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RESEARCH ARTICLE

Minimum Degree Energy of Graphs and Pebbling Graphs

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Abstract

Objectives: The minimum degree energy concept is applied to pebbling graphs. This study establishes a connection between the minimum degree energy of graphs and the minimum degree energy of pebbling graphs by applying the lowest degree energy of pebbling notion to twenty standard graphs. **Methods:** The minimum degree energy of pebbling graph with the matrix whose (i,j) was calculated using $mDEP(G) = \begin{cases} min\{v_i, v_j\} \; if \; i \neq j, v_i \; and \; v_j \text{ are adjacent} \\ 0 \qquad \text{otherwise} \end{cases}$. The characteristic polynomial of the minimum degree, $mDEP(G)$ is $det(\lambda I - mD(G))$ must be found from the matrix. Next, the eigen values of the matrix were calculated using $|M(G) - \lambda I| = 0$ and the sum of all the eigen values gives the minimum degree energy. The lower and upper bounds for the minimum degree energy of graphs are established along with the algorithm for computing the minimum degree energy of graphs. **Findings:** The lower and upper bounds were found for the minimum degree energy of pebbling graphs. For twenty standard graphs and pebbling graphs, the minimum degree energy values were calculated, and their relation was tabulated. **Novelty:** Pebbling graphs were subjected to the minimal energy idea and a relationship was found between the minimum energy of pebbling graphs and the minimum energy of graphs in general.

Keywords: Energy; Minimum degree energy; Grotzsch graph; Pebbling graph; Data mining

1 Introduction

Graph pebbling was invented in 1956 by Erdos, as a way for answering Erdos and Lemke's number theory conjecture[(1)]. When two pebbles are transformed from one vertex to another vertex in a graph G, one pebble is lost as fuel during transit and only one pebble reaches the end vertex is called pebbling. The pebbling step examines the cost of pebble loss, and it has been the topic of substantial and in-depth research in the context. This is a fascinating topic that has piqued the interest of academics all across the world. The concept of energy has been applied to many graphs such as bipolar fuzzy graphs was established by Akram M, Sarwar M, Dudek WA[(2)], whereas the energy of m-polar fuzzy digraphs and the laplacian energy of interval valued fuzzy graphs [3] was also established. The author expanded the idea of adjacency energy to include pebbling graphs [4] in 2021. Minimum covering maximum degree energy of graphs are found and also their bounds were evaluated by Raju Kotambari, Sudhir Jog [5]. Then the concept of minimum degree energy was applied to p-splitting graphs and p-shadow graphs [6], spectral radii of some graphs by Xiujun Zhang, Ahmad Bilal, Mobeen Munir, Hafiz Mutte ur Rehman [7]. Sarah and Mathew applied the concept of secure domination and found cover pebbling number of graphs [8]. Recently, study on edge pebbling number, covering cover edge pebbling number for friendship graphs was found by Paul P, Fathima SSA [9]. The authors Isaak G and Prudente, M. described the two-player pebbling game on diameter 2 graphs [10]. In 2023, Veena Mathad and Anand introduced the Maximum Hub Degree Energy for double star graphs [11]. The author in this paper used the concept of minimum degree energy on pebbling graphs and evaluated their energy values. Standard graphs, fuzzy graphs, intuitionistic fuzzy graphs, and hyper graphs were all given the energy concept's application; pebbling graphs, as documented in the body of current research, were not taken into consideration. This article establishes a new notion of the lowest degree energy of pebbling graphs, and [Table 1] discusses a comparison of these concepts.

2 Methodology

2.1 Preliminaries

2.1.1 The minimum degree energy of a graph

Let G be an n-dimensional graph. The minimum degree matrix, $mDE(G)$, if its $(i,j)^{th}$ entry is $min\{d_i,d_j\}$ whenever $i\neq j$, and zero otherwise, where d_i and d_j are the degrees of i^{th} and j^{th} vertices of G.

$$mDE(G)=\begin{cases} min\{d_i,d_j\} & if\ i\neq j, v_i\ and\ v_j\ are\ adjacent \\ 0 & \text{otherwise} \end{cases}$$

2.1.2 Minimum Degree Energy of pebbling graphs

The minimum degree energy of pebbling graph is given by,

$$mDEP(G)=\begin{cases} min\{v_i,v_j\} & if\ i\neq j, v_i\ and\ v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

2.2 Algorithm for finding Minimum degree energy of pebbling graphs

Step 1: To obtain the minimum degree energy of pebbling graph matrix, use any graph G with vertices and edges.

Step 2: The minimum degree of each vertex is then determined, and the matrix is formed.

Step 3: Now the minimum degree energy of pebbling graph matrix whose (i,j) element is

$$mDEP(G)=\begin{cases} min\{v_i,v_j\} & if\ i\neq j, v_i\ and\ v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

Step 4: The characteristic polynomial of the minimum degree energy of pebbling graph $mDEP(G)$ is $det(\lambda I - mDEP(G))$ must be found from the matrix.

Step 5: Next, we calculate the eigen values of the matrix, using $|M(G)-\lambda I)|=0$.

Step 6: The sum of all the eigen values gives the minimum degree energy of pebbling graph

$$mDEP(G)=\sum_{i=1}^{n}|\lambda_i|$$

3 Results and Discussion

3.1 Basic Properties of minimum degree energy of pebbling graph

Define the number H as

$$L=min(v_i,v_j)\ if\ i\ and\ j \text{ are adjacent.}$$

Preposition 1

The polynomial's first three coefficients $\phi_{mDP}(G,\lambda)$ are as follows.
(i) $a_0 = 1$
(ii) $a_1 = 0$
(ii) $a_2 = -L$
Proof:
(i) By definition of $\phi_{mDP}(G,\lambda) = det[\lambda I - mDP(G)]$, we get $a_0 = 1$.
(ii) The trace of $R(G)$ equals the sum of determinants of all 1×1 principal sub matrices of $R(G)$ meaning that $a_1 = (-1)^1 \times$ the trace of $R(G) = 0$.
(iii) By the definition of $\phi_{mDP}\ (G,\lambda)$,

$$(-1)^2 a_2 = \sum_{1\le i\le j\le n} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix}$$

$$= \sum_{1\le i\le j\le n}\ a_{ii}a_{jj} - a_{ji}a_{ij}$$

$$= \sum_{1\le i\le j\le n}\ a_{ii}a_{jj} - \sum_{1\le i\le j\le n}\ a_{ji}a_{ij}$$

$$= -L$$

Hence, $a_2 = -L$
Preposition 2
If $\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n$ be the reciprocal eigen values of $mDP(G)$, then

$$\sum_{i=1}^{n} \lambda_i^2 = 2L$$

Proof:
It follows that,

$$\sum_{i=1}^{n}\ \lambda_i^2 = \sum_{i=1}^{n}\ \sum_{j=1}^{n}\ a_{ij}a_{ji}$$

$$= 2\sum_{i<j}\ (a_{ij})^2 + \sum_{i=1}^{n}\ (a_{ii})^2$$

$$= 2\sum_{i<j}\ (a_{ij})^2$$

$$= 2L$$

The lower and upper bounds on the minimum degree energy of pebbling graphs are obtained using the above result.
Theorem 3
Let G be an n-vertices pebbling graph. Then $mDEP(G) \le \sqrt{2nL}$.
Proof:
Let $\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n$ be the eigen values of $mDP(G)$. From the Cauchy-Schwartz inequality,

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

Let $a_i = 1, b_i = |\lambda_i|$. Then

$$\left(\sum_{i=1}^{n} |\lambda_i|\right)^2 \le \left(\sum_{i=1}^{n} 1\right)\left(\sum_{i=1}^{n} |\lambda_i|^2\right)$$

Implying that

$$[mDEP(G)]^2 \le n.2L$$

Hence, we get

$$mDEP(G) \leq \sqrt{2nL}$$

This is the upper limit.

Theorem 4

Let G be $n-verticed$ pebbling graph. If $S = det(mDEP(G))$, then

$$mDEP(G) \geq \sqrt{2L + n(n-1)S^{\left(\frac{2}{n}\right)}}$$

Proof:

By definition, we have

$$(mDEP(G))^2 = \left(\sum_{i=1}^{n} |\lambda_i|\right)^2$$

$$= \sum_{i=1}^{n} |\lambda_i| \sum_{j=1}^{n} |\lambda_j|$$

$$= \left(\sum_{i=1}^{n} |\lambda_i|^2\right) + \sum_{i \neq j} |\lambda_i| |\lambda_j|$$

By the arithmetic-geometric mean inequality,

$$\frac{1}{n(n-1)} \sum_{i \neq j} |\lambda_i| |\lambda_j| \geq \left(\prod_{i \neq j} |\lambda_i| |\lambda_j|\right)^{\frac{1}{n(n-1)}}$$

Hence,

$$(R(G))^2 \geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1) \left(\prod_{i \neq j} |\lambda_i| |\lambda_j|\right)^{\frac{1}{n(n-1)}}$$

$$\geq \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1) \left(\prod_{i=1}^{n} |\lambda_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}}$$

$$= \sum_{i=1}^{n} |\lambda_i|^2 + n(n-1)S^{\left(\frac{2}{n}\right)}$$

$$= 2L + n(n-1)S^{\left(\frac{2}{n}\right)}$$

$$mDPE(G) \geq \sqrt{2L + n(n-1)S^{\left(\frac{2}{n}\right)}}$$

Thus,

3.2 Relation between minimum degree energy of graphs and minimum degree energy of pebbling graphs

Theorem 5

Prove that in the Grotzsch graph, the minimum degree energy is greater than the minimum degree energy of pebbling graph. (i.e) $mDE(G) > mDEP(G)$.

Proof:

The Grotzsch graph is $G(V,E)$. It is a triangle-free graph and named after Herbert Grotzsch, a German mathematician. The minimum degree matrix is created and the eigen values are determined using the notion of minimum degree energy. Similarly, the least degree energy matrix is generated and eigen values are determined using the notion of minimum degree energy of pebbling graph. When comparing the two results, $mDE(G) > mDEP(G)$.

Example

Prove that in the Grotzsch graph. $mDE(G) > mDEP(G)$.

Solution:

[Figure 1] shows the Grotzsch graph with vertices $= \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}\}$. The matrix energy, and it (i, j) is

$$mDE(G) = \begin{cases} min\{d_i, d_j\} \; if\ i \neq j, v_i\ and\ v_j \text{ are adjacent} \\ \text{0 otherwise} \end{cases}$$

Then the characteristic polynomial is

$$M(G) = \lambda^{11} - 215\lambda^9 + 13025\lambda^7 - 24008\lambda^6 - 270225\lambda^5 + 1077120\lambda^4 - 462105\lambda^3 - 2566080\lambda^2 + 1568079\lambda + 2361960$$

Adding all eigen values, $\sum_{i=1}^{11} |\lambda_i| = 56.1138$.

The minimum degree energy of pebbling graph is given by,

$$mDEP(G) = \begin{cases} min\{v_i, v_j\} & if\ i \neq j, v_i\ and\ v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

The pebbling Grotzsch graph with eleven vertices (i.e., $'V_1'$ through $'V_{11}'$) is laid out in [Figure 2]. The transit is originates at $'V_1'$ and should terminate at $'V_2'$, $'V_5'$, ' $'V_7'$ and $'V_{10}'$. At $'V_5'$, the graph diverges into three paths each to terminate at $'V_4'$, $'V_6'$ and $'V_9'$. At $'V_2'$, the graph diverges into three paths each to terminate at $'V_3'$, $'V_6'$ and $'V_8'$. At $'V_6'$, the graph terminates at $'V_{11}'$. Despite there is a connection between $'V_3'$ and $'V_4'$, yet there is no transit between them.

Initially twenty-eight pebbles are present at vertex $'V_1'$. The transit takes place from $'V_1'$ to $'V_5'$ losing six pebbles as fuel and reaches $'V_5'$ with six pebbles, likewise, the transit takes place from $'V_1'$ to $'V_2'$ losing six pebbles as fuel and reaches $'V_2'$ with six pebbles and also the vertices $'V_{10}'$ and $'V_7'$, each gets one pebble after losing one pebble as fuel respectively. During transit at $'V_5'$, the graph diverges into three paths each to terminate at $'V_4'$, $'V_6'$ and $'V_9'$, each gets one pebble after losing one pebble as fuel respectively. Similarly, at $'V_2'$, the graph diverges into three paths each to terminate at $'V_3'$, $'V_6'$ and $'V_8'$, the vertices $'V_6'$ and $'V_8'$ each gets one pebble after losing one pebble as fuel, but from $'V_2'$ to $'V_3'$, two pebbles are lost during transit and the vertex $'V_3'$ get two pebbles. Finally, the vertex $'V_{11}'$ gets one pebble from vertex $'V_6'$ one pebble is lost as fuel.

The pebbling Grotzsch graph is depicted in [Figure 2]. The minimum degree energy of pebbling matrix is shown as

$$M = \begin{bmatrix} 0 & 6 & 0 & 0 & 6 & 0 & 1 & 0 & 0 & 1 & 0 \\ 6 & 0 & 2 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 6 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Then the characteristic polynomial is

$$M(G) = \lambda^{11} - 98\lambda^9 + 1066\lambda^7 - 590\lambda^6 - 2999\lambda^5 + 3516\lambda^4 - 464\lambda^3 - 598\lambda^2 + 105\lambda + 36$$

Adding all the eigen values, $\sum_{i=1}^{11} |\lambda_i| = 30.2793$. Comparing both the results, it is observed that $mDE(G) > mDEP(G)$.

[Table 1] presents the comparison of the minimum degree energy values and minimum degree pebbling energy values of twenty standard graphs. The concept of energy was applied to standard graphs, fuzzy graphs, intuitionistic fuzzy graphs, hyper graphs; however, it was not addressed to pebbling graphs as reported in the existing literature. A novel concept of minimum degree energy of pebbling graphs has been established in this article and its comparison was discussed in [Table 1].

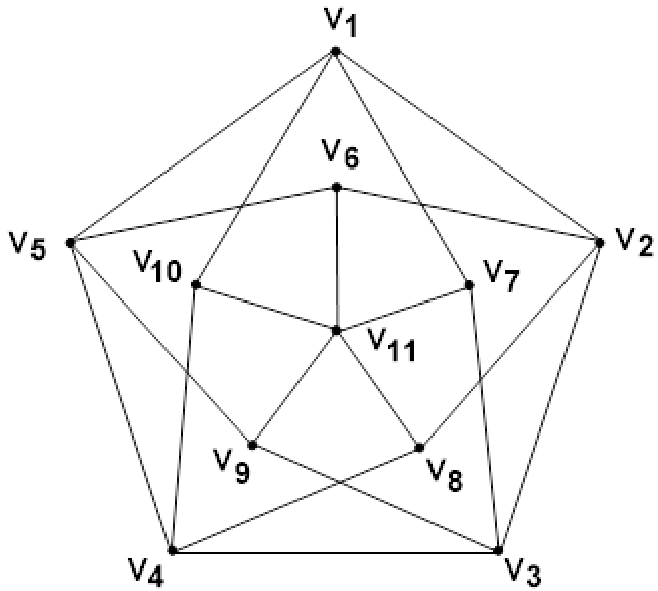

Fig 1. Grotzsch graph

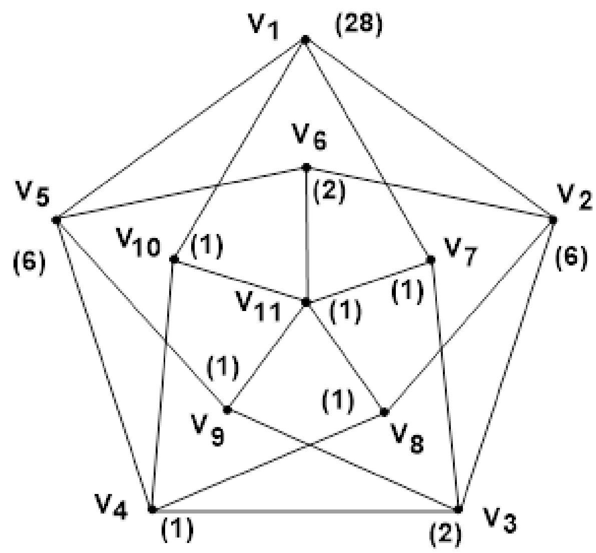

Fig 2. Pebbling Grotzsch graph

Table 1. Minimum degree energy value of graphs and pebbling graphs

S. No	Name of Graphs	Minimum Degree Energy, mDE(G)	Minimum Degree Energy of Pebbling graphs, mDEP(G)
1	Chavtal graph	76.4924	34.5708
2	Grotzsch graph	56.1138	30.2793
3	Johnson Solid graph	47.0612	24.5335
4	(3,3) King's tour graph	55.4340	33.1029
5	Herschel graph	51.6981	57.0351
6	Cub octahedral graph	80	41.9983
7	Pent tope graph	32	10.7446

Continued on next page

Table 1 continued

8	Bull graph	10.3138	9.1530
9	Golomb graph	52.0549	25.8480
10	Diamond graph	11.5440	6.5461
11	Square of cycle graph	54.6276	20.6263
12	Hexahedral graph	38.5696	17.6437
13	Wagner graph	34.4377	21.5601
14	Lattice graph	48	17.6462
15	Seifert graph	66.0245	23.5086
16	Octahedral graph	32	9.7482
17	Prism graph	43.0880	29.9076
18	Planar graph	24.7332	8.5942
19	Fan graph	14.5830	9.2112
20	Moore graph	33.5270	18.5162

4 Conclusion

This study presents the study of minimum degree energy of pebbling graph. It was found that the upper and lower constraints exist for the minimum degree of pebbling graphs. The relationship between the minimum degree energy of graphs and the minimum degree energy of pebbling graphs was discovered when this notion was applied to twenty standard graphs. The result, $mDE(G) > mDEP(G)$ is true for 19 graphs as shown in [Table 1]; however, in Herschel graph $mDE(G) < mDEP(G)$. This study can also be expanded to numerous sorts of pebbling graph energies. In theoretical computer science, the term "energy pebbling" refers to the movement of energy pebbles on graphs, each of which carries a specific quantity of energy. Resonance energy, enthalpies of compound formation, and enthalpies of reaction are all calculated using bond energies. Weighted graphs are extensively utilized in computer science to build data mining, software testing, image processing, communication networks, and information security. Further, algorithm design, distributed computing, and communication difficulty are just a few of the computer science domains where energy pebbling has significance. Its interconnections to other theoretical ideas, such reversible computing and space-time trade offs in computation may be investigated. As a useful tool in theoretical computer science research, energy pebbling offers a framework for examining the computational and resource constraints of specific graph-based issues.

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