INDIAN JOURNAL OF SCIENCE AND TECHNOLOGY

RESEARCH ARTICLE

Runge-Kutta Method Based on the Linear Combination of Arithmetic Mean, Geometric Mean, and Centroidal Mean: A Numerical Solution for the Hybrid Fuzzy Differential Equation

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OPEN ACCESS

Received: 09-01-2024

Accepted: 07-04-2024

Published: 30-04-2024

Citation: Rajakumari PED, Sharmila RG (2024) Runge-Kutta Method Based on the Linear Combination of Arithmetic Mean, Geometric Mean, and Centroidal Mean: A Numerical Solution for the Hybrid Fuzzy Differential Equation. Indian Journal of Science and Technology 17(18): 1860-1867. https://doi.org/10.17485/IJST/v17i18.75

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Funding: None

Competing Interests: None

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Published By Indian Society for Education and Environment (iSee)

ISSN
Print: 0974-6846
Electronic: 0974-5645

Abstract

Objectives: To find the absolute error of the Hybrid fuzzy differential equation with triangular fuzzy number since its membership function has a triangular form. **Methods:** The third order runge-kuttta method based on the linear combination of Arithmetic mean, Geometric mean and Centroidal mean is introduced to solve Hybrid fuzzy differential equation. Seikkala's derivatives are considered, and Numerical example is given to demonstrate the effectiveness of the proposed method. **Findings:** The comparative study have been done between the proposed method and the existing third order runge-kutta method based on Arithmetic mean. The proposed method gives better error tolerance than the third order runge-kutta method based on Arithmetic mean. **Novelty:** In this study a new formula has been developed by combining three means Arithmetic Mean, Geometric Mean and Centroidal Mean using Khattri's formula and it is solved by the third order runge-kutta method for the first order hybrid fuzzy differential equation. The error tolerance has been calculated for the proposed method and the Arithmetic mean and it is seen that the error tolerance is better for the proposed method.

Keywords: Hybrid fuzzy differential equations; Triangular fuzzy number; Seikkala's derivative; Third order runge-kutta method; Initial value problem

1 Introduction

Control systems that are capable of controlling complex systems which have discrete time dynamics and continuous time dynamics can be modeled by hybrid systems. The differential systems containing fuzzy valued functions and interaction with a discrete time controller are named as hybrid fuzzy differential systems.

A brief analysis and interpretation on arithmetic operations of fuzzy numbers has been done[1]. A Third order Runge-kutta method based on linear combination of three means of various combinations have been proposed before[2–4]. A method is proposed in[5] to solve fuzzy Initial value problem using explicit runge-kutta method. In[6], compared various derivatives of fuzzy functions and arrived at the conclusion that Seikkala's solution is the most general solution to the fuzzy initial value problem. Arithmetic operations of Trigonal fuzzy numbers using Alpha cuts has been proposed in[7]. Iterative methods have been proposed in[8]. The Analysis and interpretation of Malaria disease model in crisp and fuzzy environment has been done in[9]. Numerical Solutions of Fuzzy Multiple Hybrid Single Retarded Delay Differential Equations have been reported in[10]. In[11], proposed a method to find the Numerical Solutions of Fuzzy Pure Multiple Neutral Delay Differential Equations. Understanding the Unique Properties of Fuzzy concept in Binary Trees has been done in[12]. A Fuzzy Fractional Power Series Approximation and Taylor Expansion for Solving Fuzzy Fractional Differential Equation has been done in[13]. In[14], proposed a technique to resolve transportation problem by Trapezoidal fuzzy numbers. In[15], proposed a Type-2 fuzzy differential inclusion for solving type-2 fuzzy differential equation.

In this study we have used Seikkala's derivative to find the most general solution to the fuzzy initial value problem and a new equation has been developed by combining three means Arithmetic Mean, Geometric Mean and Centroidal Mean using Khattri's formula and it is solved by the third order runge-kutta method for the first order hybrid fuzzy differential equation. The comparative study has been done between the proposed method and the existing third order runge-kutta method based on Arithmetic mean. The proposed method gives better error tolerance than the third order runge-kutta method based on Arithmetic mean and it has been shown in Table 1 and Figure 1.

2 Preliminaries

Definition: 2.1 (Fuzzy Set)

If X is a collection of objects denoted generally by x, then a fuzzy set $\widetilde{A}$ in X is a set of ordered pairs $\widetilde{A} = \left\{ \left(x, \mu_{\widetilde{A}(x)} \right) / x \in X \right\}$, where $\mu_{\widetilde{A}(x)}$ is called the membership function or grade of membership of x in $\widetilde{A}$.

Definition: 2.3 (Triangular fuzzy number)

It is a fuzzy number represented with three points as follows: $\widetilde{A} = (a_1, a_2, a_3)$, this representation is interpreted as membership functions and holds the following condition:

(i) a_1 to a_2 is increasing function
(ii) $a_2$to $a_3$is decreasing function
(iii) $a_1 \leq a_2 \leq a_3$

$$\mu_{\widetilde{A}} \quad (x) = \begin{cases} 0 & , x < a_1 \\ \frac{x-a_1}{a_2-a_1} & , a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2} & , a_2 \leq x \leq a_3 \\ 0 & , x > a_3 \end{cases}$$

Now, if you get crisp interval by α - cut operation, interval $\widetilde{A}_\alpha$ shall be obtained as follows,$\forall \alpha \in [0,1]$ from

$$\frac{a_1^{(\alpha)} - a_1}{a_2 - a_1} = \alpha, \ \frac{a_3 - a_3^{(\alpha)}}{a_3 - a_2} = \alpha$$

The Third order runge-kutta method based on a linear combination of Arithmetic mean (AM), Geometric mean (GeM) and Centroidal mean (CeM) for IVP was introduced by Khattri as follows:

$$Y_{n+1}(k_1,k_2) = \frac{14AM(k_1,k_2) - GeM(k_1,k_2) + 32CeM(k_1,k_2)}{45}$$

3 The Hybrid Fuzzy Differential Equations

The hybrid fuzzy differential equation

$$\begin{aligned} &\dot{y}(x) = f(t, y(t), \lambda_k(y_k)), t \in [t_k, t_{k+1}] \ k = 0,1,2,\ldots \\ &y(t_0) = y_0 \end{aligned} \quad (3.1)$$

where $t_k = 0$ is strictly increasing and unbounded, y_k denotes $y(t_k)$, $f : (t_0, \infty) \times R \times R \to R$ is continuous, and each $\lambda_k : R \to R$ is a continuous function. A solution to (Equation (3.1)) will be a function $y : (t_0, \infty) \to R$ satisfying (Equation (3.1)). For

$k = 0,1,2,3,\ldots$, let $f_k : (t_k, t_{k+1}) \times R \to R$ where, $f(t, y_k(t)) = f(t, y_k(t), \lambda_k(y_k))$. The hybrid fuzzy differential equation in (Equation (3.1)) can be written in expanded form as

$$\dot{y}(t) = \begin{cases} \dot{y}_0(t) = f(t, y_0(t), \lambda_0(y_0)) = f_0(t, y_0(t)), y_0(t_0) = y_0, t_0 \le t \le t_1 \\ \dot{y}_1(t) = f(t, y_1(t), \lambda_1(y_1)) = f_1(t, y_1(t)), y_1(t_1) = y_1, t_1 \le t \le t_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{y}_k(t) = f(t, y_k(t), \lambda_k(y_k)) = f_k(t, y_k(t)), y_k(t_k) = y_k, t_k \le t \le t_{k+1} \\ \cdot \\ \cdot \end{cases}$$

and a solution of (Equation (3.1)) can be expressed as

$$y(t) = \begin{cases} y_0(t) = y_0, \quad t_0 \le t \le t_1 \\ y_0(t) = y_0, \quad t_0 \le t \le t_1 \\ \cdot \\ \cdot \\ \cdot \\ y_0(t) = y_0, \quad t_0 \le t \le t_1 \\ \cdot \\ \cdot \end{cases}$$

we note that the solution of (Equation (3.1)) is continuous and piecewise differentiable over (t_0, ∞) and differentiable on each interval (t_k, t_{k+1}) for any fixed $y_k \in R$ and $k = 0,1,2,\ldots$.

4 A Third order runge-kutta method based on linear combination of Arithmetic mean, Geometric mean and Centroidal mean for hybrid fuzzy differential equation

Consider the IVP (Equation (3.1)) with crisp initial condition $y(t_0) = y_0 \in R$ and $t \in [t_0, T]$. Let the exact solution $[Y(t)]_r = [\underline{Y}(t;r), \overline{Y}(t;r)]$ is approximated by some $[y(t)]_r = [\underline{y}(t;r), \overline{y}(t;r)]$ from the equation $y_{n+1} = y_n + \frac{h}{2}\left[\sum_{i=1}^{2} Means\right]$ where means includes Arithmetic mean (AM), Geometric mean (GM), Contra - Harmonic mean (CoM), Centroidal mean (CeM), Root mean Square (RM), Harmonic mean (HaM), and Heronian mean (HeM) which involves $k_i 1 \le i \le 4$,

where,

$$k_1 = f(t_n, y_n)$$
$$k_2 = f(t_n + a_1 h, y_n + a_1 h k_1)$$
$$k_3 = f(t_n + (a_2 + a_3) h, y_n + a_2 h k_1 + a_3 h k_2)$$

where the parameters for the linear combination of Arithmetic mean, Geometric mean and Centroidal mean are

$$a_1 = \frac{2}{3},\ a_2 = \frac{203}{810},\ a_3 = \frac{743}{810}$$

The third order formulae based on runge-kutta scheme using the Combination of Arithmetic mean, Geometric mean and Centroidal mean are

$$y_{n+1} = y_n + \frac{h}{90}\left\{7(k_1 + 2k_2 + k_3) - \left(\sqrt{k_1 k_2} + \sqrt{k_2 k_3}\right) + \frac{64}{3}\left(\frac{k_1^2 + k_1 k_2 + k_2^2}{k_1 + k_2} + \frac{k_2^2 + k_2 k_3 + k_3^2}{k_2 + k_3}\right)\right\}$$

with the grid point $s\ a = t_0 \le t_1 \le \cdots \le t_N = b$ and $h = \frac{(b-a)}{N} = t_{i+1} - t_i$

We define

$$\underline{y}_{k,n+1}(r) - \underline{y}_{k,n}(r) = \sum_{i=1}^{3} w_i \underline{k}_i(t_{k,n}; y_{k,n}(r))$$
$$\overline{y}_{k,n+1}(r) - \overline{y}_{k,n}(r) = \sum_{i=1}^{3} w_i \overline{k}_i(t_{k,n}; y_{k,n}(r))$$

where w_1, w_2, w_3 are constants,

$$\begin{aligned}\underline{k}_1\left(t_{k,n};y_{k,n}(r)\right) &= min\left\{f\left(t_{k,n},u,\lambda_k\left(u_k\right)\right)\ \backslash\ u\in\left[\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)\right],u_k\in\left[\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)\right]\right\}\\ \overline{k}_1\left(t_{k,n};y_{k,n}(r)\right) &= max\left\{f\left(t_{k,n},u,\lambda_k\left(u_k\right)\right)\ \backslash\ u\in\left[\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)\right],u_k\in\left[\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)\right]\right\}\end{aligned}$$

$$\underline{k}_2\left(t_{k,n};y_{k,n}(r)\right) = min\left\{\begin{array}{c} f\left(t_{k,n}+\tfrac{2}{3}h_k,u,\lambda_k\left(u_k\right)\right)\ \backslash\ u\in\left[\underline{z}_{k_1}\left(t_{k,n};y_{k,n}(r)\right),\overline{z}_{k_1}\left(t_{k,n};y_{k,n}(r)\right)\right], \\ u_k\in\left[\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)\right]\end{array}\right\}$$

$$\overline{k}_2\left(t_{k,n};y_{k,n}(r)\right) = max\left\{\begin{array}{c} f\left(t_{k,n}+\tfrac{2}{3}h_k,u,\lambda_k\left(u_k\right)\right)\ \backslash\ u\in\left[\underline{z}_{k_1}\left(t_{k,n};y_{k,n}(r)\right),\overline{z}_{k_1}\left(t_{k,n};y_{k,n}(r)\right)\right], \\ u_k\in\left[\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)\right]\end{array}\right\}$$

$$\underline{k}_3\left(t_{k,n};y_{k,n}(r)\right) = min\left\{\begin{array}{c} f\left(t_{k,n}+\tfrac{2}{3}h_k,u,\lambda_k\left(u_k\right)\right)\ \backslash\ u\in\left[\underline{z}_{k_2}\left(t_{k,n};y_{k,n}(r)\right),\overline{z}_{k_2}\left(t_{k,n};y_{k,n}(r)\right)\right], \\ u_k\in\left[\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)\right]\end{array}\right\}$$

$$\overline{k}_3\left(t_{k,n};y_{k,n}(r)\right) = max\left\{\begin{array}{c} f\left(t_{k,n}+\tfrac{2}{3}h_k,u,\lambda_k\left(u_k\right)\right)\ \backslash\ u\in\left[\underline{z}_{k_2}\left(t_{k,n};y_{k,n}(r)\right),\overline{z}_{k_2}\left(t_{k,n};y_{k,n}(r)\right)\right], \\ u_k\in\left[\underline{y}_{k,0}(r),\overline{y}_{k,0}(r)\right]\end{array}\right\}$$

where

$$\underline{z}_{k_1}\left(t_{k,n};y_{k,n}(r)\right) = \underline{y}_{k,n}(r)+\frac{2}{3}h_k\underline{k}_1\left(t_{k,n};y_{k,n}(r)\right)$$

$$\overline{z}_{k_1}\left(t_{k,n};y_{k,n}(r)\right) = \overline{y}_{k,n}(r)+\frac{2}{3}h_k\overline{k}_1\left(t_{k,n};y_{k,n}(r)\right)$$

$$\underline{z}_{k_2}\left(t_{k,n};y_{k,n}(r)\right) = \underline{y}_{k,n}(r)-\frac{203}{810}h_k\underline{k}_1\left(t_{k,n};y_{k,n}(r)\right)+\frac{743}{810}h_k\underline{k}_2\left(t_{k,n};y_{k,n}(r)\right)$$

where

$$\underline{z}_{k_1}\left(t_{k,n};y_{k,n}(r)\right) = \underline{y}_{k,n}(r)+\tfrac{2}{3}h_k\underline{k}_1\left(t_{k,n};y_{k,n}(r)\right)$$

$$\overline{z}_{k_1}\left(t_{k,n};y_{k,n}(r)\right) = \overline{y}_{k,n}(r)+\frac{2}{3}h_k\overline{k}_1\left(t_{k,n};y_{k,n}(r)\right)$$

$$\underline{z}_{k_2}\left(t_{k,n};y_{k,n}(r)\right) = \underline{y}_{k,n}(r)-\tfrac{203}{810}h_k\underline{k}_1\left(t_{k,n};y_{k,n}(r)\right)+\tfrac{743}{810}h_k\underline{k}_2\left(t_{k,n};y_{k,n}(r)\right)$$

$$\overline{z}_{k_2}\left(t_{k,n};y_{k,n}(r)\right) = \overline{y}_{k,n}(r)-\frac{203}{810}h_k\overline{k}_1\left(t_{k,n};y_{k,n}(r)\right)+\frac{743}{810}h_k\overline{k}_2\left(t_{k,n};y_{k,n}(r)\right)$$

define

$$S_k\left[t_{k,n};\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)\right]=\left\{\begin{array}{c}\dfrac{7\left[\underline{k}_1\left(t_{k,n},y_{k,n}(r)\right)+2\underline{k}_2\left(t_{k,n},y_{k,n}(r)\right)+\underline{k}_3\left(t_{k,n},y_{k,n}(r)\right)\right]}{-\left[\sqrt{\left(\underline{k}_1\left(t_{k,n},y_{k,n}(r)\right)\underline{k}_2\left(t_{k,n},y_{k,n}(r)\right)+\sqrt{\left[\underline{k}_2\left(t_{k,n},y_{k,n}(r)\right)\underline{k}_3\left(t_{k,n},y_{k,n}(r)\right)\right]}\right.}\right]}\\ +\frac{64}{3}\left[\begin{array}{c}\frac{\underline{k}_1^2\left(t_{k,n},y_{k,n}(r)\right)+\underline{k}_1\left(t_{k,n},y_{k,n}(r)\right)\underline{k}_2\left(t_{k,n},y_{k,n}(r)\right)+\underline{k}_2^2\left(t_{k,n},y_{k,n}(r)\right)}{\underline{k}_1\left(t_{k,n},y_{k,n}(r)\right)+\underline{k}_2\left(t_{k,n},y_{k,n}(r)\right)}\\ +\frac{\underline{k}_2^2\left(t_{k,n},y_{k,n}(r)\right)+\underline{k}_2\left(t_{k,n},y_{k,n}(r)\right)\underline{k}_3\left(t_{k,n},y_{k,n}(r)\right)+\underline{k}_3^2\left(t_{k,n},y_{k,n}(r)\right)}{\underline{k}_2\left(t_{k,n},y_{k,n}(r)\right)+\underline{k}_3\left(t_{k,n},y_{k,n}(r)\right)}\end{array}\right]\end{array}\right\}$$

$$T_k\left(t_{k,n};\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)\right]=\left\{\begin{array}{c}\dfrac{7\left[\overline{k}_1\left(t_{k,n},y_{k,n}(r)\right)+2\overline{k}_2\left(t_{k,n},y_{k,n}(r)\right)+\overline{k}_3\left(t_{k,n},y_{k,n}(r)\right)\right]}{-\left[\sqrt{\overline{k}_1\left(t_{k,n},y_{k,n}(r)\right)\overline{k}_2\left(t_{k,n},y_{k,n}(r)\right)}+\sqrt{\overline{k}_2\left(t_{k,n},y_{k,n}(r)\right)\overline{\overline{k}}_3\left(t_{k,n},y_{k,n}(r)\right)}\right]}\\ +\frac{64}{3}\left[\begin{array}{c}\frac{\overline{k}_1^2\left(t_{k,n},y_{k,n}(r)\right)+\overline{k}_1\left(t_{k,n},y_{k,n}(r)\right)\overline{k}_2\left(t_{k,n},y_{k,n}(r)\right)+\overline{k}_2^2\left(t_{k,n},y_{k,n}(r)\right)}{\overline{k}_1\left(t_{k,n},y_{k,n}(r)\right)+\overline{k}_2\left(t_{k,n},y_{k,n}(r)\right)}\\ +\frac{\overline{k}_2^2\left(t_{k,n},y_{k,n}(r)\right)+\overline{k}_2\left(t_{k,n},y_{k,n}(r)\right)\overline{k}_3\left(t_{k,n},y_{k,n}(r)\right)+\overline{k}_3^2\left(t_{k,n},y_{k,n}(r)\right)}{\overline{k}_2\left(t_{k,n},y_{k,n}(r)\right)+\overline{k}_3\left(t_{k,n},y_{k,n}(r)\right)}\end{array}\right]\end{array}\right\}$$

The exact solutions at $t_{k,n+1}$ is given by

$$\underline{Y}_{k,n+1}(r)\approx\underline{Y}_{k,n}(r)+\tfrac{h_k}{90}S_k\left[t_{k,n};\underline{Y}_{k,n}(r),\overline{Y}_{k,n}(r)\right]$$

$$\overline{Y}_{k,n+1}(r)\approx\overline{Y}_{k,n}(r)+\tfrac{h_k}{90}T_k\left[t_{k,n};\underline{Y}_{k,n}(r),\overline{Y}_{k,n}(r)\right]$$

The approximate solutions at $t_{k,n+1}$ is given by

$$\underline{y}_{k,n+1}(r)=\underline{y}_{k,n}(r)+\tfrac{h_k}{90}S_k\left[t_{k,n};\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)\right]$$

$$\overline{y}_{k,n+1}(r)=\overline{y}_{k,n}(r)+\tfrac{h_k}{90}T_k\left[t_{k,n};\underline{y}_{k,n}(r),\overline{y}_{k,n}(r)\right]$$

5 Numerical Example

Consider the fuzzy Initial Value Problem

$$\left\{\begin{array}{c}y'(t)=y(t),\ t\in[0,1]\\ y(0;r)=[0.75+0.25r,\ 1.125-0.125r],\ 0\le r\le 1\end{array}\right.$$

The exact solution is given by

$$Y(t;r)=\left[(0.75+0.25r)e^t,\ (1.125-0.125r)e^t\right],\ 0\le r\le 1$$

at $t=1$ we get

$$Y(1;r)=\left[(0.75+0.25r)e^1,\ (1.125-0.125r)e^1\right],\ 0\le r\le 1$$

By the third order runge-kutta method based on the combination of Arithmetic mean, Geometric mean and Centroidal mean with $N=2$, gives

$$y(1.0;r)=\left[(0.75+0.25r)\left(C_{0,1}\right)^2,\ (1.125-0.125r)\left(C_{0,1}\right)^2\right],\ 0\le r\le 1$$

Where

$$C_{0,1}=1+\frac{h}{90}\left\{\begin{array}{c}7\left[4+2h+\frac{743}{1215}h^2\right]-\left[\sqrt{\left(1+\frac{2}{3}h\right)}+\sqrt{\left(1+\frac{2}{3}h\right)\left(1+\frac{2}{3}h+\frac{743}{1215}h^2\right)}\right]\\ +\frac{64}{3}\left[\frac{1+\left(1+\frac{2}{3}h\right)+\left(1+\frac{2}{3}h\right)^2}{2+\frac{2}{3}h}+\frac{\left(1+\frac{2}{3}h\right)^2+\left(1+\frac{2}{3}h\right)\left(1+\frac{2}{3}h+\frac{743}{1215}h^2\right)+\left(1+\frac{2}{3}h+\frac{743}{1215}h^2\right)^2}{2+\frac{4}{3}h+\frac{743}{1215}h^2}\right]\end{array}\right]$$

Now consider the Hybrid fuzzy initial value problem

$$\begin{cases} y^{'}(t)=y(t)+m(t)\lambda_k\left(y\left(t_k\right)\right),\ t\in\left[t_k,t_{k+1}\right],\ t_k=k,\ k=0,1,2,\ldots \\ y(t;r)=\left[(0.75+0.25r)e^t,\ (1.125-0.125r)e^t\right],\ 0\leq r\leq 1 \end{cases}$$

Where

$$m(t)=\begin{cases} 2(t(mod1)),\ if\ t(mod1)\leq 0.5, \\ 2(1-t(mod1)),\ if\ t(mod1)\geq 0.5 \end{cases}$$

$$\lambda_k\left(y\left(t_k\right)\right)=\begin{cases} 0 & ,\ if\ k=0, \\ \mu & ,\ if\ k\in 1,2,\ldots \end{cases}$$

hybrid fuzzy Initial value problems is equivalent to the following system of fuzzy Initial value problems.

The exact solution for $t\in[0,2]$ is given by

$$\begin{cases} y^{'}(t)=y(t),\ t\in[0,1] \\ y_0(0;r)=[(0.75+0.25r),\ (1.125-0.125r)],\ 0\leq r\leq 1 \\ y_i^{'}(t)=y_i(t)+m(t)y_{i-1}(t),\ t\in\left[t_i,t_{i+1}\right],\ y_i(t)=y_{i-1}\left(t_i\right),\ i=1,2,\ldots \end{cases}$$

$y(t)+m(t)\lambda_k\left(y\left(t_k\right)\right)$ is continuous function of t,x and $\lambda_k\left(y\left(t_k\right)\right)$, for each $k=0,1,2,\ldots$, the fuzzy Initial value problem

$$\begin{cases} y^{'}(t)=y(t)+m(t)\lambda_k\left(y\left(t_k\right)\right),\ t\in\left[t_k,t_{k+1}\right],\ t_k=k \\ y\left(t_k\right)=y_{t_k} \end{cases}$$

has a unique solution on $\left[t_{k,}t_{k+1}\right]$. To numerically solve the hybrid fuzzy IVP the modified third order runge-kutta method based on Arithmetic mean, Geometric mean and Centroidal mean is applied for solving hybrid fuzzy differential equation with $N=2$ to obtain, $y_{1,2}(r)$ approximating $x(2.0;r)$.

Let $f:(0,\infty)\times R\times R\rightarrow R$ be given by

$f\left(t,y,\lambda_k\left(y\left(t_k\right)\right)\right)=y(t)+m(t)\lambda_k\left(y\left(t_k\right)\right),\ t_k=k,k=0,1,2,\ldots,$

Where $\lambda_k:R\rightarrow R$ is given by

$$\lambda_k(y)=\begin{cases} 0,\ if\ k=0 \\ y,\ if\ k\in\{1,2,\ldots,\} \end{cases}$$

Since the exact solution for $t\varepsilon[1,1.5]$ is

$$Y(t;r)=Y(1;r)\left(3e^{t-1}-2t\right),\ 0\leq r\leq 1,\ Y(1.5;r)=Y(1;r)(3\sqrt{e}-3),\ 0\leq r\leq 1$$

Then $Y(1.5;r)$ is approximately 5.290222 and the exact solution for $t\varepsilon[1.5,2]$ is

$$Y(t;r)=Y(1;r)\left(2t-2+e^{t-1.5}(3\sqrt{e}-4)\right),\ 0\leq r\leq 1$$

Therefore,

$$Y(2.0;r)=Y(1;r)(2+3e-4\sqrt{e})$$

Then $Y(2.0;r)$ is approximately 9.676976.

The absolute error of the third order runge-kutta method based on a linear combination of Arithmetic mean (RK3AM), Geometric mean (RK3GeM) and Centroidal mean (RK3CeM) is compared with the absolute error by Arithmetic mean (RK3AM) for the r-level set with $h=0.1$ and $t=2$ of the example is given below.

Figure 1 is the graphical representation for the absolute error of the third order runge-kutta method based on the linear combination of RK3AM, RK3GM, RK3CeM and RK3AM for the above example is given below, where $t=1.5,\ h=0.1$.

Table 1. Absolute error of third order runge-kutta method based on a linear combination of RK3AM, RK3GeM, RK3CeM and RK3AM for the$r$-level set with $h = 0.1$ and $t = 2$

r	T	A bsolute error		A bsolute error	
		HRK3AMGeMCeM		HRK3AM	
0	2	4.55E-04	6.82E-04	5.94E-04	8.91E-04
0.1	2	4.70E-04	6.75E-04	6.14E-04	8.81E-04
0.2	2	4.85E-04	6.67E-04	6.34E-04	8.71E-04
0.3	2	5.00E-04	6.59E-04	6.54E-04	8.61E-04
0.4	2	5.15E-04	6.52E-04	6.73E-04	8.52E-04
0.5	2	5.31E-04	6.44E-04	6.93E-04	8.42E-04
0.6	2	5.46E-04	6.37E-04	7.13E-04	8.32E-04
0.7	2	5.61E-04	6.29E-04	7.33E-04	8.22E-04
0.8	2	5.76E-04	6.22E-04	7.53E-04	8.12E-04
0.9	2	5.91E-04	6.14E-04	7.72E-04	8.02E-04
1	2	6.06E-04	6.06E-04	7.92E-04	7.92E-04

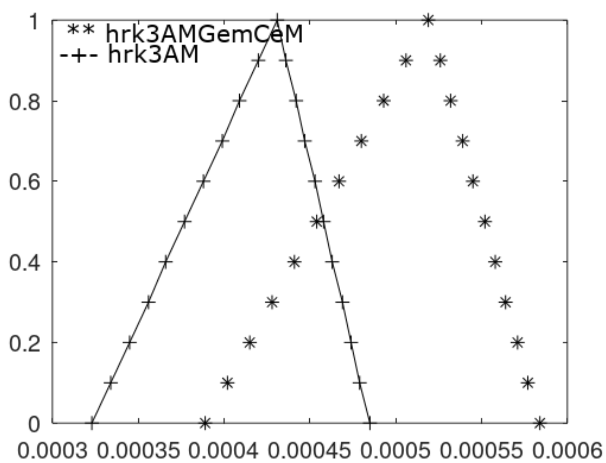

Fig 1. 1 $(h = 0.1,\ t = 1.5)$

6 Conclusion

In this study, the first order hybrid fuzzy differential equation is solved using the third order runge-kutta method based on the linear combination of Arithmetic Mean, Geometric Mean and Centroidal Mean. The absolute error of the proposed method is compared with the absolute error of the Arithmetic mean method, and it has been noted that the proposed method gives the better error tolerance and it is suitable for solving hybrid fuzzy initial value problem. Likewise, the approximate and exact solutions can also be calculated for this proposed method.

References

1) Mukherjee AK, Gazi KH, Salahshour S, Ghosh A, Mondal SP. A Brief Analysis and Interpretation on Arithmetic Operations of Fuzzy Numbers. *Results in Control and Optimization*. 2023;13:1–42. Available from: https://doi.org/10.1016/j.rico.2023.100312.
2) Rajakumari PED, Sharmila RG. A Third order Runge-Kutta method based on linear combination of Arithmetic mean, Harmonic mean and Heronian mean for hybrid fuzzy differential equation. *Journal of Algebraic statistics*. 2022;13(2):1049–1057. Available from: https://doi.org/10.52783/jas.v13i2.261.
3) Rajakumari PED, Sharmila RG. A Third order Runge-Kutta method based on linear combination of Arithmetic mean, Heronian mean and Contra Harmonic mean for hybrid fuzzy differential equation. *Journal of Algebraic statistics*. 2022;13(3):1454–1461. Available from: https://publishoa.com/index.php/journal/article/view/767/658.
4) Sharmila RG, Suvitha S, Sunithy MS. A third order Runge-Kutta method based on a linear combination of arithmetic mean, geometric mean and centroidal mean for first order differential equation. In: PROCEEDINGS OF INTERNATIONAL CONFERENCE ON ADVANCES IN MATERIALS RESEARCH (ICAMR - 2019);vol. 2261, Issue 1 of AIP Conference Proceedings. AIP Publishing. 2020;p. 42–46. Available from: https://doi.org/10.1063/5.0017103.

5) Jeyaraj T, Rajan D. Explicit Runge Kutta method in solving fuzzy Initial Value Problem. *Advances and Applications in Mathematical Sciences*. 2021;20(4):663–674. Available from: https://www.mililink.com/upload/article/1124438296aams_vol_204_feb_2021_a19_p663-674_t._jeyaraj_and_d._rajan.pdf.
6) Buckley JJ, Feuring T. Fuzzy differential equations. *Fuzzy Sets and Systems*. 2000;110(1):43–54. Available from: https://dx.doi.org/10.1016/s0165-0114(98)00141-9.
7) Kalaiarasi K, Swathi S. Arithmetic Operations of Trigonal Fuzzy Numbers Using Alpha Cuts: A Flexible Approach to Handling Uncertainty. *Indian Journal Of Science And Technology*. 2023;16(47):4585–4593. Available from: https://doi.org/10.17485/IJST/v16i47.2654.
8) Kanade GD, Dhaigude RM. Iterative Methods for Solving Ordinary Differential Equations: A Review. *International Journal of Creative Research Thoughts (IJCRT)*. 2022;10(2):266–270. Available from: https://www.ijcrt.org/papers/IJCRTL020078.pdf.
9) Singh P, Gor B, Mukherjee S, Mahata A, Mondal SP. Analysis and interpretation of Malaria disease model in crisp and fuzzy environment. *Results in Control and Optimization*. 2023;12:1–22. Available from: https://doi.org/10.1016/j.rico.2023.100257.
10) Bharathi DP, Jayakumar T, Vinoth S. Numerical Solutions of Fuzzy Multiple Hybrid Single Retarded Delay Differential Equations. *International Journal of Recent Technology and Engineering (IJRTE)*. 2019;8(3):1946–1949. Available from: https://www.ijrte.org/wp-content/uploads/papers/v8i3/C4479098319.pdf.
11) Bharathi DP, Jayakumar T, Vinoth S. Numerical Solutions of Fuzzy Pure Multiple Neutral Delay Differential Equations. *International Journal of Advanced Scientific Research and Management*. 2019;4(1):172–178. Available from: https://ijasrm.com/wp-content/uploads/2019/01/IJASRM_V4S1_1133_172_178.pdf.
12) Sahu R, Sharma AK. Understanding the Unique Properties of Fuzzy concept in Binary Trees. *Indian Journal of Science and Technology*. 2023;16(33):2631–2636. Available from: https://doi.org/10.17485/IJST/v16i33.874.
13) Singh P, Gazi KH, Rahaman M, Salahshour S, Mondal SP. A Fuzzy Fractional Power Series Approximation and Taylor Expansion for Solving Fuzzy Fractional Differential Equation. *Decision Analytics Journal*. 2024;10:1–16. Available from: https://dx.doi.org/10.1016/j.dajour.2024.100402.
14) Srinivasan R, Karthikeyan N, Jayaraja A. A Proposed Technique to Resolve Transportation Problem by Trapezoidal Fuzzy Numbers. *Indian Journal of Science and Technology*. 2021;14(20):1642–1646. Available from: https://dx.doi.org/10.17485/ijst/v14i20.645.
15) Tude S, Gazi KH, Rahaman M, Mondal SP, Chatterjee B, Alam S. Type-2 fuzzy differential inclusion for solving type-2 fuzzy differential equation. *Annals of Fuzzy Mathematics and Informatics*. 2023;25(1):33–53. Available from: http://www.afmi.or.kr/papers/2023/Vol-25_No-01/PDF/AFMI-25.1(33-53)-H-220122R2.pdf.