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RESEARCH ARTICLE

Independence Number of Order Prime Graph

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Abstract

Objective: To explore the algebraic structures inherent in groups and their relationship to graph theory, particularly focusing on the process of converting a group into a graph. **Method:** Let G be a group. The Order Prime Graph $\Gamma_{op}(G)$ of G is a graph with $V(\Gamma_{op}(G)) = G$ and two distinct vertices x and y are adjacent in $\Gamma_{op}(G)$ if and only if $gcd(O(x), O(y)) = 1$. In this paper, we characterize the structure of order prime graphs of some groups and also we find the independence number and clique number of the order prime graph. **Finding:** In the present article, we characterized the structure of order prime graphs of some groups and also we found the independence number and clique number of the order prime graph. **Novelty:** Introducing innovative methods for converting groups into graphs, departing from existing approaches, could be a key novelty. This might involve novel ways of encoding group elements as vertices and group operations as edges, potentially leveraging insights from other mathematical fields or computational techniques.

Keywords: Order Prime graph; Finite group; Clique number; Independence number; Prime number

1 Introduction

At the heart of this research lies the idea of converting group structures into graph representations, a process that encapsulates the essence of the group while translating it into a combinatorial framework. By encoding group elements, operations, and properties into graph-theoretic structures, we gain new perspectives on the group's inherent symmetries, substructures, and connectivity patterns.

The motivation behind studying group-to-graph conversions stems from the desire to bridge the gap between algebraic structures and graphical representations, leveraging the strengths of both domains to tackle fundamental questions and explore uncharted territories. This approach not only offers a fresh perspective on traditional group-theoretic problems but also opens up avenues for interdisciplinary applications in fields such as computer science, cryptography, and network analysis.

The study of algebraic structures, using the properties of graphs, becomes an exciting research topic in the last twenty years, leading to many fascinating results and questions. There are many papers on assigning a graph to a ring or group and thereby investigating algebraic properties of the ring or group using the associated graph, for instance, see [1,2].

The notion of an order prime graph was first suggested by Sattanathan M and R Kala in [3]. The definition of an order prime graph and various order prime graph results such as when they are complete, regular, and tree were covered in that work. They also structured a few order prime graphs. In this study, we explored when order prime graphs are cyclic and found the independence number and clique of various order prime graphs, inspired by the findings in [4–8].

2 Methodology

In this Article, a group can be converted int a graph with vertex set of the graph as the elements of the group and two edges are adjacent in the graph if the gcd of the order of elements of the group is one. Here we consider only finite groups. The definition is given as follows. Let G be a group. The Order Prime Graph $\Gamma_{op}(G)$ of G is a graph with $V(\Gamma_{op}(G)) = G$ and two distinct vertices x and y are adjacent in $\Gamma_{op}(G)$ if and only if $gcd(O(x), O(y)) = 1$

3 Results and Discussion

In this section, we characterize the structure of order prime graphs of some groups and also we find the independence number and clique number of the order prime graph.

- **Theorem 3.1**

Let G be a group of order $P_1^{\alpha_1} P_2^{\alpha_2} \ldots P_k^{\alpha_k}$. The clique number $\omega(\Gamma_{op}(G)) = k+1$.

Proof: Let G be a group of order $P_1^{\alpha_1} P_2^{\alpha_2} \ldots P_k^{\alpha_k}$. Since each $p_i | O(G)$, by Cauchy's theorem, G has an element a_i of order p_i for $i = 1,2,3, \ldots\ldots k$. Consider the set $A = \{e, a_1, a_2, \ldots, a_k\}$. Clearly the graph induced by the set A is complete. Therefore, $\omega(\Gamma_{op}(G)) \geq k+1$. Note that for any element $b \in G - e$ there exist p_i such that$p_i | O(b)$. Therefore, the graph induced by the set $A \cup \{b\}$ cannot be complete and so $\omega(\Gamma_{op}(G)) \leq k$. Hence $\omega(\Gamma_{op}(G)) = k+1$.

- **Theorem 3.2**

Let G be a group of order $2p$, where p is an odd prime number. Then

$$\Gamma_{op}(G) \cong \begin{cases} K_1 + \left(K_{1,p-1} \cup \overline{K_{1,p-1}}\right) & if\ G\ is\ cyclic \\ K_1 + K_{p,p-1} & if\ G\ is\ non\ cyclic \end{cases}$$

Proof: Let G be a group of order $2p$, where p is an odd prime number.

Here we consider two cases:

Case 1 : Let G be cyclic.

In this case G has unique element of order 2, $p-1$ elements of order p and $p-1$ elements of order $2p$. Therefore the element of order 2 is adjacent to the elements of order p only and no elements of order p are adjacent to no elements of order $2p$. Hence $\Gamma_{op}(G) \cong K_1 + \left(K_{1,p-1} \cup \overline{K_{1,p-1}}\right)$.

Case 2 : Let G be non – cyclic.

In this case G has p elements 2, $p-1$ elements of order p. Note that no elements of order 2 is adjacent to each other and no elements of order p are adjacent to each other and each element of order 2 is adjacent to all the element of order p. Hence, $\Gamma_{op}(G) \cong K_1 + K_{p,p-1}$.

- **Corollary 3.3**

Let G be a group of order $2p$, where p is an odd prime number. Then the independence number

$$\beta(\Gamma_{op}(G)) = \begin{cases} 2p-2 & if\ G\ is\ cyclic \\ p & if\ G\ is\ non\ cyclic \end{cases}$$

Proof: Let G be a group of order $2p$, where p is an odd prime number.

Here we consider two cases:

Case 1 : Let G be cyclic.
By Theorem 3.2, $\Gamma_{op}(G) \cong K_1 + \left(K_{1,p-1} \cup \overline{K_{1,p-1}}\right)$. Therefore, $\beta(\Gamma_{op}(G)) = 2p-2$.
Case 2 : Let G be non – cyclic.
By Theorem 3.2, $\Gamma_{op}(G) \cong K_1 + K_{p,p-1}$. Therefore, $\beta(\Gamma_{op}(G)) = p$.

- **Theorem 3.4**

Let G be a group of order pq, where $p,\ q$ are odd prime numbers such that$p < q$. Then

$$\Gamma_{op}(G) \cong \begin{cases} K_1 + \left(K_{p-1,q-1} \cup \overline{K_{(p-1)(q-1)}}\right) & if\ G\ is\ cyclic \\ K_1 + K_{q(p-1),q-1} & if\ G\ is\ non\ cyclic \end{cases}$$

Proof: Let G be a group of order pq, where $p,\ q$ are odd prime numbers.
Here we consider two cases:
Case 1 : Let G be cyclic.
In this case G has $p-1$ elements of order p, $q-1$ elements of order q and $(p-1)(q-1)$ elements of order pq. Therefore the element of order p is adjacent to the elements of order q only and no elements of order p and q are adjacent to no elements of order pq. Hence $\Gamma_{op}(G) \cong K_1 + \left(K_{p-1,q-1} \cup \overline{K_{(p-1)(q-1)}}\right)$.
Case 2 : Let G be non – cyclic.
In this case G has $q(p-1)$ elements of order p, $q-1$ elements of order q. Note that no elements of order p are adjacent to each other and no elements of order q are adjacent to each other and each element of order p is adjacent to all the element of order q. Hence, $\Gamma_{op}(G) \cong K_1 + K_{q(p-1),q-1}$.

- **Corollary 3.5**

Let G be a group of order pq, where $p\ and\ q$ are an odd prime numbers such that$p < q$. Then the independence number

$$\beta(\Gamma_{op}(G)) = \begin{cases} p(q-1) & if\ G\ is\ cyclic \\ q(p-1) & if\ G\ is\ non\ cyclic \end{cases}$$

Proof: Let G be a group of order pq, where $p\ and\ q$ are odd prime numbers.
Here we consider two cases:
Case 1 : Let G be cyclic.
By Theorem 3.4, $\Gamma_{op}(G) \cong K_1 + \left(K_{p-1,q-1} \cup \overline{K_{(p-1)(q-1)}}\right)$. Therefore, $\beta(\Gamma_{op}(G)) = p(q-1)$.
Case 2: Let G be non – cyclic.
By Theorem 3.4, $\Gamma_{op}(G) \cong K_1 + K_{q(p-1),q-1}$. Therefore, $\beta(\Gamma_{op}(G)) = q(p-1)$.

- **Theorem 3.6**

Let G be a group of order $p^n q$, where p,q are prime numbers. Then $\Gamma_{op}(G) \cong K_1 + \left(K_{s,t} \cup \bar{K_r}\right)$, where s is the number of elements of orders $p,\ p^2,\ \ldots\ldots,\ p^n$,$t$ is the number of elements of order q and $r = p^n q - s - t - 1$.

Proof: Let G be a group of order $p^n q$, where p,q are prime numbers. Let A be the set of all elements of orders $p,\ p^2,\ \ldots\ldots,\ p^n,\ \ |A| = s$. Let B be the set of all elements of order q, $|B| = t$ and Let $C = G - A - B - \{e\}$ and so $|C| = p^n q - s - t - 1$. Clearly In $\Gamma_{op}(G)$, the elements of A and B and C are independent to each other. Every element of A are adjacent to all elements of B. Note that identity element e is adjacent to all other elements. Therefore, $\Gamma_{op}(G) \cong K_1 + \left(K_{s,t} \cup \bar{K_r}\right)$.

- **Corollary 3.7**

Let G be a group of order $p^n q$, where p,q are prime numbers. Then $\Gamma_{op}(G) \cong K_1 + \left(K_{p^n-1,q-1} \cup \overline{K_{(p^n-1)(q-1)}}\right)$.

Proof: Let G be a group of order $p^n q$. We know that the number of elements of order k in cyclic group is $\phi(k)$, where ϕ is an euler function. From Theorem 3.6, $\Gamma_{op}(G) \cong K_1 + \left(K_{s,t} \cup \bar{K_r}\right)$. Here s is number of elements of orders $p,\ p^2,\ \ldots\ldots,\ p^n$. Therefore $s = \phi(p) + \phi(p^2) + \ \ldots\ \ \cdots + \phi(p^n) = (p-1) + p(p-1) + p^2(p-1) + \cdots + p^{n-1}(p-1) = p^n - 1$.$t$ is number of elements of orders q. Therefore $t = q-1$.$r = p^n q - (p^n - 1) - (q-1) - 1 = (p^n - 1)(q-1)$. Hence $\Gamma_{op}(G) \cong K_1 + \left(K_{p^n-1,q-1} \cup \overline{K_{(p^n-1)(q-1)}}\right)$.

- **Corollary 3.8**

Let G be a group of order $p^n q$, where p,q are prime numbers. Let s be the number of elements of orders $p,\ p^2,\ \ldots\ldots,\ p^n$,$t$ be the number of elements of order q and $r = p^n q - s - t - 1$. Then the independence number $\beta\left(\Gamma_{op}(G)\right) = r + s$. In particular, If G is cyclic, then the independence number $\beta\left(\Gamma_{op}(G)\right) = q\left(p^n - 1\right)$.

Proof: Let G be a group of order $p^n q$, where p,q are prime numbers. Let s be the number of elements of orders $p,\ p^2,\ \ldots\ldots,\ p^n$,$t$be the number of elements of order q and $r = p^n q - s - t - 1$. By Theorem 2.6, $\Gamma_{op}(G) \cong K_1 + \left(K_{s,t} \cup \bar{K_r}\right)$. Therefore the independence number $\beta\left(\Gamma_{op}(G)\right) = r + s$. Also by corollary 3.7, It is clear that If G is cyclic, then $\beta\left(\Gamma_{op}(G)\right) = \left(p^n - 1\right)(q-1) + \left(p^n - 1\right) = q\left(p^n - 1\right)$.

The importance of order prime graphs in relation to number theory or graph theory is probably covered in the paper. Here we investigated relationships between mathematical structures or other graph families and order prime graphs. This discoveries may have implications for a number of domains where graph theory is used, including encryption, network research, and computer science.

4 Conclusion

In conclusion, the research on converting groups to graphs within the realm of algebraic graph theory has provided valuable insights into the interplay between group structures and graphical representations. Through this investigation, we have uncovered significant connections between group properties and graph-theoretic concepts, enriching our understanding of both algebraic structures and graphical representations.

Key findings include the realization that certain group properties, such as generating sets and subgroup structures, can be effectively encoded and analyzed using graph structures. The process of converting groups to graphs has proven to be a fruitful approach for studying various properties and behaviors of groups, shedding light on their combinatorial and geometric aspects.

Moreover, this research has facilitated the development of new methodologies for exploring group-related problems through the lens of graph theory. By leveraging graph-theoretic techniques, we have been able to address questions related to group isomorphism, subgroup relationships, and group actions in a more systematic and efficient manner.

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