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* **Corresponding author**.

prati_manohar@yahoo.co.in

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Solution of Fractional Differential Equations Involving Hilfer-Hadamard Fractional Derivatives

Lata Chanchlani[1], Pratibha Manohar[2*], Ajay Sharma[3], Sangeeta Choudhary[4]

1 Assistant Professor, Department of Mathematics, University of Rajasthan, Jaipur, 302004, Rajasthan, India
2 Assistant Professor, Department of Statistics, Mathematics and Computer Science, SKN Agriculture University, Jobner, Jaipur, 303329, Rajasthan, India
3 Assistant Professor, Department of Science and Humanities, Government Polytechnic College, Sikar, 332001, Rajasthan, India
4 Associate Professor, Department of Mathematics, Swami Keshvanand Institute of Technology Management and Gramothan, Jaipur, 302017, Rajasthan, India

Abstract

Objectives: The aim is to establish prerequisite properties for the Hilfer-Hadamard fractional derivatives and address boundary value problems related to fractional polar Laplace and fractional Sturm-Liouville equations involving Hilfer-Hadamard fractional derivatives. **Methods:** Existing definitions and findings are utilized to obtain the properties for fractional derivatives, and the Adomian decomposition method is employed to solve the fractional differential equations. **Findings:** Validity conditions for the law of exponents are determined, and the study investigates the fractional differential equations and their corresponding solutions, possessing the capacity to replace the traditional polar Laplace and Sturm-Liouville boundary value problems to effectively represent real-world phenomena. **Novelty:** The study introduces the substitution of two consecutively operated Hilfer-Hadamard fractional derivatives with a corresponding single Hilfer-Hadamard fractional derivative using the law of exponents. Additionally, the polar Laplace and Sturm-Liouville boundary value problems are extended to their respective fractional counterparts, expressed in a concise format using HilferHadamard fractional derivatives.

Keywords: Adomian decomposition method; Hilfer-Hadamard fractional derivative; Fractional polar Laplace equation; Fractional Sturm-Liouville boundary value problem

1 Introduction

Fractional calculus, a mathematical analysis tool, extends the concepts of derivatives and integrals to non-integer orders. One of the main advantages of fractional calculus is its ability to uncover the complex dynamics of systems, making it a more realistic framework for modeling various phenomena. It has been widely applied in science, engineering, geology, rheology, finance, and biology[(1,2)]. Fractional integrals play a crucial role in defining most fractional derivatives, with the Riemann-Liouville (RL) fractional integral being particularly significant in the definition of the Riemann-Liouville and Caputo fractional derivatives. In 2000, Hilfer[(3)] introduced the composite fractional derivative based on fractional time evolution, which combines both the RL and Caputo fractional derivatives. This derivative provides an extremely flexible framework for describing complex processes. In parallel, Kilbas et al.[(4)] developed and studied the Hadamard fractional integral, derivative, and their properties. Zafar et al.[(5)] investigated the Caputo-type modification of Hadamard fractional derivatives and established a Taylor expansion based on it. Building on these concepts, Promsakon et al.[(6)] defined the left-sided Hilfer-Hadamard fractional derivative $D_{a+}^{\alpha,\beta}$ of order $\alpha \geq 0, n = -[-\alpha]$ and type $\beta, 0 \leq \beta \leq 1$.

The differential equations involving these derivatives are called fractional differential equations (FDE) and have wide-ranging applications in physics, chemistry, biophysics, biology, medical sciences, financial economics, ecology, and bioengineering, among others[(7)]. Vu et al.[(8)] introduced a general form of random FDE with the concept of a Caputo-type fractional derivative. Tarasov[(9)] defined new discrete maps with memory derived from fractional differential equations with Hilfer derivatives. However, finding analytic solutions for these equations is often challenging, and several efficient methods have been proposed to solve them. These methods include the Elzaki and Sumudu transform[(10)], inverse fractional Shehu transform[(11)], double Shehu transform[(12)], Unified predictor-corrector method[(13)], Adomian Decomposition Method[(14)], Fractional Decomposition Method[(15)], and many others. Bachir et al.[(16)] discussed the existence and stability of solutions for a class of Hilfer-Hadamard FDEs. Additionally, Vivek et al.[(17)] investigated the existence of solutions for integrodifferential equations with Hilfer-Hadamard fractional derivative.

The Adomian decomposition method (ADM) is a convergent methodology that employs a semianalytical approach to decompose the nonlinear operator equations into a series of functions. The method has been used to solve various types of linear and nonlinear fractional differential equations, such as fractional diffusion equations[(18)] and impulsive fractional differential equations (IFDEs)[(19)]. Additionally, the ADM has been employed to analyze the distribution of thermodynamic variables in protoplanets[(20)] and is particularly effective in studying Fractional Order Chaotic Systems[(21)].

In this paper, we establish composition formulas for the Hadamard fractional integrals with the Hilfer-Hadamard fractional derivatives. Finally, we solve a fractional polar Laplace equation and a fractional Sturm-Liouville boundary value problem involving Hilfer-Hadamard fractional derivatives using the Adomian decomposition method.

2 Methodology

In this section, we present several definitions that are used throughout the paper.

Definition 2.1: The left-sided and right-sided Hadamard fractional integrals of the order $\alpha \in \Box, R(\alpha) > 0$ are respectively defined as[(4)]

$$J_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(log\frac{t}{u}\right)^{\alpha-1} f(u)\frac{du}{u}, (a < t) \tag{1}$$

and

$$J_{b-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(log\frac{u}{t}\right)^{\alpha-1} f(u)\frac{du}{u}, (t < b). \tag{2}$$

The Hadamard fractional integral J_{a+}^{α} of the logarithmic functions $\left(log\frac{t}{a}\right)^{\sigma-1}$ is given by[(4)]

$$J_{a+}^{\alpha}\left(log\frac{t}{a}\right)^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma+\alpha)}\left(log\frac{t}{a}\right)^{\sigma+\alpha-1}, t > a, \Re(\sigma) > 0. \tag{3}$$

Definition 2.4: $AC[a,b]$ is the space of absolutely continuous functions on $[a,b]$ and $AC_{\delta}^{n}[a,b]$ is the space of complex-valued functions, such that

$AC_{\delta}^{n}[a,b] = \left\{f : [a,b] \to \Box : \delta^{n-1}[f(t)] \in AC[a,b], \delta \equiv t\frac{d}{dt}\right\}.$

In particular, $AC_{\delta}^{1}[a,b] = AC[a,b]$ then for $n = -[-\Re(\alpha)]$ and $f \in AC_{\delta}^{n}[a,b]$, left-sided Hadamard fractional derivatives exist almost everywhere $[a,b]$ and can be represented as [4]

$$\left(Df_{a+}^{\alpha}f\right)(t) = \sum_{k=0}^{n-1} \frac{\left(\delta^{k}f\right)(a)}{\Gamma(1+k-\alpha)}\left(log\frac{t}{a}\right)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}\left(log\frac{t}{u}\right)^{n-\alpha-1}\left(\delta^{n}f\right)(u)du \tag{4}$$

If $\Re(\gamma) > \Re(\alpha) > 0, 1 \le p < \infty$ and $0 < a < b < \infty$, then for $f \in L^{p}(a,b)$ [4]

$$\left(D_{a+}^{\alpha}J_{a+}^{\gamma}\right)f(t) = J_{a+}^{\gamma-\alpha}f(t) \tag{5}$$

Definition 2.6: The left-sided Hilfer-Hadamard fractional derivative of order $\alpha, \alpha \in \square, \Re(\alpha) \ge 0, n = -[-\Re(\alpha)]$ and type $\beta, 0 \le \beta \le 1$ is given by [6]

$$\left(D_{a+}^{\alpha,\beta}f\right)(t) = J_{a+}^{\beta(n-\alpha)}\left(D_{a+}^{\alpha+\beta(n-\alpha)}f\right)(t) = J_{a+}^{\beta(n-\alpha)}\delta^{n}\left(J_{a+}^{(1-\beta)(n-\alpha)}f\right)(t), (a < t < b), \tag{6}$$

Similarly, the right-sided Hilfer-Hadamard fractional derivative of order $\alpha, \alpha \in \square, \Re(\alpha) \ge 0, n = -[-\Re(\alpha)]$ and type $\beta, 0 \le \beta \le 1$ can be defined as

$$\left(D_{b-}^{\alpha,\beta}f\right)(t) = J_{b-}^{\beta(n-\alpha)}\left(D_{b-}^{\alpha+\beta(n-\alpha)}f\right)(x) = J_{b-}^{\beta(n-\alpha)}(-\delta)^{n}\left(J_{b-}^{(1-\beta)(n-\alpha)}f\right)(t), (a < t < b). \tag{7}$$

For $\beta = 0$ and $\beta = 1$, Equations (6) and (7) represent the Hadamard fractional derivatives and Caputotype Hadamard fractional derivatives of order α [6] respectively. For $0 < \beta < 1$, these equations interpolate continuously between the above-mentioned fractional derivatives respectively.

For $\Re(\alpha) \ge 0,\ n = -[-\Re(\alpha)], 0 \le \beta \le 1$ and $f \in L(a,b),\ J_{a+}^{(n-\alpha)(1-\beta)}f \in AC_{\delta}^{n}[a,b]$ $J_{b-}^{(n-\alpha)(1-\beta)}f \in AC_{\delta}^{n}[a,b]$, the composition of the left-sided Hadamard fractional integral J_{a+}^{α} with the left-sided Hilfer-Hadamard fractional derivatives $D_{a+}^{\alpha,\beta}$ is given by [6]

$$\left(J_{a+}^{\alpha}D_{a+}^{\alpha,\beta}\right)f(t) = f(t) - \sum_{k=1}^{n}\frac{\left(log\frac{t}{a}\right)^{\alpha+\beta(n-\alpha)-k}}{\Gamma(\alpha+\beta(n-\alpha)-k+1)}\lim_{x\to a}\delta^{n-k}\left(J_{a+}^{(n-\alpha)(1-\beta)}f\right)(t), \tag{8}$$

Similarly, the composition of the right-sided Hadamard fractional integral J_{b-}^{α} with the rightsided Hilfer-Hadamard fractional derivative $D_{b-}^{\alpha,\beta}$ can be given as

$$\left(J_{b-}^{\alpha}D_{b-}^{\alpha,\beta}\right)f(t) = f(t) - \sum_{k=1}^{n}\frac{\left(log\frac{b}{t}\right)^{\alpha+\beta(n-\alpha)-k}}{\Gamma(\alpha+\beta(n-\alpha)-k+1)}\lim_{x\to b}\delta^{n-k}\left(J_{b-}^{(n-\alpha)(1-\beta)}f\right)(t). \tag{9}$$

3 Results and Discussion

3.1 Properties for Hilfer-Hadamard Fractional Derivatives

We will now establish certain properties of the left-sided and right-sided Hilfer-Hadamard fractional derivatives of order $\alpha, \alpha \in \square, \Re(\alpha) \ge 0, n = -[-\Re(\alpha)]$ and type $\beta, 0 \le \beta \le 1$, given by Equations (6) and (7) respectively. Similar to the Caputo-Hadamard and Hadamard fractional derivatives [3,5], the Hilfer-Hadamard fractional derivatives of the power and logarithmic functions provide an identical function with a constant multiplication factor.

The Hilfer-Hadamard fractional derivatives [Equations (6) and (7)] of the power function t^{σ}, can be expressed as

$$D_{0+}^{\alpha,\beta}t^{\sigma} = \sigma^{\alpha}t^{\sigma}, \Re(\alpha) \ge 0, n = -[-\Re(\alpha)], 0 \le \beta \le 1, \Re(\sigma) > 0 \tag{10}$$

and

$$D_{0-}^{\alpha,\beta}t^{\sigma} = (-\sigma)^{\alpha}t^{\sigma}, \Re(\alpha) \ge 0,\ n = -[-\Re(\alpha)], 0 \le \beta \le 1,\ \Re(\sigma) < 0. \tag{11}$$

For $\Re(\alpha) \geq 0, n = -[-\Re(\alpha)], 0 \leq \beta \leq 1$ and $R(\sigma) > 0$, the Hilfer-Hadamard fractional derivatives [Equations (6) and (7)] of the logarithmic functions $\left(log\frac{t}{a}\right)^{\sigma-1}\left(log\frac{b}{t}\right)^{\sigma-1}$ can be expressed as

$$D_{a+}^{\alpha,\beta}\left(log\frac{t}{a}\right)^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\alpha)}\left(log\frac{t}{a}\right)^{\sigma-\alpha-1}, t > a, \tag{12}$$

and

$$D_{b-}^{\alpha,\beta}\left(log\frac{b}{t}\right)^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\alpha)}\left(log\frac{b}{t}\right)^{\sigma-\alpha-1}, t < b. \tag{13}$$

We will now establish some theorems for the Hilfer-Hadamard fractional derivatives $D_{a+}^{\alpha,\beta}$ and $D_{b-}^{\alpha,\beta}$.

Theorem 3.1.1: For $\Re(\alpha) \geq 0, n = -[-\Re(\alpha)], 0 \leq \beta \leq 1$ and $f \in AC_\delta^n[a,b]$, the Hadamard fractional derivatives $D_{a+}^{\alpha,\beta} D_{b-}^{\alpha,\beta}$ exist almost everywhere $[a,b]$ and can be represented as

$$\left(Df_{a+}^{\alpha,\beta} f\right)(t) = \Sigma_{k=0}^{n-1} \frac{\left(\delta^k f\right)(a)}{\Gamma(1+k-\alpha)}\left(log\frac{t}{a}\right)^{k-\alpha} + \frac{1}{\Gamma(n-\alpha)}\int_a^t \left(log\frac{t}{u}\right)^{n-\alpha-1}(\delta^n f)(u)du \tag{14}$$

$$\left(D_{b-}^{\alpha,\beta} f\right)(t) = \Sigma_{k=0}^{n-1} \frac{(-1)^k\left(\delta^k f\right)(a)}{\Gamma(1+k-\alpha)}\left(log\frac{b}{t}\right)^{k-\alpha} + \frac{(-1)^n}{\Gamma(n-\alpha)}\int_t^b \left(log\frac{u}{t}\right)^{n-\alpha-1}(\delta^n f)(u)du \tag{15}$$

Proof: For $\Re(\alpha) \geq 0, n = -[-\Re(\alpha)]$ and $0 \leq \beta \leq 1$, the Hilfer-Hadamard fractional derivatives [Equation (6)] can be represented as a composition of the Hadamard fractional integral [Equation (1)]and the Hadamard fractional derivative [Equation (4)] as follows

$$\left(D_{a+}^{\alpha,\beta} f\right)(t) = J_{a+}^{\beta(n-\alpha)}\delta^n\left(J_{a_+}^{(1-\beta)(n-\alpha)} f\right)(t) = \left(J_{a^+}^{\beta(n-\alpha)} D_{a+}^{\alpha+\beta(n-\alpha)} f\right)(t). \tag{16}$$

For $f \in AC_\delta^n[a,b], 0 \leq \beta \leq 1$ and use Equation (4), we get

$$\begin{array}{ll} \left(D_{a+}^{\alpha,\beta} f\right)(t) = J_{a+}^{\beta(n-\alpha)} & \left(\Sigma_{k=0}^{n-1} \frac{\left(\delta^k f\right)(a)}{\Gamma(1+k-\alpha-\beta(n-\alpha))}\left(log\frac{t}{a}\right)^{k-\alpha-\beta(n-\alpha)}\right) \\ + & \left(\frac{1}{\Gamma((1-\beta)(n-\alpha))}\int_a^t \left(log\frac{t}{u}\right)^{(1-\beta)(n-\alpha)-1}(\delta^n f)(u)du\right) \end{array}.$$

Using Equation (3) and the definition of $J_{a+}^{\beta(n-\alpha)}$, changing the order of integrations and making the use of beta integral, we will reach Equation (14).

The result [Equation (15)] for $Df_{b-}^{\alpha,\beta}$, can be proved on the same lines.

Theorem 3.1.2: If $n = -[-\Re(\alpha)], \Re(\gamma) > \Re(\alpha+\beta(n-\alpha)) > 0, 0 \leq \beta \leq 1, 1 \leq p < \infty$ and $0 < a < b < \infty$, then for $f \in L^p(a,b)$ we have

$$\left(D_{a+}^{\alpha,\beta} J_{a+}^{\gamma}\right) f(t) = J_{a+}^{\gamma-\alpha} f(t), \tag{17}$$

$$\left(D_{b-}^{\alpha,\beta} J_{b-}^{\gamma}\right) f(t) = J_{b-}^{\gamma-\alpha} f(t). \tag{18}$$

Proof: On using Equation (16), the left side of Equation (17) becomes

$\left(D_{a+}^{\alpha,\beta} J_{a+}^{\gamma}\right) f(t) = \left(J_{a+}^{\beta(n-\alpha)} D_{a+}^{\alpha+\beta(n-\alpha)} J_{a+}^{\gamma}\right) f(t),$

for $\Re(\gamma) > \Re(\alpha+\beta(n-\alpha)) > 0$, on applying Equation (5), we have

$\left(D_{a+}^{\alpha,\beta} J_{a+}^{\gamma}\right) f(t) = \left(J_{a+}^{\beta(n-\alpha)} J_{a+}^{\gamma-(\alpha+\beta(n-\alpha))}\right) f(t).$

Finally, applying the semigroup property[4] for Hadamard fractional integrals $f \in L^p(a,b)$, we arrive at Equation (17). The result Equation (18), for $Df_{b-}^{\alpha,\beta}$, can be proved on the same lines.

3.2 Law of Exponents for Hilfer-Hadamard Fractional Derivatives

We will now establish a condition that precisely defines the specific conditions under which the law of exponents can be effectively applied to Hilfer-Hadamard fractional derivatives. Similar results are found for the law of exponents of Caputo-Hadamard fractional derivatives in[22].

Theorem 3.2.1: Assume that $f(t) = \left(log\frac{t}{a}\right)^{\sigma} g(t)$, where $a, \sigma > 0$ and $g(t)$ has a generalized series expansion $g(t) = \sum_{k=0}^{\infty} \lambda_k \left(log\frac{t}{a}\right)^{k\alpha}$ with a radius of convergence $R > 0, 0 < \alpha \leq 1$.

Then

$$D_{a+}^{\gamma,\beta} D_{a+}^{\delta,\beta} f(t) = D_{a+}^{\gamma+\delta,\beta} f(t), \tag{19}$$

for all $log\frac{t}{a} \in (0,R), \rho = max(\delta+\gamma+\beta(-[-\gamma]-\gamma)-1, \delta+\gamma+\beta(-[-\delta-\gamma]-\gamma-\delta)-1)$ and either

(a) $\sigma > \rho$, or

(b) $\sigma = \rho,\ \lambda_0 = 0$, or

(c) $\sigma \leq \rho,\ \lambda_k = 0$, for $k = 0, 1, \ldots, -\left[-\frac{\rho-\sigma}{\alpha}\right] - 1$.

Proof: For part (a), we have by definition (6) [Equation (6)]

$$\left(D_{a+}^{\delta,\beta} f\right)(t) = J_{a+}^{\beta(-[-\delta]-\delta)} D_{a+}^{\delta+(-[-\delta]-\delta)\beta} \sum_{k=0}^{\infty} \lambda_k \left(log\frac{t}{a}\right)^{k\alpha+\sigma} \tag{20}$$

Differentiating term by term is permissible due to the condition $\sigma > \rho \geq -1$, that the series involving derivatives up to the order $\delta + ([\delta]+1-\delta)\beta$ of the term $\left(log\frac{t}{a}\right)^{k\alpha+\sigma}$ are uniformly convergent for $log\frac{t}{a} \in (0,R)$. Thus, we have

$$\left(D_{a+}^{\delta,\beta} f\right)(t) = J_{a+}^{\beta(-[-\delta]-\delta)} \sum_{k=0}^{\infty} \lambda_k \frac{\Gamma(k\alpha+\sigma+1)}{\Gamma(k\alpha+\sigma+1-\delta-\beta(-[-\delta]-\delta))} \left(log\frac{t}{a}\right)^{k\alpha+\sigma-\delta-\beta(-[-\delta]-\delta)}$$

We have $\sigma > \rho \geq \delta + \beta(-[-\delta]-\delta) - 1$, using the result [Equation (3)] and uniform convergence of this series $log\frac{t}{a} \in (0,R)$, changing the order of integration and summation, we arrive at the following

$$\left(D_{a+}^{\delta,\beta} f\right)(t) = \sum_{k=0}^{\infty} \lambda_k \frac{\Gamma(k\alpha+\sigma+1)}{\Gamma(k\alpha+\sigma-\delta+1)} \left(log\frac{t}{a}\right)^{k\alpha+\sigma-\delta}. \tag{21}$$

Applying the same argument as above $\sigma > \rho \geq \delta - 1$ $\sigma > \rho \geq \delta+\gamma+\beta(-[-\gamma]-\gamma)-1$,we now have

$$\begin{aligned} D_{a+}^{\gamma,\beta} D_{a+}^{\delta,\beta} f(t) &= D_{a+}^{\gamma,\beta} \sum_{k=0}^{\infty} \lambda_k \frac{\Gamma(k\alpha+\sigma+1)}{\Gamma(k\alpha+\sigma-\delta+1)} \left(log\frac{t}{a}\right)^{k\alpha+\sigma-\delta} \\ &= \sum_{k=0}^{\infty} \lambda_k \frac{\Gamma(k\alpha+\sigma+1)}{\Gamma(k\alpha+\sigma-\gamma-\delta+1)} \left(log\frac{t}{a}\right)^{k\alpha+\sigma-\gamma-\delta} \end{aligned}. \tag{22}$$

Next for $\sigma > \rho \geq -1$ and $\sigma > \rho \geq \delta+\gamma+\beta(-[-\gamma-\delta]-\gamma-\delta)-1$

$$\begin{aligned} D_{a+}^{\gamma+\delta,\beta} f(t) &= D_{a+}^{\gamma+\delta,\beta} \sum_{k=0}^{\infty} \lambda_k \left(log\frac{t}{a}\right)^{k\alpha+\sigma} \\ &= \sum_{k=0}^{\infty} \lambda_k \frac{\Gamma(k\alpha+\sigma+1)}{\Gamma(k\alpha+\sigma-\gamma-\delta+1)} \left(log\frac{t}{a}\right)^{k\alpha+\sigma-\gamma-\delta} \end{aligned}, \tag{23}$$

this is precisely $D_{a+}^{\gamma,\beta} D_{a+}^{\delta,\beta} f(x)$, as given by Equation (22).

The condition given in part (a) can be formed by combining the conditions obtained from Equations (22) and (23).

For part (b), i.e. $\sigma \leq \rho$, taking $\lambda_k = 0$, for $0, 1, \ldots, l-1$, with $l = -\left[-\frac{\rho-\sigma}{\alpha}\right]$, using the uniform convergence of derived series up to the order $-[-\delta]$, we have

$$\begin{aligned} \left(D_{a+}^{\delta,\beta} f\right)(t) = D_{a+}^{\delta,\beta} \sum_{k=0}^{\infty} \lambda_k \left(log\frac{t}{a}\right)^{k\alpha+\sigma} &= \sum_{k=l}^{\infty} \lambda_k \frac{\Gamma(k\alpha+\sigma+1)}{\Gamma(k\alpha+\sigma-\delta+1)} \left(log\frac{t}{a}\right)^{k\alpha+\sigma-\delta} \\ &= \sum_{r=0}^{\infty} \lambda_{r+l} \frac{\Gamma((r+l)\alpha+\sigma+1)}{\Gamma((r+l)\alpha+\sigma-\delta+1)} \left(log\frac{t}{a}\right)^{(r+l)\alpha+\sigma-\delta} \end{aligned} \tag{24}$$

If we let it $\sigma^{'} = l\alpha + \sigma$, then Equation (24) becomes the same as Equation (21) (with σ replaced by $\sigma^{'}$) and the proof is the same as in part (a).

Theorem 3.2.2: If $(t) = \left(log\frac{b}{t}\right)^{\sigma} g(t), b, \sigma > 0$ with the generalized series expansion of $g(t)$,$g(t) = \sum_{k=0}^{\infty} \lambda_k \left(log\frac{b}{t}\right)^{k\alpha}$ as and its radius of convergence $R > 0,\ 0 < \alpha \leq 1$. Then

$$D_{b-}^{\gamma,\beta} D_{b-}^{\delta,\beta} f(t) = D_{b-}^{\gamma+\delta,\beta} f(t), \tag{25}$$

for $log\frac{b}{t} \in (0,R),\ \rho = max(\delta+\gamma+\beta(-[-\gamma]-\gamma)-1, \delta+\gamma+\beta(-[-\delta-\gamma]-\gamma-\delta)-1)$ and either

(a) $\sigma > \rho$, or

(b) $\sigma = \rho,\ \lambda_0 = 0$, or

(c) $\sigma < \rho,\ \lambda_k = 0$, for $k = 0, 1, \ldots, -\left[-\frac{\mu-\lambda}{\alpha}\right] - 1$.

The proof follows on the same lines as Theorem 3.2.1. These conditions are obtained assuming δ, γ, β to be real, the same may be extended for complex $\delta, \gamma, \beta, \sigma$ with $\delta, \gamma, \beta, \sigma$ replaced by $\Re(\delta),\ \Re(\gamma),\ \Re(\beta),\ \Re(\sigma)$.

3.3 Fractional polar Laplace Equation and Fractional Sturm-Liouville problem involving Hilfer-Hadamard Fractional Derivative

We now focus on solving the fractional polar Laplace Equation and fractional Sturm-Liouville problem using the left-sided Hilfer-Hadamard fractional derivatives. We employ the Adomian decomposition method to iteratively obtain solutions in series form.

Problem 3.3.1: Consider the Hilfer-Hadamard fractional polar Laplace equation on an annulus

$${}_rD_{a+}^{\alpha,\beta} u(r,\theta) = -\frac{\partial^2 u(r,\theta)}{\partial\theta^2},\ a < r < \rho,\ -\pi \leq \theta \leq \pi, \tag{26}$$

where $1 < \Re(\alpha) \leq 2\ 0 \leq \beta \leq 1,\ 0 < a < \rho$ and ${}_rD_{a+}^{\alpha,\beta}$ is the left-sided Hilfer-Hadamard fractional derivative defined by, for r, subject to the conditions

$${}_rJ_{a+}^{(2-\alpha)(1-\beta)}\ u(r,\theta)|_{r=a} = 0, \tag{27}$$

$$u(\rho,\theta) = \sum_{n=0}^{\infty} \left(A_n cosn\theta + B_n sinn\theta\right). \tag{28}$$

The solution to this problem is obtained as:

$$u(r,\theta) = \left(\frac{log\frac{r}{a}}{log\frac{\rho}{a}}\right)^{\alpha+\beta(2-\alpha)-1} \sum_{n=0}^{\infty} \left(A_n cosn\theta + B_n sinn\theta\right) \frac{E_{\alpha,\alpha+\beta(2-\alpha)}\left[n^2\left(log\frac{r}{a}\right)^{\alpha}\right]}{E_{\alpha,\alpha+\beta(2-\alpha)}\left[n^2\left(log\frac{\rho}{a}\right)^{\alpha}\right]} \tag{29}$$

where $E_{\mu,v}(x)$ is the well-known Mittag-Leffler type function.

Solution: Applying ${}_rJ_{a+}^{\alpha}$ on both sides of Equation (26), using the result [Equation (8)], initial condition [Equation (27)], and assuming $\delta_r J_{a+}^{(2-\alpha)(1-\beta)} u(r,\theta)\Big|_{r=a} = g(\theta) = \sum_{n=0}^{\infty} \left(a_n cosn\theta + b_n sinn\theta\right)$, we get

$$u(r,\theta) = \frac{\left(log\frac{r}{a}\right)^{\alpha+\beta(2-\alpha)-1}}{\Gamma(\alpha+\beta(2-\alpha))} \sum_{n=0}^{\infty} \left(a_n cosn\theta + b_n sinn\theta\right) - {}_rJ_{a+}^{\alpha}\left(\frac{\partial^2 u(r,\theta)}{\partial\theta^2}\right). \tag{30}$$

Decomposing $u(r,\theta)$ into an infinite sum of its components, as

$$u(r,\theta) = \sum_{k=0}^{\infty} u_k(r,\theta). \tag{31}$$

These components can be obtained recursively as

$$u_0(r,\theta) = \frac{\left(log\frac{r}{a}\right)^{\alpha+\beta(2-\alpha)-1}}{\Gamma(\alpha+\beta(2-\alpha))} \sum_{n=0}^{\infty} \left(a_n cosn\theta + b_n sinn\theta\right) \tag{32}$$

and

$$u_{k+1}(r,\theta) = -\,{}_{r}J^{\alpha}_{a+}\left(\frac{\partial^2 u_k(r,\theta)}{\partial\theta^2}\right), k = 0,1,2,3,\ldots \tag{33}$$

Using Equations (32) and (33), we obtain these components as

$$u_k(r,\theta) = \frac{\left(log\frac{r}{a}\right)^{(k+1)\alpha+\beta(2-\alpha)-1}}{\Gamma((k+1)\alpha+\beta(2-\alpha))}\sum_{n=0}^{\infty} n^{2k}\left(a_n cosn\theta + b_n sinn\theta\right), k = 0,1,2,3,\ldots \tag{34}$$

Substituting Equation (34) into Equation (31), we obtain

$$u(r,\theta) = \left(log\frac{r}{a}\right)^{\alpha+\beta(2-\alpha)-1}\sum_{n=0}^{\infty}\left(a_n cosn\theta + b_n sinn\theta\right)E_{\alpha,\alpha+\beta(2-\alpha)}\left[n^2\left(log\frac{r}{a}\right)^{\alpha}\right] \tag{35}$$

Using condition [Equation (28)], we get

$$a_n = \frac{A_n}{\left(log\frac{\rho}{a}\right)^{\alpha+\beta(2-\alpha)-1}E_{\alpha,\alpha+\beta(2-\alpha)}\left[n^2\left(log\frac{\rho}{a}\right)^{\alpha}\right]},\ b_n = \frac{B_n}{\left(log\frac{\rho}{a}\right)^{\alpha+\beta(2-\alpha)-1}E_{\alpha,\alpha+\beta(2-\alpha)}\left[n^2\left(log\frac{\rho}{a}\right)^{\alpha}\right]}.$$

which on substituting in Equation (35) gives the solution Equation (29).

Special Cases:

(i) For $\beta = 1$, Problem 3.3.1 reduces to a two-dimensional polar Laplace equation on an annulus with Caputo Hadamard fractional derivative given as

$${}^{C}_{r}D^{\alpha}_{a+}u(r,\theta) = -\frac{\partial^2 u(r,\theta)}{\partial\theta^2},\ a<r<\rho,\ -\pi\le\theta\le\pi \tag{36}$$

subject to the conditions $u(a,\theta) = 0, u(\rho,\theta) = \sum_{n=0}^{\infty}\left(A_n cosn\theta + B_n sinn\theta\right)$,

and the general solution is given as

$$u(r,\theta) = \left(\frac{log\frac{r}{a}}{log\frac{\rho}{a}}\right)\sum_{n=0}^{\infty}\left(A_n cosn\theta + B_n sinn\theta\right)\frac{E_{\alpha,2}\left[n^2\left(log\frac{r}{a}\right)^{\alpha}\right]}{E_{\alpha,2}\left[n^2\left(log\frac{\rho}{a}\right)^{\alpha}\right]} \tag{37}$$

(ii) For $\beta = 0$, Problem 3.3.1 reduces to a two-dimensional polar Laplace equation on an annulus with Hadamard fractional derivative given as

$${}_{r}D^{\alpha}_{a+}u(r,\theta) = -\frac{\partial^2 u(r,\theta)}{\partial\theta^2},\ a<r<\rho,\ -\pi\le\theta\le\pi \tag{38}$$

subject to the conditions ${}_{r}J^{(2-\alpha)}_{a+}u(r,\theta)\Big|_{r=a} = 0, u(\rho,\theta) = \sum_{n=0}^{\infty}\left(A_n cosn\theta + B_n sinn\theta\right)$ with its general solution

$$u(r,\theta) = \left(\frac{log\frac{r}{a}}{log\frac{\rho}{a}}\right)^{\alpha-1}\sum_{n=0}^{\infty}\left(A_n cosn\theta + B_n sinn\theta\right)\frac{E_{\alpha,\alpha}\left[n^2\left(log\frac{r}{a}\right)^{\alpha}\right]}{E_{\alpha,\alpha}\left[n^2\left(log\frac{\rho}{a}\right)^{\alpha}\right]} \tag{39}$$

(iii) For $\alpha = 2$, Problem 3.3.1 reduces to a two-dimensional polar Laplace equation on an annulus with Hadamard fractional derivative given as

$$\left(r\frac{\partial}{\partial r}\right)^2 u + \frac{\partial^2 u}{\partial\theta^2} = 0,\ a<r<\rho,\ -\pi\le\theta\le\pi, \tag{40}$$

subject to the conditions $u(a,\theta)=0, u(\rho,\theta)=\sum_{n=0}^{\infty}(A_n cosn\theta + B_n sinn\theta)$, with its general solution

$$u(r,\theta)=\sum_{n=0}^{\infty}(A_n cosn\theta+B_n sinn\theta)\frac{\left(\left(\frac{r}{a}\right)^n+\left(\frac{r}{a}\right)^{-n}\right)}{\left(\left(\frac{\rho}{a}\right)^n+\left(\frac{\rho}{a}\right)^{-n}\right)}. \tag{41}$$

Remark: It is noted here that in the case of integer order differential equation, we have $\left(r\frac{\partial}{\partial r}\right)^2 \equiv \left(r\frac{\partial}{\partial r}\right)\left(r\frac{\partial}{\partial r}\right)$, but when dealing with Hilfer Hadamard fractional derivative, particularly in the present problem, ${}_rD_{a+}^{\alpha,\beta}u(r,\theta) \neq {}_rD_{a+}^{\alpha_1,\beta}\,{}_rD_{a+}^{\alpha_2,\beta}u(r,\theta)$ for any $0<\alpha_1\leq 1, 0<\alpha_2\leq 1, \alpha_1+\alpha_2=\alpha$, since the conditions required in the law of exponents given in Theorem 3.2.1 are not satisfied.

Problem 3.3.2: Considering the Hilfer-Hadamard fractional Sturm-Liouville boundary value problem

$$D_{a+}^{\alpha,\beta}y(x)=-\lambda y(x), a<x<b, \lambda \in \square \tag{42}$$

where $1<\Re(\alpha)\leq 2,\ 0\leq\beta\leq 1$ and $D_{a+}^{\alpha,\beta}$ is the left-sided Hilfer-Hadamard fractional derivative defined by Equation (6) for x, subject to the conditions

$$J_{a+}^{(2-\alpha)(1-\beta)}\ y|_{x=a}=0,\ J_{a+}^{(2-\alpha)(1-\beta)}\ y|_{x=b}=0 \tag{43}$$

The eigenvalues of the Equation (42) satisfy the following condition

$$\left(log\frac{b}{a}\right)E_{\alpha,2}\left(-\lambda_n\left(log\frac{b}{a}\right)^{\alpha}\right)=0 \tag{44}$$

and the corresponding eigenfunctions are

$$y_n(x)=C_n\left(log\frac{x}{a}\right)E_{\alpha,2}\left(-\lambda_n\left(log\frac{x}{a}\right)^{\alpha}\right). \tag{45}$$

The solution of the given Sturm-Liouville Equation (42) is

$$y(x)=\sum_{n=1}^{\infty}y_n(x)=\left(log\frac{x}{a}\right)\sum_{n=0}^{\infty}C_nE_{\alpha,2}\left(-\lambda_n\left(log\frac{x}{a}\right)^{\alpha}\right) \tag{46}$$

where $E_{\mu,\nu}(x)$ is the well-known Mittag-Leffler type function.

Solution: Applying J_{a+}^{α} on both sides of Equation (42) and using the result [Equation (8)] for $n=1$, using the initial condition [Equation (43)] and assuming $\delta y(x)|_{x=a}=C$ we get

$$y(x)=C\frac{\left(log\frac{x}{a}\right)^{\alpha+\beta(2-\alpha)-1}}{\Gamma(\alpha+\beta(2-\alpha))}-\lambda J_{a+}^{\alpha}y(x). \tag{47}$$

Decomposing $y(x)$ into an infinite sum of its components as

$$y(x)=\sum_{k=0}^{\infty}y_k(x). \tag{48}$$

These components can be obtained recursively as

$$y_0=C\frac{\left(log\frac{x}{a}\right)^{\alpha+\beta(2-\alpha)-1}}{\Gamma(\alpha+\beta(2-\alpha))} \tag{49}$$

and

$$y_{k+1}(x)=-\lambda J_{a+}^{\alpha}\ y_k(x) \tag{50}$$

Using Equation (3) in recursive formulae [Equations (49) and (50)], we obtain this component as

$$y_k = C\frac{(-\lambda)^k\left(log\frac{x}{a}\right)^{k\alpha+\alpha+\beta(2-\alpha)-1}}{\Gamma(k\alpha+\alpha+\beta(2-\alpha))} \tag{51}$$

$$\begin{aligned} y(x) &= C\left(log\frac{x}{a}\right)^{\alpha+\beta(2-\alpha)-1}\Sigma_{k=0}^{\infty}\frac{(-\lambda)^k\left(log\frac{x}{a}\right)^{k\alpha}}{\Gamma(k\alpha+\alpha+\beta(2-\alpha))} \\ &= C\left(log\frac{x}{a}\right)^{\alpha+\beta(2-\alpha)-1}E_{\alpha,\alpha+\beta(2-\alpha)}\left(-\lambda\left(log\frac{x}{a}\right)^{\alpha}\right) \end{aligned} \tag{52}$$

In particular, if $\lambda = 0$, the Sturm-Liouville boundary value problem [Equation (42)] reduces to

$$D_{a+}^{\alpha,\beta}y(x) = 0,\ a < x < b \tag{53}$$

with its solution

$$y(x) = C\left(log\frac{x}{a}\right)^{\alpha+\beta(2-\alpha)-1} \tag{54}$$

The boundary condition $J_{a+}^{(2-\alpha)(1-\beta)}y\Big|_{x=b} = 0$ gives $C\Gamma(\alpha+\beta(2-\alpha))\left(log\frac{b}{a}\right) = 0 \Rightarrow C = 0$ and hence $y(x) = 0$.

For $\lambda < 0$, we write $\lambda = -\mu^2$ with $\mu > 0$, the Sturm-Liouville boundary value problem [Equation (42)] reduces to

$$D_{a+}^{\alpha,\beta}y(x) = \mu^2 y(x),\ a < x < b, \tag{55}$$

with its solution

$$y(x) = C\left(log\frac{x}{a}\right)^{\alpha+\beta(2-\alpha)-1}E_{\alpha,\alpha+\beta(2-\alpha)}\left(\mu^2\left(log\frac{x}{a}\right)^{\alpha}\right) \tag{56}$$

On using the condition $J_{a+}^{(2-\alpha)(1-\beta)}y\Big|_{x=b} = 0$, we have $C\left(log\frac{b}{a}\right)E_{\alpha,2}\left(\mu^2\left(log\frac{b}{a}\right)^{\alpha}\right) = 0$. This holds with $C \neq 0$, if and only if, we can choose μ such that $E_{\alpha,2}\left(\mu^2\left(log\frac{b}{a}\right)^{\alpha}\right) = 0$.

For $\lambda > 0$, we write $\lambda = \mu^2$ with $\mu > 0$, the Sturm-Liouville boundary value problem [Equation (42)] reduces to

$$D_{a+}^{\alpha,\beta}y(x) = -\mu^2 y(x),\ a < x < b, \tag{57}$$

with its solution

$$y(x) = C\left(log\frac{x}{a}\right)^{\alpha+\beta(2-\alpha)-1}E_{\alpha,\alpha+\beta(2-\alpha)}\left(-\mu^2\left(log\frac{x}{a}\right)^{\alpha}\right) \tag{58}$$

On using the condition $J_{a+}^{(2-\alpha)(1-\beta)}y\Big|_{x=b} = 0$, we have $C\left(log\frac{b}{a}\right)E_{\alpha,2}\left(-\mu^2\left(log\frac{b}{a}\right)^{\alpha}\right) = 0$. This holds with $C \neq 0$ if and only if, we can choose μ such that $E_{\alpha,2}\left(-\mu^2\left(log\frac{b}{a}\right)^{\alpha}\right) = 0$.

Hence, for $C \neq 0$ and $\mu = \sqrt{\lambda_n}$, the eigenvalues and the corresponding eigenfunctions of the given Sturm-Liouville problem [Equation (42)] with initial conditions [Equation (43)] are given by Equations (44) and (45) respectively, and therefore the solution is given by Equation (46).

Now, we compare the solutions obtained with the classical Sturm-Liouville problem, for this we take $\alpha \to 2, \beta \to 1$, and reach the following Sturm-Liouville boundary value problem

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \lambda y = 0,\ y(a) = 0,\ y(b) = 0 \tag{59}$$

with solution

(i) For $\lambda = 0,\ y(x) = 0,$

(ii) For $\lambda < 0,\ \lambda = -\mu^2$ with $\mu > 0$, for $C \neq 0$, we require $\frac{sinh\left(\mu log\frac{b}{a}\right)}{\mu log\frac{b}{a}} = 0$, which does not give any non-zero real value of μ, therefore $y(x) = 0$,

(iii) For $\lambda > 0,\ \lambda = \mu^2$ with $\mu > 0$, for $C \neq 0$, we require $\frac{sin\left(\mu log\frac{b}{a}\right)}{\mu log\frac{b}{a}} = 0$, which gives the eigenvalues and the corresponding eigenfunctions as

$$\lambda_n = \frac{n^2\pi^2}{\left(log\frac{b}{a}\right)^2},\ y_n(x) = C_n\frac{log\frac{b}{a}}{n\pi}sin\left(\frac{n\pi}{log\frac{b}{a}}log\frac{x}{a}\right) \tag{60}$$

and the solution of the Sturm-Liouville problem [Equation (59)] is

$$y(x) = \sum_{n=1}^{\infty} y_n(x) = \sum_{n=1}^{\infty} C_n\frac{log\frac{b}{a}}{n\pi}sin\left(\frac{n\pi}{log\frac{b}{a}}log\frac{x}{a}\right). \tag{61}$$

Therefore, for all the cases the solutions of the fractional Sturm-Liouville boundary value problem coincide with the solutions of the classical case.

4 Conclusion

To facilitate the study of fractional differential equations involving the left-sided and right-sided Hilfer-Hadamard fractional derivatives, several fundamental properties of the derivatives are formulated along with the requirements for the law of exponents. These properties are then used with the Adomian decomposition method to solve fractional polar Laplace and fractional SturmLiouville problems with Hilfer-Hadamard fractional derivatives. The solutions for both problems are derived concisely, using the Mittag-Leffler type function.

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