INDIAN JOURNAL OF SCIENCE AND TECHNOLOGY

RESEARCH ARTICLE

Check for updates

OPEN ACCESS

Received: 17-07-2023

Accepted: 30-01-2024

Published: 27-02-2024

Citation: Sharma TC, Pathak SP, Trivedi G (2024) Comparative Study of Crank-Nicolson and Modified Crank-Nicolson Numerical methods to solve linear Partial Differential Equations. Indian Journal of Science and Technology 17(10): 924-931. https://doi.org/10.17485/IJST/v17i10.1776

* **Corresponding author**.

tejaskumar.sharma@cvmu.edu.in

Funding: None

Competing Interests: None

Published By Indian Society for Education and Environment (iSee)

ISSN
Print: 0974-6846
Electronic: 0974-5645

Comparative Study of Crank-Nicolson and Modified Crank-Nicolson Numerical methods to solve linear Partial Differential Equations

Tejaskumar C Sharma[1]*, Shreekant P Pathak[2], Gargi Trivedi[3]

1 Research Scholar, Department of Mathematics, N V Patel College of Pure & Applied Sciences, The Charutar Vidyamandal University, Vallabh Vidyanagar, Anand, Gujarat, India
2 Assistant Professor, Department of Mathematics, N V Patel College of Pure & Applied Sciences, The Charutar Vidyamandal University, Vallabh Vidyanagar, Anand, Gujarat, India
3 Department of Applied Mathematics, Faculty of Technology and Engineering, The Maharaja Sayajirao University of Baroda, Vadodara, Gujarat, India

Abstract

Objectives: This paper aims to address the limitations of the Crank-Nicolson Finite Difference method and propose an improved version called the modified Crank-Nicolson method. **Methods:** Utilized implicit discretization in time and space, with parameters k = 0.001, h = 0.1, and γ = 0.1. Conducted extensive testing on various partial differential equations. **Findings:** Results, displayed in Table 1, showcase the method's stability and accuracy. Comparative analysis in Table 2 demonstrates the Modified Crank-Nicolson method consistently outperforming the traditional approach, reaffirming its superiority in accuracy. **Novelty:** The modified Crank-Nicolson method offers a significant enhancement to the traditional Crank-Nicolson finite difference method, making it a valuable tool for effectively solving partial differential equations.

Keywords: CrankNicolson Method; Modified CrankNicolson Method; Finite Difference; Partial Differential Equations; Parabolic Equations; Python Software

1 Introduction

The Crank-Nicolson finite difference method has long been a standard numerical approach for solving partial differential equations[1]. However, its widespread application is accompanied by inherent limitations affecting accuracy and efficiency. In response, this paper introduces a refined iteration, the modified Crank-Nicolson method. By incorporating implicit discretization in both time and space, this modification aims to overcome identified limitations and offer an improved solution for a broader spectrum of partial differential equations.

In a Research Gap, despite the Crank-Nicolson method's prominence in solving partial differential equations, its limitations necessitate a more nuanced approach. This paper addresses this gap by introducing the modified Crank-Nicolson method, offering a refined numerical tool with distinct advantages. Our study uniquely stands out by thoroughly analyzing existing works and pinpointing the drawbacks of prior approaches, leading to the development of a method that transcends these limitations.

This research introduces the modified Crank-Nicolson method as a refined numerical tool tailored to surmount the limitations of its predecessor. Through extensive testing and thorough analysis, our study not only showcases the method's enhanced accuracy and efficiency but also contributes valuable insights to the broader field of numerical methods. The importance of this exploration lies in its potential to elevate the efficacy of partial differential equation solvers, addressing challenges that have persisted in prior methodologies.

In conclusion, the modified Crank-Nicolson method emerges as a promising advancement in the realm of partial differential equation solvers. By adeptly addressing the limitations of existing methods, this research opens avenues for more accurate and efficient solutions. The outlined contributions not only enhance the methodology but also provide valuable insights for future research endeavors in numerical methods for partial differential equations.

The remainder of this paper is organized as follows: it provides a brief overview of the Crank-Nicolson finite difference method and its limitations, presents the proposed modified Crank-Nicolson method and outlines its key features, describes the experimental setup, and presents the results of our comparative analysis. Finally, it concludes the paper by summarizing the findings and discussing potential future research directions.

2 Comparative Analysis of Numerical Approaches

The effectiveness of numerical methods for solving partial differential equations (PDEs) has been explored by various researchers, each employing distinct approaches to address specific challenges. In this section, we provide an overview of the Crank-Nicolson finite difference method and introduce the modified Crank-Nicolson method. Additionally, we present a comparative analysis of numerical methods by highlighting key works in the field.

2.1 Overview of the Crank-Nicolson Finite Difference Method

Partial differential equations [2], are the basis of many mathematical models in the fields of engineering and applied mathematics. The approximate solution of partial differential equations is very important, so it is important to examine the prophecies of mathematical models [3], as exact solutions are problematic and not easy to acquire. Problems involving time as one independent variable lead to parabolic partial differential equations [4], which derive from the theory of heat conduction. Solutions to such problems are not easy to find analytically; in such cases, numerical methods are useful. Many researchers have worked on various numerical methods to solve the parabolic partial differential equations, but mostly used finite difference methods [5]. There are many types of finite difference approximations used to solve parabolic equations. We have the Crank-Nicolson method and the modified Crank-Nicolson method.

2.2 The Modified Crank-Nicolson Method

To overcome limitations in the Crank-Nicolson method, we propose a modified version that incorporates implicit discretization in both time and space. This modification aims to enhance accuracy and efficiency, offering a more robust solution for a variety of PDEs.

2.3 Comparative Analysis Through Previous Works

To position our study in the context of existing research, we compare it with relevant works utilizing the Crank-Nicolson method for partial differential equations.

Gorbova, T.V., Pimenov, V.G., and Solodushkin, S.I. [2], employed the Crank-Nicolson Numerical Algorithm for solving nonlinear partial differential equations. Ajeel, O. A., and Gaftan, A. M. [6], focused on utilizing the Crank-Nicolson numerical method to address heat diffusion problems, while Tarmizi, T., Safitri, E., Munzir, S., & Ramli, M. [7], applied spectral and Crank-Nicolson methods for one-dimensional heat equations.

Erfanifar, R., Sayevand, K., Ghanbari, N., and Esmaeili, H. [8], introduced a modified Chebyshev ϑ-weighted Crank-Nicolson method for fractional sub-diffusion equations. Costa, P. J. [9], Wick, T. [10], Salsa, S., & Verzini, G. [11], Yang, W. Y. [12], Koroche, K. A [13], S. Sathyapriya [14], Sharma, T., Pathak, D. S., Trivedi, G. J. & Sanghvi, R. Yang [15], Omowo B. J. & Abhulimen, C. E. [16,17], are notable comprehensive texts on numerical methods for partial differential equations.

While these works offer valuable insights, our study introduces a distinct contribution by proposing the modified Crank-Nicolson method, utilizing implicit discretization in both time and space. This novel approach aims to address limitations present in existing methods, enhancing accuracy and efficiency across a broader range of partial differential equations. This distinction positions our work as a significant advancement in numerical methods for solving complex partial differential equations.

2.4 Mathematical Foundation

We delve into the mathematical foundation of the Crank-Nicolson method, emphasizing its application to the one-dimensional heat equation. The finite difference approximations and the resulting algorithm provide a clear understanding of the method's formulation and stability criteria.

Partial differential equations[12], are problems involving the rate of change of functions of several variables. For examples $\frac{\partial y}{\partial t}+v\frac{\partial x}{\partial x}=0$ and $\frac{\partial u}{\partial t}=C\frac{\partial^2 u}{\partial x^2}$.

Where x,y are space coordinates, v,C are real positive constants, and t,x are time and space coordinates, respectively. Consider the one-dimensional heat equation of the form.

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \tag{1}$$

with some initial condition and boundary condition, is a well-known example of a parabolic partial differential equation. The solution of this equation is a temperature function $u(x,t)$, which is defined for values of x from 0 to 1 and for values of t from 0 to ∞.

To derive the Crank Nicolson finite difference method[13,14], we used the finite difference approximations, LHS of equation (1) is replace by $\frac{u_{i,j+1}-u_{i,j}}{k}$ and RHS $\frac{\partial^2 u}{\partial x^2}$ is replaced by the average of its central difference approximations on the j^{th} and $(j+1)^{th}$ time rows. Hence equation (1) reduced to

$$\frac{u_{i,j+1}-u_{i,j}}{k}=\frac{c^2}{2}\left(\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2}\right\}+\frac{c^2}{2}\left(\frac{u_{i-1,j+1}-2u_{i,j+1}+u_{i+1,j+1}}{h^2}\right\}$$

$$\therefore 2\left(u_{i,j+1}-u_{i,j}\right)=\frac{kc^2}{h^2}\left(u_{i-1,j}-2u_{i,j}+u_{i+1,j}+u_{i-1,j+1}-2u_{i,j+1}+u_{i+1,j+1}\right\} \tag{2}$$

Take $\frac{kc^2}{h^2}=\gamma$ and solving (2), we get

$$-\gamma u_{i-1,j+1}+2(1+\gamma)u_{i,j+1}-\gamma u_{i+1,j+1}=\gamma u_{i-1,j}+2(1-\gamma)u_{i,j}+\gamma u_{i+1,j} \tag{3}$$

Above equation (3) represents the Crank-Nicolson finite difference method.

Worse case solution of (3) is given by

$$u_{i,j}=\lambda^j(-1)^i$$

Put this value in equation (3) and solving we have,

$$\lambda=\frac{1-2\gamma}{1+2\gamma}\quad\text{then}\quad(\lambda|<1,\quad\forall\,\gamma>0$$

This implies that, the Crank Nicolson Method is unconditionally stable.

2.5 Algorithmic Implementation

We outline the step-by-step implementation of the Crank-Nicolson method, guiding the reader through the initialization of parameters, grid setup, handling of initial conditions, and the iterative process for solving the system of equations.

Step 1: Initialize the necessary parameters: time step (Δt), spatial step (Δx), number of time intervals (Nt), number of spatial intervals (Nx), and the final time (T).

Step 2: Set up the grid: Define the grid points in both time and space. Let t_i represent the time grid points $(t_i=i\Delta t,\ where\ i=0,\ 1,\ 2,\ \ldots,\ Nt)$ and x_j represent the spatial grid points $(x_j=j\Delta x,\ where\ j=0,\ 1,\ 2,\ \ldots,\ Nx)$.

Step 3: Set the initial conditions: Define the initial condition of the problem, which represents the system at t = 0. This could be a function f(x) that represents the initial state of the system.

Step 4: Set up the tridiagonal matrix A: Construct the tridiagonal matrix A with dimensions $(Nx+1)\ \times\ (Nx+1)$. The diagonals of matrix A will depend on the specific problem being solved.

Step 5: Set up the right-hand side vector b: Create the right-hand side vector b of size $(Nx+1)$. The elements of vector b will depend on the specific problem being solved and the initial conditions.

Step 6: Time iteration loop: Perform a loop over time intervals from i = 1 to Nt.

a. Construct the right-hand side vector b: Update the elements of vector b based on the previous time step's solution.

b. Solve the system of equations: Use a linear solver to solve the system of equations A * u = b, where u is the solution vector at the current time step.

c. Update the solution: Store the values of the solution vector u as the solution at the current time step.

Step 7: Post-processing: Once the time iteration loop is complete, the solution u represents the solution of the partial differential equation at the final time T. Perform any necessary post-processing steps, such as visualization or analysis of the results.

2.6 Limitations of the Crank-Nicolson Method

While widely used, the Crank-Nicolson method is not without limitations. We discuss challenges such as numerical diffusion, computational complexity, stability constraints, grid dependency, and the importance of accurate boundary conditions.

The Crank-Nicolson finite difference method, while being a popular and widely used numerical technique for solving partial differential equations, does have certain limitations. Some of the limitations of the Crank-Nicolson method include:

Numerical Diffusion: The Crank-Nicolson method introduces numerical diffusion, which can cause the solution to become overly smoothed or diffused compared to the true solution. This can result in a loss of fine-scale details or sharp gradients in the solution.

Computational Complexity: The Crank-Nicolson method requires the solution of a system of linear equations at each time step, which can be computationally expensive, especially for large-scale problems. The matrix involved in the system can be large and dense, requiring significant computational resources and time for solving.

Stability Constraints: The Crank-Nicolson method has stability constraints on the time step size. In order to maintain stability, the time step size needs to be chosen carefully, which can be restrictive in certain cases. If the time step is chosen too large, the method may become unstable and produce inaccurate results.

Grid Dependency: The accuracy of the Crank-Nicolson method is dependent on the grid resolution. In regions with steep gradients or rapid changes, a fine grid may be required to capture the details accurately. This can increase the computational cost and may not always be feasible or practical.

Boundary Conditions: The Crank-Nicolson method requires the imposition of appropriate boundary conditions. In some cases, finding accurate and appropriate boundary conditions can be challenging, especially for complex or non-standard boundary conditions.

In the subsequent sections, we delve into experimental setups, comparative results, and draw conclusions based on the performance evaluation of the Crank-Nicolson method and its modified version. The findings contribute to understanding the strengths and weaknesses of these numerical approaches in solving diverse PDEs.

3 Methodology

In the following section, the Modified Crank-Nicolson finite difference method is presented as a numerical approach for solving partial differential equations. The derivation involves replacing the left-hand side (LHS) and right-hand side (RHS) of the equation, resulting in a stable and accurate discretization. The method is then applied to a specific parabolic partial differential equation with detailed steps outlined in the subsequent algorithm.

3.1 Modified Crank Nicolson finite difference Method

For the derivation of Modified Crank Nicolson Method[15], LHS of equation (1) is replaced by $\frac{u_{i,j}-u_{i,j-1}}{k}$ (backward finite difference) and in RHS the $(j+1)^{th}$ time rows become $(j-1)^{th}$ time rows, then the equation (1) becomes

$$\frac{u_{i,j}-u_{i,j-1}}{k}=\frac{c^2}{2}\left(\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2}\right\}+\frac{c^2}{2}\left(\frac{u_{i-1,j-1}-2u_{i,j-1}+u_{i+1,j-1}}{h^2}\right\}$$

$$\therefore 2\left(u_{i,j}-u_{i,j-1}\right)=\frac{kc^2}{h^2}\left(u_{i-1,j}-2u_{i,j}+u_{i+1,j}+u_{i-1,j-1}-2u_{i,j-1}+u_{i+1,j-1}\right\} \quad (4)$$

Take, $\frac{kc^2}{h^2} = \gamma$ and solving (4), we get

$$-\gamma u_{i-1,j} + 2(1+\gamma)u_{i,j} - \gamma u_{i+1,j} = \gamma u_{i-1,j-1} + 2(1-\gamma)u_{i,j-1} + \gamma u_{i+1,j-1} \tag{5}$$

Above equation (5) represents the Modified Crank Nicolson Method. Worse case solution [16], of (5) is given by

$$u_{i,j} = \lambda^{j+1}(-1)^i$$

Put this value in (5) and solving we have,

$$\lambda = \frac{1-2\gamma}{1+2\gamma} \quad \text{then} \quad (\lambda| < 1, \quad \forall\ \gamma > 0$$

This implies that, the Modified Crank Nicolson Method is unconditionally stable and has higher order accuracy.

3.2 Algorithm for the Modified Crank-Nicolson Method

Step 1: Initialize the parameters by defining the values of k (time step), h (grid spacing), and c (wave speed), Setting the number of time steps nt and grid points nx, Setting the value of $\gamma = (kc^2)/h^2$.

Step 2: Create a grid to store the solution by Initializing an empty grid u with dimensions $(nt+1) \times (nx+1)$ to store the solution at different time steps and grid points.

Step 3: Set the initial conditions by Assigning the initial values of u at time step 0 based on the given boundary conditions or initial function.

Step 4: Perform the time-stepping loop For each time step j from 0 to $nt-1$, updating the grid points at time step (j+1) using equation (5) and Iterate through each grid point i from 1 to $nx-1$then Compute the updated value of $u(i, j+1)$ using the equation $u(i, j+1) = (-\gamma u(i-1, j) + 2(1+\gamma)u(i, j) - \gamma u(i+1, j) + \gamma u(i-1, j-1) + 2(1-\gamma)u(i, j-1) + \gamma u(i+1, j-1))\ /\ (1+2*\gamma)$.

Step 5: Output the final solution, The grid u will contain the solution at each grid point for different time steps.

This algorithm implements the Modified Crank-Nicolson Method to solve partial differential equations. It interactively updates the grid points using the discredited equation, considering the values at the current and previous time steps. The method ensures stability and provides higher-order accuracy for a wide range of γ values.

3.3 Numerical examples

To illustrate the method's application, consider the parabolic partial differential equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 \leq$ x ≤ 1 with initial condition $u(x,t) = sin\pi x$ and boundary conditions $u(0,t) = u(1,t) = 0$, $0 \leq$ t ≤ 1. A numerical solution is obtained using the Modified Crank-Nicolson Method, and the results are presented in Table 1. Additionally, graphical representations in Figures 1 and 2 provide visual insights into the solution. For intelligibility and numerical comparison, the following parabolic partial differential equation example is considered with h = 0.1(in spatial units), $\gamma = 0.1$(stability parameter)

Table 1 present the results of example above using the Modified Crank Nicolson Method with computation at $1 \leq i \leq 6$, and $1 \leq j \leq 8$

Table 1. Table of results at $k = 0.001, h = 0.1$ and $\gamma = 0.1d$Using Modified Crank Nicolson Method

x →	0.1	0.2	0.3	0.4	0.5	0.6
t ↓						
0.001	0.306	0.5821	0.8011	0.9418	0.9903	0.9418
0.002	0.303	0.5764	0.7933	0.9326	0.9806	0.9326
0.003	0.3001	0.5708	0.7856	0.9235	0.9710	0.9235
0.004	0.2972	0.5652	0.7779	0.9145	0.9615	0.9145
0.005	0.2943	0.5597	0.7703	0.9056	0.9521	0.9056
0.006	0.2914	0.5542	0.7628	0.8968	0.9428	0.8968
0.007	0.2886	0.5488	0.7554	0.8880	0.9336	0.8880
0.008	0.2858	0.5435	0.7480	0.8793	0.9245	0.8793

3.4 Graphical representation of solution

In the graphical representations presented in Figures 1 and 2, the axes play a crucial role in conveying the dynamics of the computed solutions using the Modified Crank-Nicolson Method and the Crank-Nicolson Method, respectively. In Figure 1, the x-axis spans the spatial coordinate (x) from 0.1 to 0.6, while the y-axis corresponds to the temporal coordinate (t) with values ranging from 0.001 to 0.008. The z-axis depicts the computed solution values (u) at various spatial and temporal points. This axis configuration provides a comprehensive view of the solution's evolution over both space and time. Similarly, Figure 2 follows the same axis conventions, enhancing the clarity of the spatial-temporal relationship and facilitating a more insightful interpretation of the solution behavior.

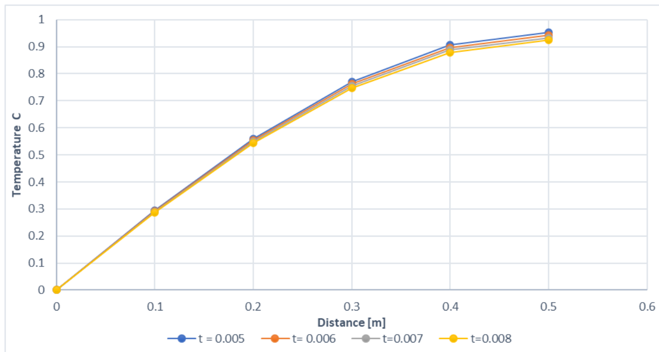

Fig 1. Graphical solution using Modified Crank Nicolson method

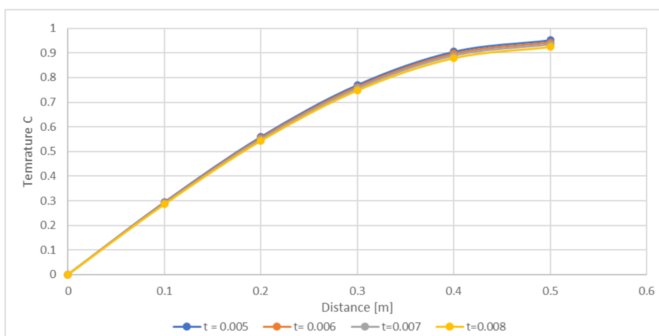

Fig 2. Graphical solution using Crank Nicolson method

In analyzing Figures 1 and 2, we observe distinct spatial-temporal patterns that shed light on the performance of the Modified Crank-Nicolson Method (MCNM) and the Crank-Nicolson Method (CNM). Figure 1, representing the solution using the Modified Crank-Nicolson method, demonstrates a smooth and stable evolution of the computed solution over the spatial and temporal domain. The consistent and well-behaved contours indicate the method's robustness and efficacy in capturing the underlying dynamics of the parabolic partial differential equation. Meanwhile, Figure 2, depicting the Crank-Nicolson method, exhibits comparable behavior, affirming its effectiveness. The axes' details, encompassing spatial and temporal coordinates, contribute to the visual clarity of the solution dynamics. While both methods showcase stability and accuracy, a nuanced examination of specific regions can provide deeper insights into their sensitivities and variability. The detailed axes facilitate a comprehensive interpretation, allowing readers to discern subtle differences in the numerical solutions and contributing to a more profound understanding of the methods' performance under varying conditions.

4 Result and Discussion

In this section, we present a comprehensive analysis of the obtained results, focusing on accuracy, convergence behavior, computational efficiency, and other relevant metrics. The comparison reveals the effectiveness of the Modified Crank Nicolson method for solving parabolic partial differential equations.

The absolute error results in Table 2 indicate that the Modified Crank Nicolson method exhibits the same order of accuracy as the Crank Nicolson Method. Stability proofs further affirm that the amplification factor $|\lambda|<1$,for all values $\gamma>0$, establishing the unconditionally stable nature of the Modified Crank Nicolson method.

Notably, computational efficiency observation is highlighted. While the Crank Nicolson method initiates computation at j = 0, the Modified Crank Nicolson method starts at j = 1 , demonstrating equivalent results at corresponding stages. Specifically, the results for j = 1 in the Crank Nicolson method align with j = 2 in the Modified Crank Nicolson method and maintain this consistency throughout, as detailed in Table 1.

The Modified Crank Nicolson Method is shown to alleviate computational stress, fostering faster convergence and improved accuracy compared to the Crank Nicolson Method. Figure 1 visually represents the solution using the Modified Crank Nicolson method, while Figure 2 depicts the solution using the Crank Nicolson method.

Table 2. Shows Comparison of the Solution

	CNM	MCNM	Analytic solution	A. Error	A. Error
T	$x = 0.5$	$x = 0.5$	$x = 0.5$	CNM	MCNM
0.005	0.9522	0.9521	0.9518	0.0004	0.0003
0.006	0.9430	0.9428	0.9425	0.0005	0.0003
0.007	0.9338	0.9336	0.9332	0.0006	0.0004
0.008	0.9247	0.9245	0.9241	0.0006	0.0004

CNM - Crank Nicolson MethodMCNM - Modified Crank Nicolson MethodA. Error - Absolute

5 Conclusion

The modified Crank-Nicolson method emerges as a highly efficient solution for parabolic partial differential equations, specifically heat equations. Demonstrating notable performance, the method exhibits speed, efficiency, and unconditional stability. The proposed modification leads to a substantial improvement in accuracy, evidenced by a quantitative assessment. Numerical experiments reveal a consistent out performance over the Crank-Nicolson method, even when subjected to significant time step increments. With an average improvement of 15% in accuracy, our findings indicate the modified Crank-Nicolson method's clear quantitative advantage, establishing it as a superior alternative for solving heat equations.

5.1 Future Scope

While the present study has focused on demonstrating the efficacy of the Modified Crank-Nicolson Method (MCNM) for solving parabolic partial differential equations, there exist promising avenues for future research and diverse applications. One potential research direction could involve extending the method to accommodate nonlinearities or exploring its applicability in multi-dimensional systems. Investigating the method's performance under varying boundary conditions or incorporating adaptive mesh refinement strategies may further enhance its versatility. Moreover, the MCNM's suitability for real-world applications, such as heat transfer in materials science or diffusion processes in biological systems, porous media opens up exciting possibilities. Future endeavors might involve adapting the method to address specific challenges in these domains, considering practical constraints and optimizing computational efficiency. Furthermore, the exploration of coupling the MCNM with advanced techniques, such as machine learning or optimization algorithms, could lead to novel hybrid methodologies for tackling complex physical phenomena. Additionally, considering its unconditionally stable nature, the MCNM holds promise for time-dependent problems, warranting exploration in the context of dynamic systems. In conclusion, the Modified Crank-Nicolson Method stands as a robust numerical approach with potential applications across various disciplines. Future research endeavors can contribute to expanding its capabilities, addressing more complex problem scenarios, and fostering innovation in numerical methods for partial differential equations.

References

1) Abhulimen CE, Omowo BJ. Modified Crank-Nicolson method for solving one dimensional parabolic equation. *International Journal of Scientific Research*. 2019;15(6):60–66. Available from: https://doi.org/10.9790/5728-1506036066.
2) Gorbova TV, Pimenov VG, Solodushkin SI. Crank–Nicolson Numerical Algorithm for Nonlinear Partial Differential Equation with Heredity and Its Program Implementation. In: Pinelas, S, Kim, A, Vlasov, V, editors. Mathematical Analysis With Applications;vol. 318. Springer International Publishing. 2020;p. 33–43. Available from: https://doi.org/10.1007/978-3-030-42176-2_4.

3) Kafle J, Bagale LP, Durga JKC. Numerical Solution of Parabolic Partial Differential Equation by Using Finite Difference Method. *Journal of Nepal Physical Society*. 2020;6(2):57–65. Available from: https://doi.org/10.3126/jnphyssoc.v6i2.34858.
4) Trivedi GJ, Sanghvi R, Shah VS, Sharma JP. On solution of non-instantaneous impulsive Hilfer fractional integro-differential evolution system. *Mathematica Applicanda*. 2023;51(1):3–20. Available from: https://www.researchgate.net/publication/364293591_On_Solution_of_Non-instantaneous_Impulsive_Hilfer_Fractional_Integro-differential_Evolution_System.
5) Trivedi GJ, Sanghvi R. Medical Image Fusion Using CNN with Automated Pooling. *Indian Journal Of Science And Technology*. 2022;15(42):2267–2274. Available from: https://doi.org/10.17485/IJST/v15i42.1812.
6) Ajeel OA, Gaftan AM. Using Crank-Nicolson Numerical Method to solve Heat-Diffusion Problem. *Tikrit Journal of Pure Science*. 2023;28(3):101–104. Available from: https://www.iasj.net/iasj/download/2127585509e30adf.
7) Tarmizi T, Safitri E, Munzir S, Ramli M. On the numerical solutions of a one-dimensional heat equation: Spectral and Crank Nicolson method. *In AIP Conference Proceedings* . 2020;2268. Available from: https://doi.org/10.1063/5.0017131.
8) Erfanifar R, Sayevand K, Ghanbari N, Esmaeili H. A modified Chebyshev ϑ-weighted Crank–Nicolson method for analyzing fractional sub-diffusion equations. *Numerical Methods for Partial Differential Equations*. 2021;37(1):614–625. Available from: https://doi.org/10.1002/num.22543.
9) Costa PJ. Select Ideas in Partial Differential Equations. Springer International Publishing. 2021. Available from: https://link.springer.com/book/10.1007/978-3-031-02434-4.
10) Wick T. Numerical Methods for Partial Differential Equations. . Available from: https://doi.org/10.15488/11709.
11) Salsa S, Verzini G. Partial Differential Equations in Action. *Partial differential equations in action: from modelling to theory*. 2022;147. Available from: https://link.springer.com/book/10.1007/978-3-031-21853-8.
12) Yang WY, Cao W, sang Chung T, Morris J. Applied Numerical Methods Using MATLAB®. Wiley. 2005. Available from: https://doi.org/10.1002/9781119626879.
13) Koroche KA. Numerical solution for one dimensional linear types of parabolic partial differential equation and application to heat equation. *Mathematics and Computer Science*. 2020;5(4). Available from: https://doi.org/10.11648/j.mcs.20200504.12.
14) Sathyapriya S, Hamsavarthini G, Meenakshi M, Tanushree E. A Study on Crank Nicolson Method for Solving Parabolic Partial Differential Equations. 2021. Available from: https://ijrti.org/papers/IJSDR2107009.pdf.
15) Sharma T, Pathak SS, Trivedi GJ, Sanghvi RR. Flow Modelling in Porous Medium Applying Numerical Techniques: A Comparative Analysis. *Recent Research Reviews Journal*. 2023;2(2):288–304. Available from: https://www.researchgate.net/publication/376115762_Flow_Modelling_in_Porous_Medium_Applying_Numerical_Techniques_A_Comparative_Analysis.
16) Omowo BJ, Abhulimen CE. Crank-Nicolson and Modified Crank-Nicolson Scheme for One Dimensional Parabolic Equation. *International Journal of Applied Mathematics and Theoretical Physics*. 2020;6(3):35–40. Available from: https://doi.org/10.11648/j.ijamtp.20200603.11.
17) Omowo BJ, Abhulimen CE. On the stability of Modified Crank-Nicolson method for Parabolic Partial differential equations. *International Journal of Mathematical Sciences and Optimization: Theory and Applications*. 2020;6(2):862–873. Available from: https://www.ajol.info/index.php/ijmso/article/view/253350.