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Mathematical Study of the Pulmonary Gas Exchange for the Respiratory System under Normal and Abnormal Conditions

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Abstract

Objectives: To find the relationship of the concentration of Oxygen and Carbon dioxide in the respiratory system. Our objective is to establish the relationship between partial pressure and saturation of oxygen. Also, to evaluate the Oxygen diffusion capacity into the capillary at different levels of hemoglobin. **Methods:** We follow the classical equations solved using analytic and numerical techniques for evaluating the diffusion capacity of oxygen from the alveolus to the pulmonary capillary by MATLAB software simulation. **Findings:** Establish the relationship between partial pressure and saturation of oxygen at various situations. Evaluate the diffusion capacity behaviour of the Oxygen into the capillary by different levels of hemoglobin. **Novelty:** This study presents a parametric model of the concentration of Oxygen and Carbon dioxide in the respiratory system that incorporates partial pressure and saturation relation concerning the exchange of gases. We also solved the classical relationship numerically.

Keywords: Oxygen concentration; Gas exchange; Partial Pressure; Saturation of gases; Mathematical model and simulation

1 Introduction

Raffe Marc and Strickland et al., create a simple mathematical model of gas exchange across the pulmonary membrane, including the oxygen and carbon dioxide exchanges of gases. The most severe physical processes in the human body are those involved. The model depicts the progression of capillary partial pressure change over time along the capillary in terms of structural factors related to the lungs and transport variables related to gas. [1,2]

A slope of the tangent lines to the oxygen saturation dissociation curve and the blood's physically dissolved form of oxygen were examined to create a mathematical model of respiratory gas exchange by Tetiana semchyk, Saproval Bernard [3,4]. Si, X. A, the study of the total oxygen diffusion capacity of the pulmonary membranes has been divided into two components, which correspond to diffusion related to oxygen binding to hemoglobin, and diffusion across the blood-gas barrier. The mathematical model of the gas exchange between the alveoli and blood capillaries for healthy humans was proposed using cylindrical blood capillaries. They demonstrated the pulmonary membrane thickness, surface area, and cardiac output impacted gas diffusion. [5]

Hung et al., Patel K and Patel N on work a mathematical model calculates the variables that impact an artery's oxygen and carbon dioxide partial pressures. Explain the integrated lumped mechanical model for ventilation and the integrated model for oxygen delivery to the blood. The study researched how much oxygen enters the blood during exercise and sleep[6,7]. Michael Jaeger develop a mathematical model to predict how systemic inflammatory stresses affect ventilation/perfusion distribution and resulting pulmonary venous partial pressure of oxygen. Blood oxygen saturation (SO_2), partial pressure, and blood oxygen concentration all impact how much oxygen is delivered to human tissues by arterial blood[8].

Previous papers on the same subject has derived the graphs on normal conditions, but in this paper work has been carried with additional situations to understand the exact results. Those additional situations were observed in real time and captured the data. The Oxygen partial pressure and oxygen saturation are related, as shown by the hemoglobin-oxygen dissociation curve. When a gas diffuses over a membrane, it needs an equilibrium time long enough to allow capillary and alveolar equilibrium and enough alveoli to allow a sufficient volume of gas exchange. Transfer gradients between the alveoli and the capillary equalized in 0.25 seconds by Wittenstein and Ashan Huq et al.[9,10].

2 Methodology

The relationship between the partial pressure of oxygen and the saturation of oxygen into the haemoglobin is explained as follows.

We take the following assumptions:

- The entire lung is considered a single unit with an alveolus and capillaries.
- Assuming each alveolus is the same and each capillary has the same shape and properties.
- In the alveolar space, the oxygen partial pressure is constant and uniform.
- The blood flow rate in each capillary is constant.

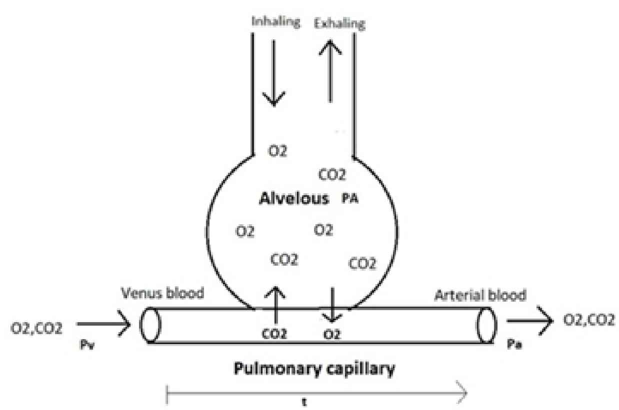

Fig 1. Presentation of gas exchange thro ugh alveolus to blood capillary

According to Graham's law, gas diffusion net from high concentration to low concentration. Both internal and external respiration are processes taken by the respiratory system. External respiration refers to the gas exchanged between the alveolus and pulmonary capillaries.

Using Fick's Law, oxygen diffusion from the alveolus to the blood capillary is described by

$$\dot{V}(t) = D_L(P_A - P(t)) \tag{1}$$

where, the partial pressures of oxygen (P_{O_2}) in the capillary tube and alveolar air are $P \& P_A$, and the amount of oxygen transferred across the pulmonary membrane in one unit of time is denoted by the symbol $\dot{V}(t)$, D_L is the pulmonary membrane's oxygen diffusion capacity, respectively.

Here, the cylinder's diffusing capacity can be expressed as a combination of the following factors:

$$D_L = K\left(\frac{2\pi rh}{T}\right)\left(\frac{\alpha}{\sqrt{M}}\right) \tag{2}$$

Where α solubility of oxygen in the blood, M is the oxygen gas's molecular weight, T is capillary thickness, and r is the capillary radius.

By applying Fick's principle to blood flow, if V_C is the entire blood volume in the capillary.

$$\dot{V}(t) = V_C \frac{dC(t)}{dt} \tag{3}$$

where C represents the amount of oxygen in the blood in capillaries.

From Equations (1) and (3), we have;

$$\frac{dC(t)}{dt} = \frac{D_L}{V_C}(P_A - P(t)) \tag{4}$$

The partial pressure P of oxygen in the capillary impacts both the chemical reaction that separates oxygen from hemoglobin and the amount of oxygen dissolved in blood plasma, which controls the oxygen concentration $C(t)$ in the capillary blood.

Henry's law of respiration states that the chemical reaction between the partial pressure of oxygen saturation in hemoglobin and the amount of oxygen dissolved in the circulation is directly proportional to the partial pressure of oxygen in alveolar air.

$$C(t) = \alpha P(t) + \beta.Hb.S \tag{5}$$

Where Hb represents the amount of hemoglobin per unit volume of blood, α represents the blood's ability to bind oxygen, and Srepresents how well the blood can saturate hemoglobin with oxygen. The amount of oxygen present per unit mass of hemoglobin when 100% of its oxygen concentration is saturated.

From Equation (5), we get;

$$\frac{dC(t)}{dt} = \propto \frac{dP(t)}{dt} + \beta Hb \frac{dS}{dt} \tag{6}$$

Blood oxygen saturation (SO_2) and the partial pressure (P_{O_2})of oxygen are two critical factors that determine how much oxygen is transferred in arterial blood to the body's tissues.

The oxygen-hemoglobin dissociation curve (ODC) in [Figure 2] shows how oxygen saturation (SO_2) and partial pressure (P_{O_2}) relate to each other.

$$S(P(t)) = \frac{(P(t))^3 + 150P(t)}{(P(t))^3 + 150P(t) + 23400} \tag{7}$$

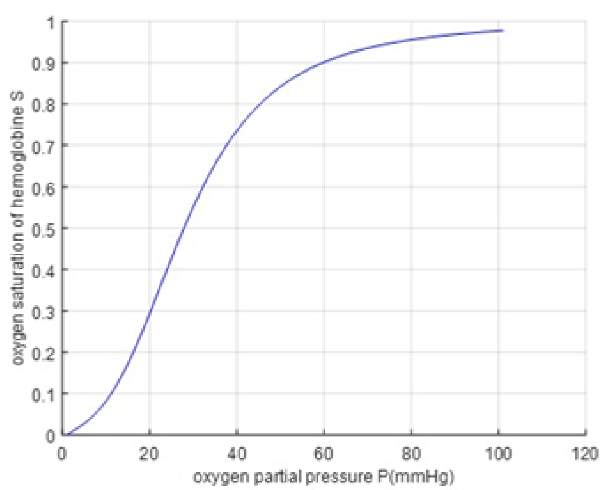

Fig 2. oxygen partial pressure vs saturation dissociation curve

The term "oxygen saturation" refers to the amount of oxygen transferred by red blood cells throughout the body. In healthy people, oxygen saturation ranges between 95% and 100%. A single pulmonary capillary cell contains millions of hemoglobin molecules which are red blood cells. All of these proteins have hemoglobin saturation which is the average oxygen saturation. Red blood cells include a protein called hemoglobin, which is in charge of transporting oxygen from the lungs to others parts of the body.[11,12]

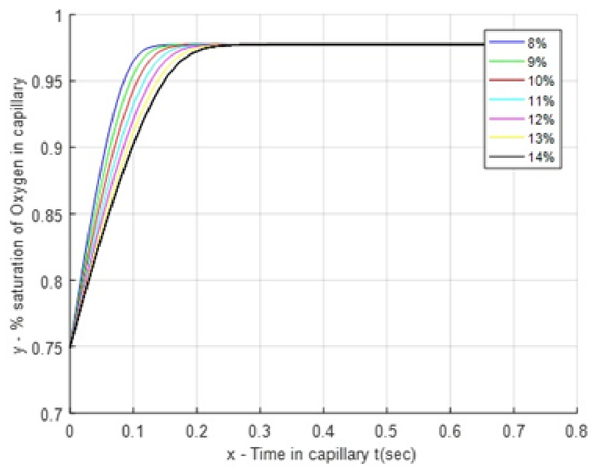

Fig 3. oxygen saturation with time in hemoglobin

The relation between the amount of hemoglobin and the oxygen saturation in human blood is shown in [Figure 3]. Hemoglobin levels affect oxygen saturation in the pulmonary capillary under different situations. Less hemoglobin means fewer red blood cells, which means it will take less time for the capillaries to get saturated with oxygen. Similarly, if the capillary hemoglobin level is high it means there are more red blood cells in the blood, and the oxygen saturation will take more time to complete.

From Equations (4), (6) and (7), we get

$$\frac{dP(t)}{dt} = \frac{\frac{D_L}{V_C}(P_A - P(t))}{\alpha + \frac{70200\beta.Hb((P(t))^2+50)}{[(P(t))^3+150P(t)+23400]^2}}, 0 \leq t \leq T \tag{8}$$

Where, T is the time taken by the blood to travel through a capillary in the lung.

0.75 seconds are required for Q = 6 mL/min cardiac output. In the mathematical model (Equation (8)), the variables $P(0) = P_V$ and P_V stand in for the oxygen partial pressure of venous blood entering the capillary, respectively. That is the initial value problem for an ordinary differential equation. $P(T) = P_a$, where P_a represents the arterial blood's oxygen partial pressure, the equation for the blood that is available in an arterial capillary [Figure 1].

Transit time is the amount of time it takes for carbon dioxide and oxygen to diffuse from the alveolus to the blood capillaries. Under normal circumstances, the alveolar-capillary system transports blood in around 0.75 seconds[13].

The rate law is a differential equation that describes the concentration (s) of reactant changes with time. Mathematicians can integrate the rate law to produce an equation that accurately represents the reactant concentrations with time. The concentration or partial pressure of oxygen and carbon dioxide with time regulates the rate of a chemical process.

$$\frac{d^2A}{dt^2} = A_0 - \frac{1}{k}\frac{dA}{dt} \tag{9}$$

where A_0 is the partial pressure of oxygen and carbon dioxide in the venous blood capillary, and k is the diffusivity of oxygen and carbon dioxide in the capillary.

Until equilibrium is reached, which takes around 0.25 seconds, oxygen and carbon dioxide will continue to diffuse. As a result, the diffusion capacity of oxygen and carbon dioxide takes up around one-third of the available time.

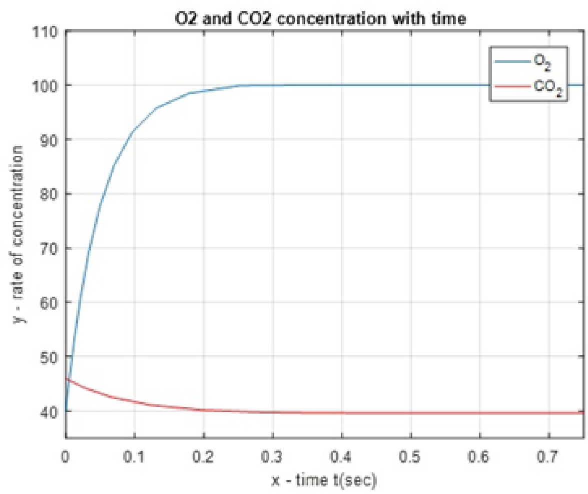

Fig 4. oxygen and carbon dioxide concentration curve for normal activity in human

Table 1. Values of the model parameters for healthy human

Parameter	Unit	Values
P_A	$mmHg$	100
P_V	$mmHg$	40
DL	$mL.min^{-1}mmHg^{-1}$	39
Q	$mL.min^{-1}$	6000
T	Sec	0.75
V_C	mL	75
α	$mL.mL^{-1}.mmHg^{-1}$	0.00003
β	$mL.g^{-1}$	134
Hb	$g.mL^{-1}$	0.15

3 Results and Discussions

3.1 Simulation result

Since the nonlinear form of the derived model makes it difficult to solve analytically, we applied the Runge-Kutta algorithm to find the solution to the differential equation and achieve a numerical solution with a step size $\triangle t = 0.001$. The numerical solution to Equation (8) is present in the following MATLAB (2016a) code, which also plots the result while considering various activities.

```
>>f=inline('(0.008889*(100-p))./(0.00003+(14636.7*(p.^2 +50)./…
((p.^3+150*p+23400).^2)))','t','p');
>>[t, p]=ode45(f,[0:0.001:.75],39);
>>plot(t, p)
>>grid on
>>axis([0 0.75 30 110])
>>xlabel('Time in capillary(sec)')
>>ylabel('Oxygen partial pressure PO2 (mmHg)')
```

When the total surface area of the respiratory membrane is decreased to about one third to one fourth normal, exchange of gases through the membrane is impeded to a significant under the exercise, normal condition, and rest condition, which is 65 ml/min/mmHg, 39 ml/min/mmHg, 21 ml/min/mmHg mathematical calculation has been done and the result obtained to the conduct simulation[14]. The simulation result show that, during the exercise the oxygen partial pressure is increases and it result into of quick oxygen diffusion in capillary, which is about 65 ml/min/mmHg, which generates the higher pulmonary

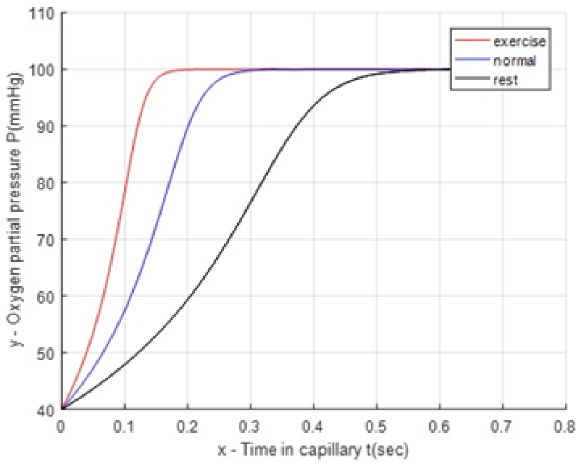

Fig 5. Oxygen partial pressure in various situations

blood flow and alveolar ventilation. In normal condition, oxygen diffuse at the standard rate, resulting in regular pulmonary blood flow and alveolar ventilation. During the rest condition the oxygen partial pressure is decrease, resultant take more time for diffusion of oxygen in the capillary[15,16].

3.2 Discussion

In this study, forms Equations (4) and (7) which give the partial pressure difference (P_A - P) between alveolar air and capillary tube in blood gas, the blood's hemoglobin content (Hb), and the blood's ability to bind oxygen (α) , the ability of hemoglobin to carry oxygen (β), the diffusion capacity of the oxygen (D_L) are all variables in the oxygen diffusion into the capillary via the pulmonary membrane. It takes about 0.25 seconds for the oxygen level in the capillaries to reach the alveoli. We solve the differential equation and simulate the results for various circumstances using MATLAB software. This study has focused on classical equation to understand and to derive the simulation results. If there are any changes within the various parameters of this model, then it can be known through the changes of model results.

Because of the difference between the partial pressures of oxygen in blood in capillaries and alveolar air, which is around 39 mmHg, oxygen will diffuse into the capillary blood on a net basis. The same is derived in result that various condition of oxygen diffusion through alveolus to capillary with partial pressures, until the alveolar and capillary concentration is reached to the balance level till that of oxygen diffusion will continue.

4 Conclusion

The relationship between oxygen saturation and partial pressure of oxygen is derived by the model, which explains the diffusion of oxygen in a capillary. In the normal condition, whenever oxygen passes through the pulmonary membrane, it binds with the red blood cells in hemoglobin. The relationship of oxygen saturation with respect to hemoglobin level is derived. The graph shows that the hemoglobin level is low then saturation of the oxygen done faster than the normal level of the hemoglobin. In normal condition, the saturation of the oxygen in the capillary reach at its maximum level at 0.25 sec while maximum time for diffusion of oxygen in the capillary is about 0.75 sec. The Runge-Kutta method (MATLAB function ode45) was used to obtain a numerical solution of the model. The oxygen diffusion from alveolar to capillary in various conditions is graphically depicted using simulations.

5 Declaration

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