\}$

With this structural information, consider the modified detour index of $I_m(Y_n)$:
$\omega^c(I_m(Y_n)) = \frac{1}{2}\sum_{\{u,v\}\subseteq V(I_m(Y_n))} l(u,v)d(u,v)$

$$\le (\frac{1}{2}\sum_{\{u,v\}\subseteq V(I_m(Y_n))} d(u,v))(\frac{1}{2}\sum_{\{u,v\}\subseteq V(I_m(Y_n))} l(u,v)). \tag{19}$$

Now we solve RHS of Equation (19) separately.

$$W(I_m(Y_n)) = \frac{1}{2}\sum_{\{u,v\}\subseteq V(I_m(Y_n))} d(u,v) \tag{20}$$

Case 1. Let $v_i, v_j \in C_i$, $1 \le i \le 4$. Then
$\frac{1}{2}\sum_{\{v_i,v_j\}\subset I_m(Y_n)} d(v_i,v_j) = W(C_i) = \frac{n^3}{8}$
Case 2. $\frac{1}{2}\sum_{\{v_i,u_j\}\subset I_m(Y_n)} d(v_i,u_j) = \frac{1}{2}\sum_{\{v_i,u_j\}\subset I_m(Y_n)} d(v_i,u_i) + d(u_i,u_j)$
$\le \frac{n}{2} + \frac{n^3}{8}$
$= \frac{n^3+4n}{8}$
Similar argument holds for $d(v_i,v'_j)$, $d(u_i,u'_j)$ and $d(v'_i,u'_j)$.
Case 3. $\frac{1}{2}\sum_{\{v_i,u'_j\}\subset I_m(Y_n)} d(v_i,u'_j) = \frac{1}{2}\sum_{\{v_i,u'_j\}\subset I_m(Y_n)} d(v_i,u_i) + d(u_i,u'_i) + d(u'_i,u'_j)$
$\le n + \frac{n^3}{8}$
$= \frac{n^3+8n}{8}$
Similar argument holds for, $d(u_i,v'_j)$. By using these cases we get

$$W(I_m(Y_n)) \le \frac{n}{4}(5n^2+16). \tag{21}$$

Since $I_m(Y_n)$ is a 4-regular bipartite graph. Therefore, $V(I_m(Y_n)) = V_1 \cup V_2$. In (14), it has been proved that $I_m(Y_n)$ is Hamiltonian-laceable graph. Therefore, the number of Hamiltonian paths between the vertices v_i and v_j at an odd distance is at most $2n(2n-1)(n-1)$. Also observe that $diam(I_m(Y_n)) = 2 + \frac{n-1}{2} = \frac{n+3}{2}$. Therefore, the longest paths between the vertices v_i and v_j at even distance is at most $n(2n-1)(n+3)$. Therefore, we conclude that

$$\frac{1}{2}\sum_{\{u,v\}\subseteq V(I_m(Y_n))} l(u,v) \le 2n(2n-1)(n-1) + n(2n-1)(n+3). \tag{22}$$

Employing Equations (21) and (22) in Equation (19) we get the desired result.

4 Conclusion

The modified detour index of certain class of Hamiltonian connected n-dimensional polytopes and Hamiltonian laceable graphs such as spider web network and image of prism graphs has been covered in this work .This study concludes by posing the following open problems:

Open Problem 1. Characterize Hamiltonian-laceable graphs with $\omega(G) = \omega'(G)$.
Open Problem 2. Characterize Hamiltonian-laceable graphs with $\omega'(G) = W(G)$.

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