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Numerical Solution of Hybrid Fuzzy Differential Equation by using Third Order Runge-Kutta Method Based on Linear Combination of Arithmetic Mean, Root Mean Square and Centroidal Mean

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Abstract

Objectives: This article develops the third order Runge-Kutta method, which uses a linear combination of the arithmetic mean, root mean square, and centroidal mean, to solve hybrid fuzzy differential equations. **Methods:** Seikkala's derivative is taken into account, and a numerical example is provided to show the efficacy of the proposed method. The outcomes demonstrate that the suggested approach is an effective tool for approximating the solution of hybrid fuzzy differential equations. **Findings:** The comparative analysis was carried out using the third order Runge-Kutta method that is currently in use and is based on arithmetic mean, root mean square, and centroidal mean. Compared to other methods, the suggested method offers a more accurate approximation. **Novelty:** In this study a new formula has been developed by combining three means Arithmetic Mean, Root Mean Square, and Centroidal Mean using Khattri's formula. And the developed formula is used to solve the third order Runge-Kutta method for the first order hybrid fuzzy differential equation. All real life problems which can be modeled in to an initial value problem can be solved using this formula.

Keywords: Hybrid fuzzy differential equations; Triangular fuzzy number; Seikkala's derivative; third order Runge-Kutta method; Arithmetic mean; Root mean square; Centroidal mean; Initial value problem

1 Introduction

Hybrid systems can be used to represent control systems that are capable of managing complicated systems with both discrete time dynamics and continuous time dynamics. Hybrid fuzzy differential systems are differential systems that interact with a discrete time controller while using fuzzy valued functions.

A Third order Runge-Kutta method based on linear combination of three means of various combinations have been proposed by Evangelin Diana Rajakumari P and Gethsi Sharmila R in [1], [2] and [3]. A method is proposed by Jeyaraj T and Rajan D to solve fuzzy Initial value problem using explicit Runge-Kutta method in [4]. Arithmetic operations of Trigonal fuzzy numbers using Alpha cuts has been proposed by Kalaiarasi k and Swathi S to handle uncertainty in [5]. Iterative methods have been proposed by Kanade G D and Dhaigude R M for solving Ordinary differential equations in [6]. Numerical Solutions of Fuzzy Multiple Hybrid Single Retarded Delay Differential Equations have been found by Prasantha Bharathi D and Jayakumar T and Vinoth S in [7]. Prasantha Bharathi D, Jayakumar T and Vinoth S have proposed a method to find the Numerical Solutions of Fuzzy Pure Multiple Neutral Delay Differential Equations in [8]. Understanding the Unique Properties of Fuzzy concept in Binary Trees is done by Renuka Sahu and Animesh Kumar Sharma in [9]. Srinivasan R, Karthikeyan N and Jayaraja A have proposed a technique to resolve transportation problem by trapezoidal fuzzy numbers in [10].

In this study a new formula has been developed by combining three means Arithmetic Mean, Root Mean Square, and Centroidal Mean using Khattri's formula. And the developed formula is used to solve the third order Runge-Kutta method for the first order hybrid fuzzy differential equation. This method is used to solve any real life problem which is modeled as initial value problem

2 Preliminaries

Definition: 2.1 (Fuzzy Set)

If X is a collection of objects denoted generally by x, then a fuzzy set $\widetilde{A}$ in X is a set of ordered pairs $\widetilde{A} = \left\{ \left(x, \mu_{\widetilde{A}(x)} \right) / x \in X \right\}$, where $\mu_{\widetilde{A}(x)}$ is called the membership function or grade of membership of x in $\widetilde{A}$.

Definition: 2.2 (Triangular fuzzy number)

It is a fuzzy number represented with three points as follows: $\widetilde{A} = (a_1, a_2, a_3)$, this representation is interpreted as membership functions and holds the following condition:

(i) a_1 to a_2 is increasing function

(ii) a_2 to a_3 is decreasing function

(iii) $a_1 \le a_2 \le a_3$

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0 & , x < a_1 \\ \frac{x-a_1}{a_2-a_1} & , a_1 \le x \le a_2 \\ \frac{a_3-x}{a_3-a_2} & , a_2 \le x \le a_3 \\ 0 & , x > a_3 \end{cases}$$

Now, if you get crisp interval by α - cut operation, interval $\widetilde{A}_\alpha$ shall be obtained as follows $\forall \alpha \in [0,1]$ from

$$\frac{a_1^{(\alpha)} - a_1}{a_2 - a_1} = \alpha \, , \; \frac{a_3 - a_3^{(\alpha)}}{a_3 - a_2} = \alpha$$

The Third order Runge-Kutta method based on a linear combination of Arithmetic mean (AM), Root mean square (RM) and Centroidal mean (CeM) for IVP was introduced by Khattri as follows

$$Y_{n+1}(k_1, k_2) = \frac{14AM(k_1, k_2) - RM(k_1, k_2) + 32CeM(k_1, k_2)}{45}$$

3 The Hybrid Fuzzy Differential Equations

The hybrid fuzzy differential equation

$$\begin{aligned} &\dot{y}(x) = f(t, y(t), \lambda_k(y_k)), t \in [t_k, t_{k+1}] \; k = 0, 1, 2, \ldots \\ &y(t_0) = y_0 \end{aligned} \tag{3.1}$$

where $t_k = 0$ is strictly increasing and unbounded, y_k denotes $y(t_k)$, $f : (t_0, \infty) \times R \times R \to R$ is continuous, and each $\lambda_k : R \to R$ is a continuous function. A solution to Equation (3.1) will be a function $y : (t_0, \infty) \to R$ satisfying Equation (3.1). For f, let $f_k : (t_k, t_{k+1}) \times R \to R$ where, $f(t, y_k(t)) = f(t, y_k(t), \lambda_k(y_k))$.

The hybrid fuzzy differential equation in Equation (3.1) can be written in expanded form as

$$\dot{y}(t)=\begin{cases}\dot{y}_0(t)=f(t,y_0\,(t),\lambda_0\,(y_0))=f_0\,(t,y_0\,(t)),y_0\,(t_0)=y_0,t_0\le t\le t_1\\ \dot{y}_1(t)=f(t,y_1\,(t),\lambda_1\,(y_1))=f_1\,(t,y_1\,(t)),y_1\,(t_1)=y_1,t_1\le t\le t_2\\ \cdot\\ \cdot\\ \cdot\\ \dot{y}_k(t)=f(t,y_k\,(t),\lambda_k\,(y_k))=f_k\,(t,y_k\,(t)),y_k\,(t_k)=y_k,t_k\le t\le t_{k+1}\\ \cdot\\ \cdot\\ \cdot\end{cases}\tag{3.2}$$

and a solution of Equation (3.1) can be expressed as

$$y(t)=\begin{cases}y_0(t)=y_0,\quad t_0\le t\le t_1\\ y_0(t)=y_0,\quad t_0\le t\le t_1\\ \cdot\\ \cdot\\ y_0(t)=y_0,\quad t_0\le t\le t_1\\ \cdot\\ \cdot\end{cases}\tag{3.3}$$

we note that the solution of Equation (3.1) is continuous and piece wise differentiable over (t_0,∞) and differentiable on each interval (t_k,t_{k+1}) for any fixed $y_k\in R$ and $k=0,1,2,\dots$

4. A Third order Runge-Kutta method based on linear combination of Arithmetic mean, Root mean square and Centroidal mean for hybrid fuzzy differential equation

Consider the IVP Equation (3.1) with crisp initial condition $y\,(t_0)=y_0\in R$ and

$t\in[t_0,T]$. Let the exact solution $[Y(t)]_r=[\underline{Y}(t;r),\overline{Y}(t;r)]$ is approximated by some $[y(t)]_r=[\underline{Y}(t;r),\overline{Y}(t;r)]$ from the equation $y_{n+1}=y_n+\frac{h}{2}\left[\sum_{i=1}^{2}Means\right]$ where means includes Arithmetic mean (AM), Geometric mean (GM), Contra - Harmonic mean (CoM), Centroidal mean (CeM), Root mean Square (RM), Harmonic mean(HaM), and Heronian mean (HeM) which involves $k_i, 1\le i\le 4$, where,

$$k_1=f\,(t_n,y_n)$$
$$k_2=f\,(t_n+a_1h,y_n+a_1hk_1)$$
$$k_3=f\,(t_n+(a_2+a_3)\,h,y_n+a_2hk_1+a_3hk_2)$$

where the parameters for the linear combination of Arithmetic mean, Root mean square and Centroidal mean are

$$a_1=\frac{2}{3},a_2=\frac{-209}{810},a_3=\frac{749}{810}$$

The third order formulae based on Runge- Kutta scheme using the Combination of Arithmetic mean, Root mean square and Centroidal mean are

$$y_{n+1}=y_n+\frac{h}{90}\left\{7\,(k_1+2k_2+k_3)-\frac{1}{\sqrt{2}}\left(\sqrt{k_1^2+k_2^2}+\sqrt{k_2^2+k_3^2}\right)+\frac{64}{3}\left(\frac{k_1^2+k_1k_2+k_2^2}{k_1+k_2}+\frac{k_2^2+k_2k_3+k_3^2}{k_2+k_3}\right)\right\}$$

with the grid points $a=t_0\le t_1\le\cdots\le t_N=b$ and $h=\frac{(b-a)}{N}=t_{i+1}-t_i$

we define

$$\underline{y}_{k,n+1}(r)-\underline{y}_{k,n}(r)=\textstyle\sum_{i=1}^{3}\;w_i\underline{k}_i\left(t_{k,n};y_{k,n}(r)\right)$$
$$\overline{y}_{k,n+1}(r)-\overline{y}_{k,n}(r)=\textstyle\sum_{i=1}^{3}\;w_i\overline{k}_i\left(t_{k,n};y_{k,n}(r)\right)$$

where w_1,w_2,w_3 are constants,

$$\underset{-1}{k}\left(t_{k,n};y_{k,n}(r)\right)=min\left\{f\left(t_{k,n},u,\lambda_k\,(u_k)\right)\backslash u\in\left[\underset{-k,n}{y}(r),\overline{y}_{k,n}(r)\right],u_k\in\left[\underset{-k,0}{y}(r),\overline{y}_{k,0}(r)\right]\right\}$$

$$\bar{k}_1\left(t_{k,n};y_{k,n}(r)\right)=max\left\{f\left(t_{k,n},u,\lambda_k\left(u_k\right)\right)\backslash u\in\left[\underline{y}_{k,n}(r),\bar{y}_{k,n}(r)\right],u_k\in\left[\underline{y}_{k,0}(r),\bar{y}_{k,0}(r)\right]\right\}$$

$$\underline{k}_2\left(t_{k,n};y_{k,n}(r)\right)=min\left\{\begin{array}{c}f\left(t_{k,n}+\frac{2}{3}h_k,u,\lambda_k\left(u_k\right)\right)\backslash u\in\left[\underline{z}_{k_1}\left(t_{k,n};y_{k,n}(r)\right),\bar{z}_{k_1}\left(t_{k,n};y_{k,n}(r)\right)\right],\\ u_k\in\left[\underline{y}_{k,0}(r),\bar{y}_{k,0}(r)\right]\end{array}\right\}$$

$$\bar{k}_2\left(t_{k,n};y_{k,n}(r)\right)=max\left\{\begin{array}{c}f\left(t_{k,n}+\frac{2}{3}h_k,u,\lambda_k\left(u_k\right)\right)\backslash u\in\left[\underline{z}_{k_1}\left(t_{k,n};y_{k,n}(r)\right),\bar{z}_{k_1}\left(t_{k,n};y_{k,n}(r)\right)\right],\\ u_k\in\left[\underline{y}_{k,0}(r),\bar{y}_{k,0}(r)\right]\end{array}\right\}$$

$$\underline{k}_3\left(t_{k,n};y_{k,n}(r)\right)=min\left\{\begin{array}{c}f\left(t_{k,n}+\frac{2}{3}h_k,u,\lambda_k\left(u_k\right)\right)\backslash u\in\left[\underline{z}_{k_2}\left(t_{k,n};y_{k,n}(r)\right),\bar{z}_{k_2}\left(t_{k,n};y_{k,n}(r)\right)\right],\\ u_k\in\left[\underline{y}_{k,0}(r),\bar{y}_{k,0}(r)\right]\end{array}\right\}$$

$$\bar{k}_3\left(t_{k,n};y_{k,n}(r)\right)=min\left\{\begin{array}{c}f\left(t_{k,n}+\frac{2}{3}h_k,u,\lambda_k\left(u_k\right)\right)\backslash u\in\left[\underline{z}_{k_2}\left(t_{k,n};y_{k,n}(r)\right),\bar{z}_{k_2}\left(t_{k,n};y_{k,n}(r)\right)\right],\\ u_k\in\left[\underline{y}_{k,0}(r),\bar{y}_{k,0}(r)\right]\end{array}\right\}$$

where

$$\begin{array}{c}\underline{z}_{k_1}\left(t_{k,n};y_{k,n}(r)\right)=\underline{y}_{k,n}(r)+\frac{2}{3}h_k\underline{k}_1\left(t_{k,n};y_{k,n}(r)\right)\\ \overline{Z}_{k_1}\left(t_{k,n};y_{k,n}(r)\right)=\bar{y}_{k,n}(r)+\frac{2}{3}h_k\bar{k}_1\left(t_{k,n};y_{k,n}(r)\right)\\ \underline{z}_{k_2}\left(t_{k,n};y_{k,n}(r)\right)=\underline{y}_{k,n}(r)-\frac{209}{810}h_k\underline{k}_1\left(t_{k,n};y_{k,n}(r)\right)+\frac{749}{810}h_k\underline{k}_2\left(t_{k,n};y_{k,n}(r)\right)\\ \bar{z}_{k_2}\left(t_{k,n};y_{k,n}(r)\right)=\bar{y}_{k,n}(r)-\frac{209}{810}h_k\bar{k}_1\left(t_{k,n};y_{k,n}(r)\right)+\frac{749}{810}h_k\bar{k}_2\left(t_{k,n};y_{k,n}(r)\right)\end{array}$$

define

$$S_k\left[t_{k,n};\underline{y}_{k,n}(r),\bar{y}_{k,n}(r)\right]=\left\{\begin{array}{c}7\left[\underline{k}_1\left(t_{k,n},y_{k,n}(r)\right)+2\underline{k}_2\left(t_{k,n},y_{k,n}(r)\right)+\underline{k}_3\left(t_{k,n},y_{k,n}(r)\right)\right]\\ -\frac{1}{\sqrt{2}}\left[\sqrt{\underline{k}_1^2\left(t_{k,n},y_{k,n}(r)\right)+\underline{k}_2^2\left(t_{k,n},y_{k,n}(r)\right)}+\sqrt{\underline{k}_2^2\left(t_{k,n},y_{k,n}(r)\right)+\underline{k}_3^2\left(t_{k,n},y_{k,n}(r)\right)}\right]\\ +\frac{64}{3}\left[\begin{array}{c}\frac{\underline{k}_1^2\left(t_{k,n},y_{k,n}(r)\right)+\underline{k}_1\left(t_{k,n},y_{k,n}(r)\right)\underline{k}_2\left(t_{k,n},y_{k,n}(r)\right)+\underline{k}_2^2\left(t_{k,n},y_{k,n}(r)\right)}{\underline{k}_1\left(t_{k,n},y_{k,n}(r)\right)+\underline{k}_2\left(t_{k,n},y_{k,n}(r)\right)}\\ +\frac{\underline{k}_2^2\left(t_{k,n},y_{k,n}(r)\right)+\underline{k}_2\left(t_{k,n},y_{k,n}(r)\right)\underline{k}_3\left(t_{k,n},y_{k,n}(r)\right)+\underline{k}_3^2\left(t_{k,n},y_{k,n}(r)\right)}{\underline{k}_2\left(t_{k,n},y_{k,n}(r)\right)+\underline{k}_3\left(t_{k,n},y_{k,n}(r)\right)}\end{array}\right]\end{array}\right\}$$

$$T_k\left[t_{k,n};\underline{y}_{k,n}(r),\bar{y}_{k,n}(r)\right]=\left\{\begin{array}{c}7\left[\bar{k}_1\left(t_{k,n},y_{k,n}(r)\right)+2\bar{k}_2\left(t_{k,n},y_{k,n}(r)\right)+\bar{k}_3\left(t_{k,n},y_{k,n}(r)\right)\right]\\ -\frac{1}{\sqrt{2}}\left[\sqrt{\bar{k}_1^2\left(t_{k,n},y_{k,n}(r)\right)+\bar{k}_2^2\left(t_{k,n},y_{k,n}(r)\right)}+\sqrt{\bar{k}_2^2\left(t_{k,n},y_{k,n}(r)\right)+\bar{k}_3^2\left(t_{k,n},y_{k,n}(r)\right)}\right]\\ +\frac{64}{3}\left[\begin{array}{c}\frac{\bar{k}_1^2\left(t_{k,n},y_{k,n}(r)\right)+\bar{k}_1\left(t_{k,n},y_{k,n}(r)\right)\bar{k}_2\left(t_{k,n},y_{k,n}(r)\right)\bar{k}_2^2\left(t_{k,n},y_{k,n}(r)\right)}{\bar{k}_1\left(t_{k,n},y_{k,n}(r)\right)+\bar{k}_2\left(t_{k,n},y_{k,n}(r)\right)}\\ +\frac{\bar{k}_2^2\left(t_{k,n},y_{k,n}(r)\right)+\bar{k}_2\left(t_{k,n},y_{k,n}(r)\right)\bar{k}_3\left(t_{k,n},y_{k,n}(r)\right)+\bar{k}_3^2\left(t_{k,n},y_{k,n}(r)\right)}{\bar{k}_2\left(t_{k,n},y_{k,n}(r)\right)+\bar{k}_3\left(t_{k,n},y_{k,n}(r)\right)}\end{array}\right]\end{array}\right\}$$

The exact solutions at $t_{k,n+1}$ is given by

$$\underset{-k,n+1}{Y}(r) \approx \underset{-k,n}{Y}(r) + \frac{h_k}{90} S_k \left[t_{k,n}; \underset{-k,n}{Y}(r), \overline{Y}_{k,n}(r) \right]$$

$$\overline{Y}_{k,n+1}(r) \approx \overline{Y}_{k,n}(r) + \frac{h_k}{90} T_k \left[t_{k,n}; \underset{-k,n}{Y}(r), \overline{Y}_{k,n}(r) \right]$$

The approximate solutions at $t_{k,n+1}$ is given by

$$\begin{array}{l} \underset{-k,n+1}{y}(r) = \underset{-k,n}{y}(r) + \frac{h_k}{90} S_k \left[t_{k,n}; \underset{-k,n}{y}(r), \overline{y}_{k,n}(r) \right] \\ \overline{y}_{k,n+1}(r) = \overline{y}_{k,n}(r) + \frac{h_k}{90} T_k \left[t_{k,n}; \underset{-k,n}{y}(r), \overline{y}_{k,n}(r) \right] \end{array}$$

4 Numerical Example

Consider the fuzzy Initial Value Problem

$$\left\{ \begin{array}{c} y'(t) = y(t), t \in [0,1] \\ y(0;r) = [0.75 + 0.25r, 1.125 - 0.125r], 0 \le r \le 1 \end{array} \right.$$

The exact solution is given by

$$Y(t;r) = \left[(0.75 + 0.25r)e^t, (1.125 - 0.125r)e^t \right], 0 \le r \le 1$$

at $t = 1$ we get

$$Y(1;r) = \left[(0.75 + 0.25r)e^1, (1.125 - 0.125r)e^1 \right], 0 \le r \le 1$$

By the third order Runge - Kutta method based on the combination of Arithmetic mean, Root mean square and Centroidal mean with $N = 2$, gives

$$y(1.0;r) = \left[(0.75 + 0.25r)(C_{0,1})^2, (1.125 - 0.125r)(C_{0,1})^2 \right], 0 \le r \le 1$$

where

$$C_{0,1} = 1 + \frac{h}{90} \left\{ \begin{array}{l} 7\left[4 + 2h + \frac{749}{1215}h^2\right] - \frac{1}{\sqrt{2}}\left[\sqrt{1 + \left(1 + \frac{2}{3}h\right)^2} + \sqrt{\left(1 + \frac{2}{3}h\right)^2 + \left(1 + \frac{2}{3}h + \frac{749}{1215}h^2\right)^2} \right] \\ \quad + \frac{64}{3}\left[\frac{1 + \left(1 + \frac{2}{3}h\right) + \left(1 + \frac{2}{3}h\right)^2}{2 + \frac{2}{3}h} + \frac{\left(1 + \frac{2}{3}h\right)^2 + \left(1 + \frac{2}{3}h\right)\left(1 + \frac{2}{3}h + \frac{749}{1215}h^2\right) + \left(1 + \frac{2}{3}h + \frac{749}{1215}h^2\right)^2}{2 + \frac{4}{3}h + \frac{749}{1215}h^2} \right] \end{array} \right\}$$

now consider the hybrid fuzzy initial value problem

$$\left\{ \begin{array}{c} y'(t) = y(t) + m(t)\lambda_k\left(y(t_k)\right), t \in (t_k, t_{k+1}], t_k = k, k = 0,1,2,\ldots \\ y(t;r) = ((0.75 + 0.25r)e^t, (1.125 - 0.125r)e^t], 0 \le r \le 1 \end{array} \right.$$

where

$$m(t) = \left\{ \begin{array}{ll} 2(t(mod1)) & if\ t(mod1) \le 0.5, \\ 2(1 - t(mod1)) & if\ t(mod1) \ge 0.5 \end{array} \right.$$

$$\lambda_k\left(y(t_k)\right) = \left\{ \begin{array}{ll} 0 & , if\ k = 0, \\ \mu & , if\ k \in 1,2,\ldots \end{array} \right.$$

The hybrid fuzzy Initial value problems is equivalent to the following system of fuzzy Initial value problems.

The exact solution for $t \in [0,2]$ is given by

$$\begin{cases} y_0^{'}(t) = y_0(t), t \in [0,1] \\ y_0(0;r) = [(0.75+0.25r),(1.125-0.125r)], 0 \le r \le 1 \\ y_i^{'}(t) = y_i(t) + m(t)y_{i-1}(t), t \in (t_i, t_{i+1}], y_i(t) = y_{i-1}(t_i), i = 1,2,\ldots \end{cases}$$

$y(t)+m(t)\lambda_k(y(t_k))$ is continuous function of t, x and $\lambda_k(y(t_k))$, for each $k = 0,1,2,\ldots$, the fuzzy Initial value problem

$$\begin{cases} y^{'}(t) = y(t)+m(t)\lambda_k(y(t_k)), t \in [t_k, t_{k+1}], t_k = k \\ y(t_k) = y_{t_k} \end{cases}$$

has a unique solution on $[t_k, t_{k+1}]$. To numerically solve the hybrid fuzzy IVP, the modified third order Runge - Kutta method based on Arithmetic mean, Root mean square and Centroidal mean is applied with $N = 2$ to obtain,

y1,2 (r) approximating $x(2.0;r)$.

Let $f : [0,\infty] \times R \times R \to R$ be given by

$$f(t, y, \lambda_k(y(t_k))) = y(t)+m(t)\lambda_k(y(t_k)), t_k = k, k = 0,1,2,\ldots,$$

Where $\lambda_k : R \to R$ is given by

$$\lambda_k(y) = \begin{cases} 0, \ if\ k = 0 \\ y, \ if\ k \in \{1,2,\ldots,\} \end{cases}$$

Since the exact solution for $t\varepsilon[1,1.5]$ is

$$Y(t;r) = Y(1;r)\left(3e^{t-1}-2t\right), 0 \le r \le 1, Y(1.5;r) = Y(1;r)(3\sqrt{e}-3), 0 \le r \le 1$$

Then $Y(1.5;r)$ is approximately 5.290656 and the exact solution for $t \in [1.5,2]$ is

$$Y(t;r) = Y(1;r)\left(2t-2+e^{t-1.5}(3\sqrt{e}-4)\right), 0 \le r \le 1$$

Therefore,

$$Y(2.0;r) = Y(1;r)(2+3e-4\sqrt{e})$$

Then $Y(2.0;$ r) is approximately 9.677457

The absolute error of the third order Runge-Kutta method based on a linear combination of Arithmetic mean (RK3AM), Root mean square (RK3RM) and Centroidal mean (RK3CeM) is compared with the absolute error by the individual means for the r-level set with $h = 0.1$ and t= 2 of the example is given below.

Table 1. Absolute error of the third order Runge-Kutta method based on a linear combination of RK3AM, RK3RM, RK3CeM and the individual means for the$r$-level set with $h = 0.1$ and $t = 2$

R	t	absolute error		absolute error		absolute error		absolute error	
		hrk3AMRMCeM		hrk3AM		hrk3RM		Hrk3CeM	
0	2	3.61E-04	5.42E-04	5.94E-04	8.91E-04	1.50E-03	2.26E-03	8.13E-04	1.22E-03
0.1	2	3.73E-04	5.36E-04	6.14E-04	8.81E-04	1.55E-03	2.23E-03	8.40E-04	1.21E-03
0.2	2	3.85E-04	5.30E-04	6.34E-04	8.71E-04	1.60E-03	2.21E-03	8.67E-04	1.19E-03
0.3	2	3.97E-04	5.24E-04	6.54E-04	8.61E-04	1.65E-03	2.18E-03	8.94E-04	1.18E-03
0.4	2	4.09E-04	5.18E-04	6.73E-04	8.52E-04	1.70E-03	2.16E-03	9.21E-04	1.16E-03
0.5	2	4.21E-04	5.12E-04	6.93E-04	8.42E-04	1.75E-03	2.13E-03	9.48E-04	1.15E-03
0.6	2	4.33E-04	5.06E-04	7.13E-04	8.32E-04	1.81E-03	2.11E-03	9.75E-04	1.14E-03
0.7	2	4.45E-04	5.00E-04	7.33E-04	8.22E-04	1.86E-03	2.08E-03	1.00E-03	1.12E-03
0.8	2	4.58E-04	4.94E-04	7.53E-04	8.12E-04	1.91E-03	2.06E-03	1.03E-03	1.11E-03
0.9	2	4.70E-04	4.88E-04	7.72E-04	8.02E-04	1.96E-03	2.03E-03	1.06E-03	1.10E-03
1	2	4.82E-04	4.82E-04	7.92E-04	7.92E-04	2.01E-03	2.01E-03	1.08E-03	1.08E-03

The following is the graphical representation for the absolute error of the third order Runge-Kutta method based on the linear combination of RK3AM, RK3RM, RK3CeM and the individual means for the above example is given below, where $h = 0.1, t = 1.5$.

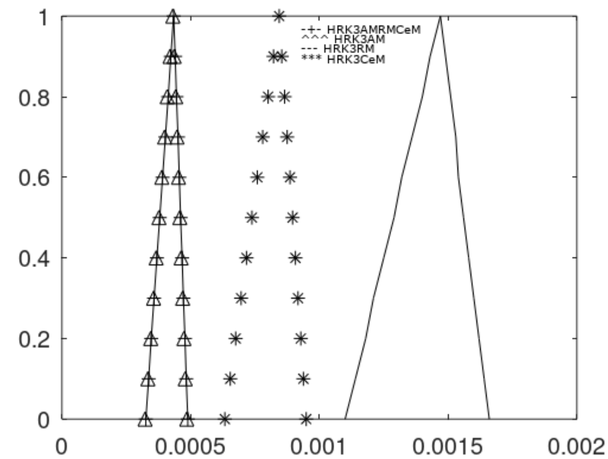

Fig 1. $(h = 0.1, t = 1.5)$

5 Conclusion

The third order Runge-Kutta approach based on a linear combination of Arithmetic Mean, Root Mean Square, and Centroidal Mean is used in this study to solve the first order hybrid fuzzy differential equation. The proposed approach's solution is compared to that of other methods, and it is observed that the proposed method provides a better result and is appropriate for handling hybrid fuzzy initial value problems.

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