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Harmony of Walled Klein-4 Brauer Diagrams and Klein-4 Vacillating Tableaux

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Abstract

Objectives: This study aims to establish the Robinson-Schensted correspondence for the walled Klein-4 Brauer algebra. We define a bijection between walled Klein-4 Brauer diagrams and pairs of Klein-4 vacillating tableaux, providing a combinatorial proof for a significant identity related to this algebra. **Method:** We construct the Robinson-Schensted correspondence using a function, d $\rightarrow$ ($P_{(\lambda,\mu)}$, $Q_{(\lambda,\mu)}$), mapping diagrams to tableaux by employing the Robinson-Schensted correspondence for the generalised Klein-4 group. This involves a pair of 4-Young standard tableaux and the paths of the Bratteli diagram. **Findings**: Our study yields a combinatorial proof of the identity $4^{(r+s)}(r+s)! = \sum_{(\lambda,\mu)\in\Lambda_4^{(r,s)}} (m^{(\lambda,\mu)})^2$, where $m^{(\lambda,\mu)}$ represents the number of paths from the 0^{th} stage of the Bratteli diagram to the shape (λ,μ), $\Lambda_4^{(r,s)}$={(λ,μ): λ is a 4-partition of (r−w), μ is a 4-partition of (s−w), $0 \leq w \leq \min(r, s)$} and r, s are non-negative integers. **Novelty:** By employing the Robinson-Schensted correspondence, we establish fundamental relations and rules, enriching the understanding of walled Klein-4 Brauer algebra in the realm of algebraic combinatorics.

Keywords: Brauer algebra; Partition algebra; 4-Partition; Walled Klein-4 Brauer algebra; Robinson-Schensted correspondence

1 Introduction

In [1], we introduced walled Klein-4 Brauer algebra, denoted by $W_{(r,s)}(l)$, where r and s are non-negative integers l is an indeterminate and described the irreducible representations of this algebra as being indexed by pairs (w, γ^L, γ^R), where $0 \leq w \leq \min(r, s)$, γ^L is the 4-partition of (r − w), and γ^R is the 4-partition of (s − w). In the present paper, our primary objective is to establish a bijection between these pairs and pairs of Klein-4 vacillating tableaux.

Halverson and Lewandowski's work, as referenced in [2], introduces a fundamental concept in algebra known as the Robinson-Schensted correspondence (RS correspondence) for partition algebra [3]. The correspondence demonstrates that for each set partition, there exists a unique pair of vacillating tableaux [4], and vice versa. This bijection has significant applications in various mathematical fields, including representation theory and combinatorics. This prior study served as inspiration for our own exploration of the RS correspondence applied to walled Klein-4 Brauer algebra.

These vacillating tableaux find versatile applications across various fields due to their ability to represent and manipulate complex data in an organised manner. In computer science and artificial intelligence, they are employed for decision-making and problem-solving, such as game tree traversal in AI gaming algorithms. In mathematics, they serve as tools for visualising and exploring combinatorial problems, particularly in combinatorial optimisation and graph theory.

This manuscript is structured as follows: In the second section, we introduce the necessary background information. In Section 3, we establish the RS correspondence for the generalised Klein-4 group. Building upon this correspondence, we expand our analysis to derive the RS correspondence for the walled Klein-4 Brauer algebra. We outline the key findings of our study in Section 4.

2 Methodology

Halverson and Lewandowski[2], utilised advanced combinatorial techniques, including Robinson-Schensted insertion and jeu de taquin to establish a connection between the number of set partitions and the sum of squared vacillating tableaux counts for specific partitions. Their work demonstrated the versatility of these methods across various diagram algebras, such as the Brauer algebra[5], and symmetric group algebra, highlighting the broader implications of their findings in the realm of algebraic structures. In this paper, we focused on investigating the Robinson-Schensted correspondence for the walled Klein-4 Brauer algebra. We developed this correspondence for the generalised Klein-4 group by employing pairs of 4-Young standard tableaux. Additionally, we constructed Klein-4 vacillating tableaux for a specific pair of partitions (λ,μ). We then established a one-to-one correspondence between walled Klein-4 Brauer diagrams and pairs of Klein-4 vacillating tableaux through a function d $\rightarrow$ $(P_{(\lambda\ ,\mu)}, Q_{(\lambda\ ,\mu)})$. Now, we provide the definitions and notations that we used in this paper.

Definition 2.1.[6] A partition of a non-negative integer (r + s) is a sequence of non-negative integers $\beta = (\beta_1, \beta_2, \ldots, \beta_l)$ such that $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_l \geq 0$ and $|\beta| = \beta_1 + \beta_2 + \cdots + \beta_l$ = (r + s). The non-zero β_i's are called the parts of β, and the number of non-zero parts is called the length of β. The notation $\beta \vdash (r+s)$ denotes that β is a partition of (r + s).

Definition 2.2.[6] Let $\beta \vdash (r+s)$. The Young diagram of β is an array of (r + s) boxes having l left justified rows, with row j containing β_j boxes for $1 \leq$ j $\leq$ l.

Definition 2.3.[6] Suppose $\beta \vdash (r+s)$. A tableau of shape β is an array t that is created by bijectively filling the numbers into the boxes of the Young diagram with the integers 1, 2, . . . , (r + s).

Definition 2.4.[6] Suppose $\beta \vdash (r+s)$. A standard tableau of shape β is an array of (r + s) boxes filled with distinct entries whose rows and columns increase. The standard Young tableau of the partition (3, 2, 1) $\vdash$ 6 is shown in the diagram below.

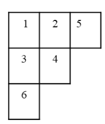

Fig 1. Standard Tableau

Definition 2.5.[1] A 4-partition of a non-negative integer (r + s) is an ordered 4-tuple of partitions $\beta = (\beta^{(1)}, \beta^{(2)}, \beta^{(3)}, \beta^{(4)})$ such that each $\beta^{(i)} \vdash k_i$ for i $\in$ {1, 2, 3, 4} and $|\beta| = |\beta^{(1)}| + |\beta^{(2)}| + |\beta^{(3)}| + |\beta^{(4)}| = k_1 + k_2 + k_3 + k_4$ = (r + s). The notation $\beta^{(i)} \vdash_4 (r+s)$ denotes that β is a 4-partition of (r + s). ((2, 2), (3, 2), (2, 1), (1^2)) is an example of the 4-partition of 14.

Notations 2.1.

- Consider the set I = {1, 2, 3, 4}.
- For i $\in$ I, we can explicitly represent the 4-partition β as: $\beta = ((\beta_1^{(1)}, \beta_2^{(1)}, \ldots, \beta_{l_1}^{(1)}), (\beta_1^{(2)}, \beta_2^{(2)}, \ldots, \beta_{l_2}^{(2)}), (\beta_1^{(3)}, \beta_2^{(3)}, \ldots, \beta_{l_3}^{(3)}), (\beta_1^{(4)}, \beta_2^{(4)}, \ldots, \beta_{l_4}^{(4)}))$ such that each $(\beta_1^{(i)}, \beta_2^{(i)}, \ldots, \beta_{l_i}^{(i)})$ is a partition of k_i.
- The i^{th} component of the 4-partition β is denoted as$(\beta_1^{(i)},\beta_2^{(i)}\ldots,\beta_{l_i}^{(i)})$.

Definition 2.6. [1] Suppose $\beta^{(i)} \vdash_4 (r+s)$. A 4-Young diagram of shape β is a collection of boxes in each component of β arranged in left-justified rows with a weakly decreasing number of boxes from top to bottom.

Definition 2.7.[1] Suppose $\beta^{(i)} \vdash_4 (r+s)$. A 4-Young tableau t (simply 4-tableau t) of shape β is an array t obtained from the 4-Young diagram of the 4-partition β by filling the numbers {1, 2, . . . , (r + s)} in the boxes.

Definition 2.8. [1] A 4-tableau t is standard if the entries in the 4-tableau t are increasing along the rows and columns in each component.

Definition 2.9. [5] A generalised permutation is represented as a two-line array of integers, denoted as:

$$\begin{pmatrix} k_1 & k_2 & \cdots & k_{(r+s)} \\ l_1 & l_2 & \cdots & l_{(r+s)} \end{pmatrix}$$

In this representation, the columns are arranged in lexicographic order, with the top entry having higher precedence, and the condition that $l_i \neq l_j$ holds for all distinct pairs of i and j. We denote the set of all such generalised permutations on (r+s) symbols as GP(r+s).

2.1 Walled Klein-4 Brauer algebra [1]

A diagram is termed a signed diagram when every edge is annotated with a sign. In the context of the walled Klein-4 Brauer algebra, we consider four types of signs: (e, e), (e, b), (a, e), and (a, b). In these signed diagrams, each signed edge is incident upon precisely two vertices, and each vertex is associated with only one signed edge. Edges connecting vertices in the top row to those in the bottom row are referred to as vertical edges, while the remaining edges are categorised as upper or lower horizontal arcs. The walled Klein-4 Brauer algebra is a diagram algebra that possesses a basis comprising these signed diagrams, where a wall is positioned between r and s in such a way that no vertical edges traverse the wall, while every upper or lower horizontal arc does intersect the wall. The following diagrams represent the walled Klein-4 Brauer diagram of the walled Klein-4 Brauer algebra with r = 3 and s = 2.

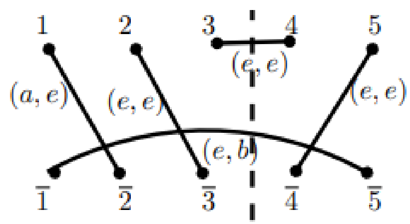

Fig 2. Walled Klein-4 Brauer Diagram

3 Results and Discussion

Our research goes beyond previous studies, specifically delving into the realm of walled Klein-4 Brauer algebras. Unlike the focus on partition algebra in Halverson and Lewandowski's work, our study takes a novel approach by establishing a connection between certain diagrams and pairs of tableaux within the context of Klein-4 vacillating tableaux. This perspective is a fresh angle that hasn't been explored in prior research. Our work also offers a clear explanation for an identity related to paths in a Bratteli diagram, providing a unique contribution to the understanding of these algebraic structures. While the study referenced in [2] examined a variety of algebraic structures, our research is more specialised, concentrating specifically on walled Klein-4 Brauer algebras. This specialization adds depth and specificity to our findings. In the following section, we present the correspondence for the generalised Klein-4 group. Subsequently, we employ this correspondence as a key tool to establish the main result of our paper.

3.1 The RS correspondence for the generalised Klein-4 group

In this section, we will develop the RSCorrespondence for the generalised Klein-4 group, employing a pair of 4-Young standard tableaux. Furthermore, we will derive the RS correspondence for the walled Klein-4 Brauer algebra.

Definition 3.1.1. A generalised Klein-4 permutation σ is a two-line array.

$$\sigma = \begin{pmatrix} i_1 & i_2 & \cdots & i_{(r+s)} \\ (\vartheta_{j_1}, j_1) & (\vartheta_{j_2}, j_2) & \cdots & (\vartheta_{j_{(r+s)}}, j_{(r+s)}) \end{pmatrix}$$

In this definition, each ϑ_{j_l} takes on one of the following values: (e, e), (a, e), (e, b), (a, b) for all l. The columns of this array are arranged in lexicographic order, with the top entry taking precedence, and it is required that $J_l \neq J_m$ for all distinct l and m. The set comprising all generalised Klein-4 permutations on (r+s) symbols is denoted as $GK_4P(r+s)$, and it is referred to as the generalised Klein-4 group. Note that the one-line notation of the above permutation is $(\vartheta_{j_1}, j_1)(\vartheta_{j_2}, j_2)\cdots(\vartheta_{j_{(r+s)}}, j_{(r+s)})$.

Definition 3.1.2 The Klein-4 tableau, denoted as P, is a 4-tableau constructed by inserting the elements J_l for all l = 1, 2,. . . , (r + s) from a generalised Klein-4 permutation based on the values of ϑ_{j_l}. If ϑ_{j_l} is equal to (e, e), then the element J_l is inserted into the first coordinate of P. If ϑ_{j_l} is equal to (a, e), the element is placed in the second coordinate of P. If ϑ_{j_l} is equal to (e, b), then it is inserted into the third coordinate, and if ϑ_{j_l} is equal to (a, b), it goes into the fourth coordinate of P.

Definition 3.1.3 A Klein-4 tableau, denoted as P, is considered standard if the entries within P increase along both the rows and columns for each coordinate in the tableau.

Proposition 3.1. The mapping $\sigma \rightarrow (P,Q)$, establishes a bijection between the set of generalised Klein-4 permutations $\sigma \in GK_4P(r+s)$ and the pair of Klein-4 standard tableaux (P,Q). In this context, we define P as $P(\sigma) = (P_1(\sigma), P_2(\sigma), P_3(\sigma), P_4(\sigma))$, and Q as $Q(\sigma) = (Q_1(\sigma), Q_2(\sigma), Q_3(\sigma), Q_4(\sigma))$, both with the shape represented by $\beta = (\beta^{(1)}, \beta^{(2)}, \beta^{(3)}, \beta^{(4)})$. Here each $\beta^{(i)} \vdash k_i$ for $i \in I$ is subject to the condition that $k_1 + k_2 + k_3 + k_4 = (r + s)$.

Proof. The proof closely resembles the proof of the RSCorrespondence for the symmetric group, as presented in[2]. In this context, we treat each $P_i(\sigma)$ and $Q_i(\sigma)$ as individual tableaux, where $i \in I$. For instance, consider σ as defined above. Here, each pair (ϑ_{j_l}, J_l) for l = 1, 2,. . . , (r + s) is inserted into the tableau P according to the sign ϑ_{j_l}. The value J_l is placed in $P_1(\sigma)$ if ϑ_{j_l} is (e, e), in $P_2(\sigma)$ if ϑ_{j_l} is (a, e), in $P_3(\sigma)$ if ϑ_{j_l} is (e, b), and in $P_4(\sigma)$ if ϑ_{j_l} is (a, b). Similarly, we construct the tableaux $Q_i(\sigma)$ using the paths of $P_i(\sigma)$, following the approach used for the symmetric group.

3.2 The RS correspondence for the walled Klein-4 Brauer algebra

In this section, we proceed with the RSCorrespondence employing the vacillating tableau approach outlined in the reference[2] for the partition algebra in order to establish the RS correspondence for the walled Klein-4 Brauer algebra.

Definition 3.2.1. A pair of 4-partitions (λ, μ) of size (r, s) is an ordered pair of 4-partitions, where λ is a 4-partition of r and μ is a 4-partition of s.

Definition 3.2.2. Let (λ, μ) be a pair of 4-partitions. A pair of 4-Young tableau of shape (λ, μ) is an array (t_1, t_2) obtained by inserting the numbers {1, 2, . . . , r} into the Young diagram of λ and the numbers {1, 2, . . . , s} into the Young diagram of μ in a bijective manner.

Definition 3.2.3. A pair of 4-tableau, denoted as (t_1, t_2), is considered standard if both 4-tableaux t_1 and t_2 are standard.

3.2.1 Bratteli diagram for the walled Klein-4 Brauer algebra

In the context of the Bratteli diagram for the walled Klein-4 Brauer algebra, we define an "up-down path" (UD_{path}) as a directed sequence of pairs of 4-partitions, starting from the 0^{th} stage $(\lambda^0, \mu^0)=((\varnothing,\varnothing, \varnothing, \varnothing), (\varnothing,\varnothing,\varnothing,\varnothing))$ and continuing through (λ^1, μ^1), (λ^2, μ^2), and so forth, until reaching $(\lambda^{(r+s)}, \mu^{(r+s)})$, which corresponds to the desired pair (λ, μ), where λ and μ are 4-partitions such that $\lambda \vdash_4 (r - w)$ and $\mu \vdash_4 (s - w)$ for $0 \leq w \leq \min(r, s)$. These sequences adhere to the following specific rules:

1. For values of i less than or equal to r, μ^i is the empty 4-partition (no boxes in any part), and λ^i is derived from $\lambda^{(i-1)}$ by adding one box to it.

2. For values of i between (r + 1) and (r + s), the sequence can take one of two actions:

- λ^i remains the same as $\lambda^{(i-1)}$, and μ^i is derived from $\mu^{(i-1)}$ by adding one box.
- μ^i remains the same as $\mu^{(i-1)}$, and λ^i is derived from $\lambda^{(i-1)}$ by removing one box.

These rules define how we can transition from the initial empty partitions to the final partitions while maintaining the specified conditions for the number of boxes in each part. Also, the UD_{path} from $(\lambda^\circ, \mu^\circ)$ to (λ, μ) is called Klein-4 vacillating tableau of shape (λ, μ), and its length is (r+s+1) is denoted by $VT(\lambda, \mu)$, where $(\lambda, \mu) \in \Lambda_4^{(r,s)}$ and $\Lambda_4^{(r,s)}=\{(\lambda, \mu)$: λ is a 4-partition of (r−w), μ is a 4-partition of (s−w), $0 \leq w \leq \min(r, s)\}$. It should be noted that the irreducible representations of the walled Brauer algebra are indexed by $\Lambda_4^{(r,s)}$. The following diagram illustrates the Bratteli diagram of the walled Klein-4 Brauer algebra $D_{1,1}(l)$.

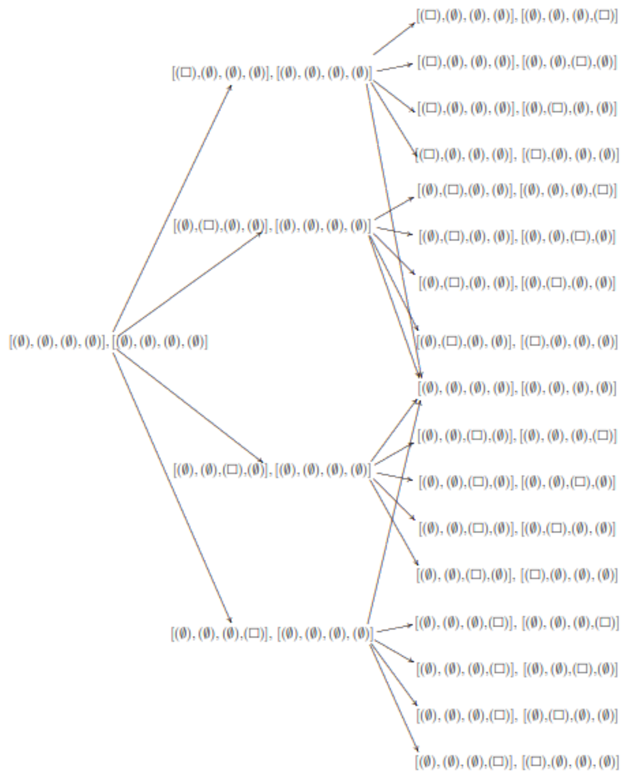

Fig 3. Bratteli Diagram

3.2.2 Single-row diagram representation of the walled Klein-4 Brauer algebra

We represent the walled Klein-4 Brauer diagram d as a single-row representation (SRP) with 2(r +s) vertices labelled from 1 to 2(r + s). To achieve this, we label the vertex j with the label (2(r + s)-j+1). The edges of the single-row representation are drawn in a specific manner, connecting vertices i and j for i $\leq$ j if and only if they are related in d. Each edge in the single-row representation of d is labelled as (x, 2(r + s) − m + 1), where the edge in d is labelled with the sign x, and m is the right vertex of that edge in the single-row representation. The possible values for x are (e, e), (a, e), (e, b), and (a, b). The following diagram illustrates the single-row diagram representation of Figure 2.

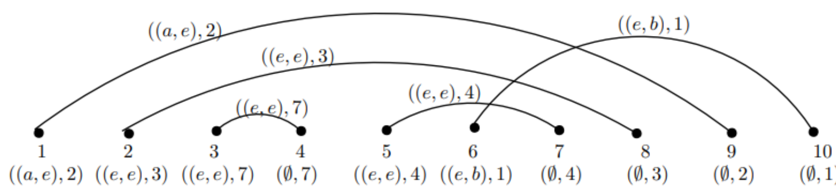

Fig 4. Single-row Representation

3.2.3 Edge Sequence

The edge sequence denoted as $E_j^{(d,(r+s))}$ of a walled Klein-4 Brauer diagram d indexed by j, where j=1,2, ... , (r +s), is defined as follows:

$$E_j^{d,(r+s)} = \begin{cases} (x, 2(r+s) - m + 1), & if\ j\ is\ left\ endpoint\ of\ an\ edge\ in\ SRP\ of\ d\ with\ labelling\ (x, 2(r+s) - m+1) \\ (\varnothing, 2(r+s) - m + 1), & if\ j\ is\ right\ endpoint\ of\ an\ edge\ in\ SRP\ of\ d\ with\ labelling\ (x, 2(r+s) - m+1) \end{cases}$$

Here, x ∈{(e, e), (a, e), (e, b), (a, b)}. Thus, the edge sequence is given by:

Table 1. Edge Sequence

J	1	2	⋯	2(r + s)
$E_j^{(d,(r+s)}$	(c_1, x_1)	(c_2, x_2)	⋯	$(c_{2(r+s)}, x_{2(r+s)})$

where $c_j \in$ {(e, e), (a, e), (e, b), (a, b), ∅}, and x_j = 2(r + s) − m + 1 if the vertex j is the endpoint of the edge with labelling (c_j,2(r + s) − m + 1), for j = 1, 2, ⋯ , 2(r + s). The following example illustrates the edge sequence of Figure 2.

Table 2. Edge Sequence of Figure 2

J	1	2	3	4	5	6	7	8	9	10
$E_j^{(d,(r+s))}$	((a, e), 2)	((e, e), 3)	((e, e), 7)	(∅, 7)	((e, e), 4)	((e, b), 1)	(∅, 4)	(∅, 3)	(∅, 2)	(∅, 1)

Proposition 3.2.1. A walled Klein-4 Brauer diagram is entirely determined by its edge sequence.
Proof. The proof follows from the fact that the edge sequence of a walled Klein-4 Brauer diagram uniquely specifies its edges.

3.2.4 Association of Klein-4 vacillating tableaux

To establish a connection between the walled Klein-4 Brauer diagram d and the edge sequence, we will construct a pairs of Klein-4 vacillating tableaux denoted as $(P_{(\lambda,\mu)}, Q_{(\lambda,\mu)})$:

This process unfolds as follows:

We start with an initial pair of empty 4-tableaux:

(t_1^0, t_2^0) = ((∅,∅ , ∅, ∅), (∅,∅,∅,∅))

Then, we iteratively define (t_1^j, t_2^j) based on (t_1^{j-1}, t_2^{j-1}) for j = 1, 2, ⋯ , 2(r + s) using the following rules:

$$(t_1^j, t_2^j) = \begin{cases} (c_j, x_j) \underset{ins}{\longrightarrow} (t_1^{j-1}, t_2^{j-1}) & if\ c_j \neq \varnothing \\ (c_j, x_j) \underset{del}{\longleftarrow} (t_1^{j-1}, t_2^{j-1}) & if\ c_j = \varnothing \end{cases}$$

Here, (t_1^j, t_2^j) represents the pair of 4-tableaux for the corresponding pair of 4-partitions (λ^j, μ^j) where 0 ≤ j ≤ (r+s). The expression$(c_j, x_j) \underset{ins}{\longrightarrow} \left(t_1^{j-1}, t_2^{j-1}\right)$ denotes the process of inserting the element x_j into the tableaux (t_1^{j-1}, t_2^{j-1}) based on the sign c_j. This insertion process adheres to the row insertion method as explained in [2]. The conditions for insertion, determined by the sign c_j, are as follows:

Case 1: When 1 ≤ j ≤ r or (r+s) < j ≤ (2r+s),

$$(t_1^j, t_2^j) = \begin{cases} x_j \underset{ins}{\longrightarrow} in\ the\ 1^{st}\ coordinate\ of\ the\ 1^{st}\ 4-partition\ of\ (t_1^{j-1}, t_2^{j-1}) & if\ c_j = (e,e) \\ x_j \underset{ins}{\longrightarrow} in\ the\ 2^{nd}\ coordinate\ of\ the\ 1^{st}\ 4-partition\ of\ (t_1^{j-1}, t_2^{j-1}) & if\ c_j = (a,e) \\ x_j \underset{ins}{\longrightarrow} in\ the\ 3^{rd}\ coordinate\ of\ the\ 1^{st}\ 4-partition\ of\ (t_1^{j-1}, t_2^{j-1}) & if\ c_j = (e,b) \\ x_j \underset{ins}{\longrightarrow} in\ the\ 4^{th}\ coordinate\ of\ the\ 1^{st}\ 4-partition\ of\ (t_1^{j-1}, t_2^{j-1}) & if\ c_j = (a,b) \end{cases}$$

Case 2: When r < j ≤ (r+s) or (2r+s) < j ≤ 2(r+s),

$$(t_1^j, t_2^j) = \begin{cases} x_j \underset{ins}{\longrightarrow} in\ the\ 1^{st}\ coordinate\ of\ the\ 2^{nd}\ 4-partition\ of\ (t_1^{j-1}, t_2^{j-1}) & if\ c_j = (e,e) \\ x_j \underset{ins}{\longrightarrow} in\ the\ 2^{nd}\ coordinate\ of\ the\ 2^{nd}\ 4-partition\ of\ (t_1^{j-1}, t_2^{j-1}) & if\ c_j = (a,e) \\ x_j \underset{ins}{\longrightarrow} in\ the\ 3^{rd}\ coordinate\ of\ the\ 2^{nd}\ 4-partition\ of\ (t_1^{j-1}, t_2^{j-1}) & if\ c_j = (e,b) \\ x_j \underset{ins}{\longrightarrow} in\ the\ 4^{th}\ coordinate\ of\ the\ 2^{nd}\ 4-partition\ of\ (t_1^{j-1}, t_2^{j-1}) & if\ c_j = (a,b) \end{cases}$$

In addition, $(c_j, x_j) \underset{del}{\longleftarrow} \left(t_1^{j-1}, t_2^{j-1}\right)$ signifies the deletion of the box containing x_j from $\left(t_1^{j-1}, t_2^{j-1}\right)$. This deletion process follows the guidelines outlined in [2]. The specific coordinate from which x_j is deleted depends on where x_j was inserted during the row insertion process. In this insertion-deletion process, it's notable that x_j is inserted into $\left(t_1^{j-1}, t_2^{j-1}\right)$ if j represents the left endpoint of the edge labelled with (c_j, x_j). Conversely, x_j is deleted from $\left(t_1^{j-1}, t_2^{j-1}\right)$ if j corresponds to the right endpoint of the edge labelled with (c_j, x_j) in the SRP of the walled Klein-4 Brauer diagram d. Following this process will ultimately result in the final shape $(\lambda^{(r+s)}, \mu^{(r+s)}) = ((\varnothing,\varnothing,\varnothing,\varnothing),(\varnothing,\varnothing,\varnothing,\varnothing))$. With these transformations completed, we can now associate a pair of Klein-4 vacillating tableaux $(P_{(\lambda,\mu)}, Q_{(\lambda,\mu)})$ with the walled Klein-4 Brauer diagram d. The summary of the association is as follows:

$Q_{(\lambda,\mu)} = ((\lambda^0, \mu^0), (\lambda^1, \mu^1), \ldots, (\lambda^{(r+s)}, \mu^{(r+s)}))$

$P_{(\lambda,\mu)} = ((\lambda^{2(r+s)}, \mu^{2(r+s)}), (\lambda^{2(r+s)-1}, \mu^{2(r+s)-1}), \ldots, (\lambda^{(r+s)}, \mu^{(r+s)}))$

$d \rightarrow (P_{(\lambda,\mu)}, Q_{(\lambda,\mu)})$

This association establishes the relationship between the walled Klein-4 Brauer diagram and the corresponding pair of Klein-4 vacillating tableaux. The subsequent example illustrates the insertion-deletion process and the pair of Klein-4 vacillating tableaux associated with the edge sequence mentioned in Table 2.

J	$E_j^{(d,(r+s))}$	(t_1^j, t_2^j)
0		$((\emptyset,\emptyset,\emptyset,\emptyset),(\emptyset,\emptyset,\emptyset,\emptyset))$
1	((a, e), 2)	$((\emptyset,\boxed{2},\emptyset,\emptyset),(\emptyset,\emptyset,\emptyset,\emptyset))$
2	((e, e), 3)	$((\boxed{3},\boxed{2},\emptyset,\emptyset),(\emptyset,\emptyset,\emptyset,\emptyset))$
3	((e, e), 7)	$((\boxed{3}\boxed{7},\boxed{2},\emptyset,\emptyset),(\emptyset,\emptyset,\emptyset,\emptyset))$
4	(∅, 7)	$((\boxed{3},\boxed{2},\emptyset,\emptyset),(\emptyset,\emptyset,\emptyset,\emptyset))$
5	((e, e), 4)	$((\boxed{3},\boxed{2},\emptyset,\emptyset),(\boxed{4},\emptyset,\emptyset,\emptyset))$
6	((e, b), 1)	$((\boxed{3},\boxed{2},\boxed{1},\emptyset)),((\boxed{4},\emptyset,\emptyset,\emptyset))$
7	(∅, 4)	$((\boxed{3},\boxed{2},\boxed{1},\emptyset),(\emptyset,\emptyset,\emptyset,\emptyset))$
8	(∅, 3)	$((\emptyset,\boxed{2},\boxed{1},\emptyset),(\emptyset,\emptyset,\emptyset,\emptyset))$
9	(∅, 2)	$((\emptyset,\emptyset,\boxed{1},\emptyset),(\emptyset,\emptyset,\emptyset,\emptyset))$
10	(∅, 1)	$((\emptyset,\emptyset,\emptyset,\emptyset),(\emptyset,\emptyset,\emptyset,\emptyset))$

Fig 5. Insertion and Deletion Process

$Q_{(\lambda,\mu)} = (\lambda^0, \mu^0), (\lambda^1, \mu^1), (\lambda^2, \mu^2), (\lambda^3, \mu^3), (\lambda^4, \mu^4), (\lambda^5, \mu^5).$

$$= ((\emptyset,\emptyset,\emptyset,\emptyset),(\emptyset,\emptyset,\emptyset,\emptyset)), ((\emptyset,\square,\emptyset,\emptyset),(\emptyset,\emptyset,\emptyset,\emptyset)), ((\square,\square,\emptyset,\emptyset),(\emptyset,\emptyset,\emptyset,\emptyset)),$$

$$((\square\square,\square,\emptyset,\emptyset),(\emptyset,\emptyset,\emptyset,\emptyset)), ((\square,\square,\emptyset,\emptyset),(\emptyset,\emptyset,\emptyset,\emptyset)), ((\square,\square,\emptyset,\emptyset),(\square,\emptyset,\emptyset,\emptyset)).$$

Fig 6. Q(λ, μ)

$P_{(\lambda,\mu)} = (\lambda^{10}, \mu^{10}), (\lambda^9, \mu^9), (\lambda^8, \mu^8), (\lambda^7, \mu^7), (\lambda^6, \mu^6), (\lambda^5, \mu^5).$

$$= ((\emptyset,\emptyset,\emptyset,\emptyset),(\emptyset,\emptyset,\emptyset,\emptyset)),((\emptyset,\emptyset,\square,\emptyset),(\emptyset,\emptyset,\emptyset,\emptyset))((\emptyset,\square,\square,\emptyset),(\emptyset,\emptyset,\emptyset,\emptyset)),$$
$$((\square,\square,\square,\emptyset,(\emptyset,\emptyset,\emptyset,\emptyset))((\square,\square,\square,\emptyset),(\square,\emptyset,\emptyset,\emptyset)),((\square,\square,\emptyset,\emptyset),(\square,\emptyset,\emptyset,\emptyset))$$

Fig 7. P(λ,μ)

Theorem 3.1. For any non-negative integers r and s, the function d $\rightarrow$ ($P_{(\lambda\ ,\mu)}$, $Q_{(\lambda\ ,\mu)}$) establishes a bijection between the set of all walled Klein-4 Brauer diagrams d and pairs of Klein-4 vacillating tableaux. This bijection provides a combinatorial proof of the following identity:

$4^{(r+s)}$(r+s)! = $\sum_{(\lambda,\mu)\in\Lambda_4^{(r,s)}} \left(m^{(\lambda,\mu)}\right)^2$

where $m^{(\lambda,\mu)}$ denotes the number of paths from 0^{th} stage of the Bratteli diagram to the shape (λ,μ).

Proof. The proof follows a similar approach as presented in [2]. The proof of the identity d $\rightarrow$ ($P_{(\lambda\ ,\ \mu)}$, $Q_{(\lambda\ ,\ \mu)}$) follows from Section 3.2.4. To complete the proof of the theorem, we will construct the inverse of this identity. First, we use $Q_{(\lambda\ ,\ \mu)}$ followed by $P_{(\lambda\ ,\mu)}$ in reverse order to construct the sequence of pairs of 4-tableaux with shapes ($\lambda^{(1)}$, $\mu^{(1)}$), ($\lambda^{(2)}$, $\mu^{(2)}$), ($\lambda^{(3)}$, $\mu^{(3)}$), …, ($\lambda^{(2(r+s))}$, $\mu^{(2(r+s))}$). We initiate the construction with ($t_1^{2(r+s)}$,$t_2^{2(r+s)}$) = ((∅,∅,∅,∅), (∅,∅,∅,∅)). We will now describe how to iteratively construct (t_1^j,t_2^j) and $E_{(j+1)}^{(d,(r+s))}$ so that ($t_1^{(j+1)}$,$t_2^{(j+1)}$) is obtained by applying $E_{(j+1)}^{(d,(r+s))}$ followed by uninsertion.

If ($\lambda^{(j+1)}$, $\mu^{(j+1)}$)\ (λ^j, μ^j) corresponds to a box 'B', we perform uninsertion on the value in 'B' to produce (t_1^j,t_2^j) and $E_{(j+1)}^{(d,(r+s))}$ according to the coordinate of the box 'B' in ($t_1^{(j+1)}$,$t_2^{(j+1)}$). Since we uninserted the value in position 'B', we can ensure that (t_1^j,t_2^j) has shape (λ^j, μ^j).

We then demonstrate how to construct (t_1^j,t_2^j) and $E_{(j+1)}^{(d,(r+s))}$such that $(t_1^{(j+1)},t_2^{(j+1)}) = E_{(j+1)}^{(d,(r+s))} \underset{del}{\longleftarrow} \left(t_1^{j-1},\ t_2^{j-1}\right)$. If ($\lambda^j$, μ^j)\ ($\lambda^{(j+1)}$, $\mu^{(j+1)}$) corresponds to a box 'B', let (t_1^j,t_2^j) be the tableau of shape (λ^j, μ^j) with the same entries as ($t_1^{(j+1)}$,$t_2^{(j+1)}$) , and having the entry (2(r + s) − j) in box 'B'. We define $E_{(j+1)}^{(d,(r+s))}$ = (∅, (2(r + s) − j)). Notably, at any given step j, 2(r + s) − j is the largest value added to the pair of 4-tableaux thus far, so (t_1^j,t_2^j) is standard. Furthermore, ($t_1^{(j+1)}$,$t_2^{(j+1)}$) =$E_{(j+1)}^{(d,(r+s))} \underset{del}{\longleftarrow} \left(t_1^j,\ t_2^j\right)$ since $E_{(j+1)}^{(d,(r+s))}$ = (∅,(2(r+s)−j)), and (2(r+s)−j) has already been inserted in a corner, making the deletion process straight forward.

This iterative process will eventually produce $E_{(2(r+s))}^{(d,(r+s))}$, $E_{(2(r+s)-1)}^{(d,(r+s))}$,…, $E_{(1)}^{(d,(r+s))}$, which completely determines d. By equating the cardinality of the basis of the walled Klein-4 Brauer algebra and the number of pairs of Klein-4 vacillating tableaux, we obtain the proof of the theorem.

4 Conclusion

This write-up presents the Robinson-Schensted correspondence for Walled Klein-4 Brauer algebras. We start by explaining the correspondence for the generalised Klein-4 group and then extend it to cover walled Klein-4 Brauer diagrams and vacillating tableaux. The article wraps up by presenting a combinatorial identity that helps calculate the dimension of this algebra. Understanding this connection provides researchers with opportunities to explore new possibilities and applications in various mathematical areas. In today's cryptography, tableaux are used to secure messages through encryption and decryption. In future research, we plan to discover mathematical applications in cryptography using this specific pair of tableaux.

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