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RESEARCH ARTICLE

Arithmetic Sequential Graceful Labeling of Star Related Graphs – II

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Abstract

Objectives: To identify a new family of arithmetic sequential graceful graphs. **Methods:** The methodology involves mathematical formulation for labeling of the vertices of a given graph and subsequently establishing that these formulations give rise to arithmetic sequential graceful labeling. **Findings:** Here, we introduce the square graph ${B_{n,n}}^2$, which shares its vertex set with G. In ${B_{n,n}}^2$, two vertices are considered adjacent if their distance in G^2 is either 1 or 2. To further elucidate, we identify the splitting graph $S'(k_{1,n})$ as an extension of G. Specifically, for each vertex v in G, we introduce a corresponding new vertex v'. Additionally, we demonstrate the construction of the W-star graph. This graph is formed by connecting the apex vertex of the 1st star graph to the apex vertex of the 2nd star graph through a path of length 3. Subsequently, we connect the apex vertex of the 2nd star graph to the apex vertex of the 3rd star graph, also via a path of length 3. **Novelty:** Here we give arithmetic sequential graceful labeling to a new family of graphs, namely W-star, Square of graph G, splitting graph, Banana tree and proved that these graphs possess arithmetic sequential graceful labeling.

Keywords: Star graph; Wstar; Square of graph G; Splitting graph; Banana tree; Graceful labeling

1 Introduction

Graph labeling is one of the most exciting areas of research in graph theory. Labeling is the process of giving values to edges or vertices. According to a recent dynamic survey [1] of graph labeling, numerous scholars contribute their work on various forms of graceful labeling, such as Skolem Graceful Labeling, Edge Even Graceful Labeling, Odd Graceful Labeling and so on. Pasaribu M, Yundari Y, Ilyas M [2] proved that U-star graph is a graceful graph and skolem graceful graph. Zeen El Deen MR, Omar NA [3] proved that the double star graph and path related graphs are r-edge even graceful graph. Amri Z, Irvan I, Maryanti I, Sumardi H [4] proved that Odd graceful labeling on the Ilalang graph (Sn, 3) is graceful graph. Angaleeswari K, Perumal M, Murugan GP, Swaminathan V [5] proved that banana tree graph obtained by connecting one leaf of each of n copies of a k-star graph with a single root vertex that is distinct from all the stars.

2 Methodology

Definition 2.1

For a simple connected graph G the square of graph G is denoted by G^2 and defined as the graph with the same vertex set as of G and two vertices are adjacent in G^2 if they are at a distance 1 or 2 apart in G.

Definition 2.2

For a graph G the splitting graph s' of G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$.

Definition 2.3

Consider a $'t'$ number of star graphs $K_{1,\,n_i}$, $1 \le i \le t$ such that $n_1 \ge 1$ and $n_i \ < \ n_{i+1}$ for $1 \le i \le t-1$. The graph $B(t,\ n)$ is one type of banana tree obtained by joining a new vertex v_0 to the i^{th} end vertex of the i^{th} star graph.

Definition 2.4

Consider 3 copies of star graph $K_{1,n}$. The graph $W - star$ is obtained by connecting the apex vertex of 1^{st} star graph to the apex vertex of 2^{nd} star graph by a path of length 3 and connecting the apex vertex of 2^{nd} star graph to the apex vertex of 3^{rd} star graph by a path of length 3.

Definition 2.5

A simple, finite, connected, undirected, non-trivial graph G is arithmetic sequential graceful graph if the function $f: V(G) \to (a, a+d, a+2d, a+3d, \ldots, 2(a+qd)\}$ where $a \ge 0$ and $d \ge 1$ is an injective function and for each edge $uv \in E(G)$, the induced function $f^*: E(G) \to \{d, 2d, 3d, 4d, \ldots, qd\}$ defined by $f^*(uv) = (f(u) - f(v)|$ is a bijective. The function, then the graph G is called arithmetic sequential graceful graph.

3 Results and Discussion

Theorem 3.1

The graph ${B_{n,n}}^2$ is an arithmetic sequential graceful graph.

Proof:

Let G be a ${B_{n,n}}^2$ graph. $V(G) = \{v\} \cup \{u\} \cup \{v_i : 1 \le i \le n\} \cup \{u_i : 1 \le i \le n\}$ and

$$E(G) = \{vv_i : 1 \le i \le n\} \cup \{uu_i : 1 \le i \le n\} \cup \{vu_i :\ 1 \le i \le n\} \cup \{uv_i : 1 \le i \le n\} \cup \{vu\}$$

Here $|V| = 2n+2$ and $|E| = 4n+1$

We define a function $f: V(G) \to \{a, a+d, a+2d, a+3d, \ldots, 2(a+qd)\}$

The vertex labeling is as follows:

$$f(v) = a$$

$$f(u) = a + (4n+1)d$$

$$f(v_i) = a + id, 1 \le i \le n$$

$$f(u_i) = a + (n+i)d, 1 \le i \le n$$

From the function $f^*: E(G) \to \{d, 2d, 3d, 4d, \ldots, qd\}$ we get the edge labels of the graph ${B_{n,n}}^2, n \ge 1$ as follows

Table 1. Edge labels of the graph $B_{n,n}{}^2, n \geq 1$

$f^*(\mathbf{u}\,\mathbf{v}), \forall\, uv \in E(G)$	**Edge labels**
$f^*(vv_i)$	$\lvert id \rvert,\ 1 \leq i \leq n$
$f^*(vu_i)$	$\lvert (n+i)d \rvert,\ 1 \leq i \leq n$
$f^*(uu_i)$	$\lvert (3n+1-i)\,d \rvert, 1 \leq i \leq n$
$f^*(uv_i)$	$\lvert (4n+1-i)d \rvert, 1 \leq i \leq n$
$f^*(uv)$	$\lvert (4n+1)d \rvert$

It is clear that the function f is injective and also Table 1 shows that $f^* : E \rightarrow \{d, 2d, 3d, 4d, \ldots, qd\}$ is bijective. Hence, f is arithmetic sequential graceful labeling and the graph $B_{n,n}{}^2$ is arithmetic sequential graceful graph.

Example 3.2: The graph $B_{3,3}{}^2$ and its arithmetic sequential graceful labeling are shown in Figure 1.

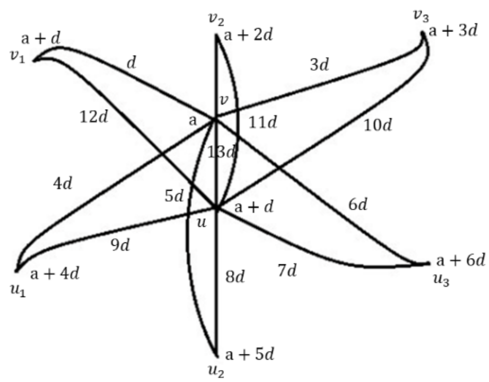

Fig 1. The graph $B_{3,3}{}^2$ and its arithmetic sequential graceful labeling

Theorem 3.3:

The graph $S'(k_{1,n})$ is an arithmetic sequential graceful graph.

Proof:

Let $V(G) = \{v\} \cup \{u\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}$ and $E(G) = \{vv_i : 1 \leq i \leq n\} \cup \{vu_i : 1 \leq i \leq n\} \cup \{uv_i : 1 \leq i \leq n\}$

Here $|V| = 2n+2$ & $|E| = 3n$

We define a function $f : V(G) \rightarrow \{a, a+d, a+2d, a+3d, \ldots, 2(a+qd)\}$

The vertex labeling is as follows:

$$f(v) = a$$

$$f(u) = a+d$$

$$f(v_i) = a+2id, 1 \leq i \leq n$$

$$f(u_i) = a+(2n+i)\,d, 1 \leq i \leq n$$

From the function $f^* : E(G) \rightarrow \{d, 2d, 3d, 4d, \ldots, qd\}$ we get the edge labels of the graph $S'(k_{1,n})$, $n \geq 1$ as follows

Table 2. Edge labels of the graph $S'(k_{1,n})$, $n \geq 1$

$f^*\ (u\,v), \forall\ uv\ \in E(G)$	Edge labels
$f^*(vv_i)$	$\lvert 2id \rvert,\ 1 \leq i \leq n$
$f^*(vu_i)$	$\lvert (2n+i)d \rvert,\ 1 \leq i \leq n$
$f^*(uv_i)$	$\lvert (2i-1)d \rvert, 1 \leq i \leq n$

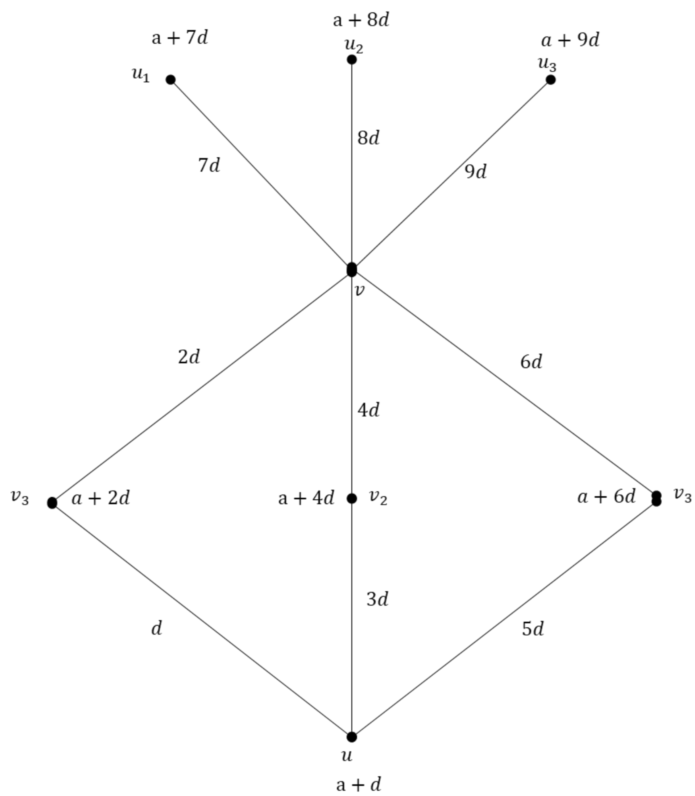

Fig 2. The graph $S'(k_{1,3})$ and its arithmetic sequential graceful labeling

It is clear that the function f is injective and also Table 2 shows that $f^* : E \to \{d, 2d, 3d, 4d, \ldots, qd\}$ is bijective. Hence, f is arithmetic sequential graceful labeling and the graph $S'(k_{1,n})$ is arithmetic sequential graceful graph.

Example 3.4: The graph $S'(k_{1,3})$ and its arithmetic sequential graceful labeling are shown in Figure 2.

Theorem 3.5:

The graph $S'(I_{n,n})$ is an arithmetic sequential graceful graph.

Proof:

Let G be a $S'(I_{n,n})$ graph. $V(G) = \{v\} \cup \{u\} \cup \left\{v'\right\} \cup \left\{u'\right\} \cup \{v_i : 1 \leq i \leq n\} \cup \left\{v_i' : 1 \leq i \leq n\right\} \cup \{u_i : 1 \leq i \leq n\} \cup \left\{u_i' : 1 \leq i \leq n\right\}$ and

$E(G) = \{vv_i : 1 \leq i \leq n\} \cup \left\{vv_i' : 1 \leq i \leq n\right\} \cup \left\{v'v_i : 1 \leq i \leq n\right\} \cup \{uu_i : 1 \leq i \leq n\} \cup \left\{uu_i' : 1 \leq i \leq n\right\} \cup \left\{u'u_i : 1 \leq i \leq n\right\} \cup \{vu\} \cup \left\{vu'\right\} \cup \{uv'\}$

Here $|V| = 4n+4$ and $|E| = 6n+3$

We define a function $f : V(G) \to \{a, a+d, a+2d, a+3d, \ldots, 2(a+qd)\}$

The vertex labeling is as follows:

Case1: when n is odd

$$f(v) = a + (2n-1)\,d, 1 \leq i \leq n$$

$$f(u) = a$$

$$f\left(v'\right) = a + 2nd$$

$$f\left(u'\right) = a + d$$

$$f\left({v_1}'\right) = a + 2d$$

$$f\left({v_{i+1}}'\right) = a + (n-1+i)\,d, 1 \le i \le n-1$$

$$f(v_i) = a + (2i+2)\,d, 1 \le i \le \frac{n-3}{2}$$

$$f(v_i) = a + (6n+1-2i)\,d, \frac{n-1}{2} \le i \le n$$

$$f\left({u_i}'\right) = a + (6n+4-i)\,d, 1 \le i \le n$$

$$f(u_1) = a + (5n+3)\,d$$

$$f(u_{i+1}) = a + (5n+3-2i)\,d, 1 \le i \le n-1$$

Example 3.6: The graph $S'(I_{3,3})$ and its arithmetic sequential graceful labeling are shown in Figure 3.

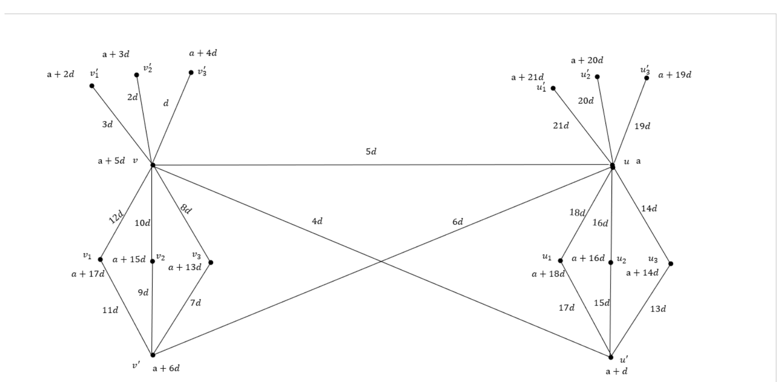

Fig 3. The graph$S'(I_{3,3})$ and its arithmetic sequential graceful labeling

Table 3. Edge labels of the graph $S'(I_{n,n})$ when n is odd

$f^*(\mathbf{u\,v}), \forall\, uv \in E(G)$	**Edge labels**
$f^*(vv_i)$	$\lvert(4n+2-2i)d\rvert, 1 \le i \le n$
$f^*\left(vv_1{}'\right)$	$\lvert(2n-3)d\rvert$
$f^*\left(vv_{i+1}{}'\right)$	$\lvert(n-i)\,d\rvert, 1 \le i \le n-1$
$f^*(uu_1)$	$\lvert(5n+3)d\rvert$
$f^*(uu_{i+1})$	$\lvert(5n+3-2i)\,d\rvert, 1 \le i \le n-1$
$f^*\left(uu_i{}'\right)$	$\lvert(6n+4-i)\,d\rvert, 1 \le i \le n$
$f^*\left(u'u_1\right)$	$\lvert(5n+2)d\rvert$
$f^*\left(u'u_{i+1}\right)$	$\lvert(5n+2-2i)\,d\rvert, 1 \le i \le n-1$
$f^*(vu)$	$\lvert(2n-1)d\rvert$
$f^*\left(vu'\right)$	$\lvert(2n-2)d\rvert$
$f^*\left(uv'\right)$	$\lvert 2nd\rvert$
$f^*(vv_i)$	$\lvert(2n-3-2i)\,d\rvert, 1 \le i \le \frac{n-3}{2},\ n \ge 5$
$f^*(vv_i)$	$\lvert(4n+2-2i)\,d\rvert, \frac{n-1}{2} \le i \le n, n \ge 5$

Case 2: when n is even

$$f(v) = a + (2n-1)\,d$$

$$f(u) = a$$

$$f\left(v'\right) = a + 2nd$$

$$f\left(u'\right) = a + d$$

$$f\left(v_1{}'\right) = a + 4nd$$

$$f\left(v_{1+i}{}'\right) = a + (n-1+i)\,d, 1 \le i \le n-1$$

$$f(v_i) = a + (1+2i)\,d, 1 \le i \le \frac{n-2}{2}$$

$$f(v_i) = a + (6n+2-2i)\,d, \frac{n}{2} \le i \le n$$

$$f\left(u_i{}'\right) = a + (6n+4-i)\,d, 1 \le i \le n$$

$$f(u_1) = a + (5n+3)d$$

$$f(u_{i+1}) = a + (5n+3-2i)d, 1 \le i \le n-1$$

Table 4. Edge labels of the graph $S'(I_{n,n})$ when n is even

$f^*(\mathbf{u\,v}), \forall\, uv \in E(G)$	**Edge labels**
$f^*(vv_i)$	$\lvert(2n-2-2i)d\rvert, 1 \le i \le \frac{n-2}{2}$
$f^*(vv_i)$	$\lvert(4n+3-2i)d\rvert, n/2 \le i \le n$
$f^*\left(vv_1{}'\right)$	$\lvert(2n+1)d\rvert$
$f^*\left(vv_{i+1}{}'\right)$	$\lvert(n-i)d\rvert, 1 \le i \le n-1$
$f^*(uu_1)$	$\lvert(5n+3)d\rvert$
$f^*(uu_{i+1})$	$\lvert(5n+3-2i)d\rvert, 1 \le i \le n-1$
$f^*\left(uu_i{}'\right)$	$\lvert(6n+4-i)d\rvert, 1 \le i \le n$
$f^*\left(u'u_1\right)$	$\lvert(5n+2)d\rvert$
$f^*\left(u'u_{i+1}\right)$	$\lvert(5n+2-2i)d\rvert, 1 \le i \le n-1$
$f^*(vu)$	$\lvert(2n-1)d\rvert$
$f^*\left(vu'\right)$	$\lvert(2n-2)d\rvert$
$f^*\left(uv'\right)$	$\lvert 2nd\rvert$

It is clear that the function f is injective and also Tables 3 and 4 shows that $f^* : E \to \{d, 2d, 3d, 4d, \ldots, qd\}$ is bijective. Hence, f is arithmetic sequential graceful labeling and the graph $S'(I_{n,n})$ is arithmetic sequential graceful graph.

Example 3.7: The graph $S'(I_{4,4})$ and its arithmetic sequential graceful labeling are shown in Figure 4.

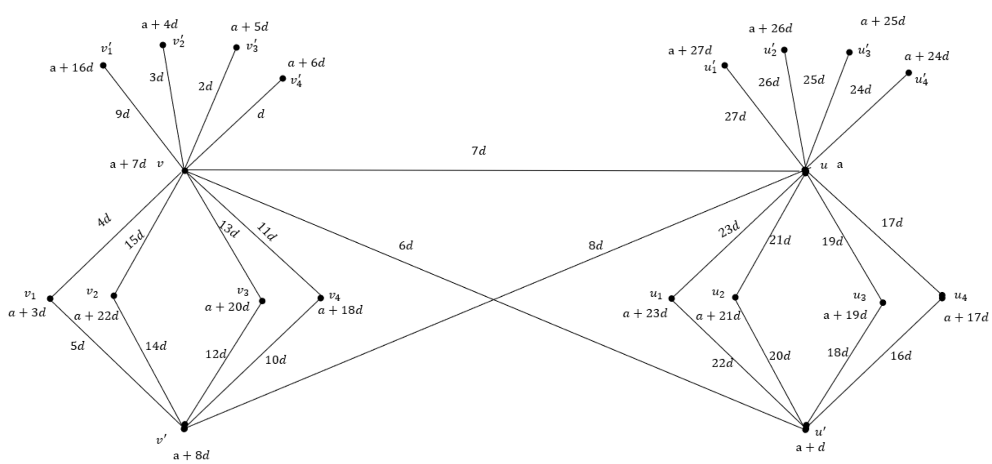

Fig 4. The graph $S'(I_{4,4})$ and its arithmetic sequential graceful labeling

Theorem 3.8:

The $W-$star graph is an arithmetic sequential graceful graph.

Proof:

Let $V(G) = \{v_1, v_2, v_3, u_1, u_2, u_3, u_4\} \cup \{v_{ij} : 1 \le i \le 3\,;\, 1 \le j \le n\}$, and $E(G) = \{v_i\, v_{ij} : 1 \le i \le 3\,;\, 1 \le j \le n\} \cup \{v_1u_1, u_1u_2, u_2v_2, v_2u_3, u_3u_4, u_4v_3\}$

Here $|V| = 3n+7$ and $|E| = 3n+6$
We define a function $f : V(G) \to \{a, a+d, a+2d, a+3d, \ldots, 2(a+qd)\}$
The vertex labeling is as follows:

$$f(v_1) = a$$

$$f(v_2) = a+2d$$

$$f(v_3) = a+d$$

$$f\left(v_{2j}\right) = a+(2+j)d, 1 \leq j \leq n-2$$

$$f(v_{2n-1}) = a+(n+3)d$$

$$f(v_{2n}) = a+(n+4)d$$

$$f(u_1) = a+(3n+6)d$$

$$f(u_2) = a+(n+1)d$$

$$f(u_3) = a+(n+2)d$$

$$f(u_4) = a+(2n+5)d$$

$$f\left(v_{3j}\right) = a+(n+4+j)d, 1 \leq j \leq n$$

$$f\left(v_{1j}\right) = a+(2n+5+j)d, 1 \leq j \leq n$$

Table 5. Edge labels of $W-$ graph, $n \geq 1$

$f^*(\mathbf{u}\,\mathbf{v}), \forall\, uv \in E(G)$	**Edge labels**
$f^*(v_1 v_{1j})$	$\lvert(2n+5+j)d\rvert, 1 \leq j \leq n$
$f^*(v_2 v_{2j})$	$\lvert jd\rvert, 1 \leq j \leq n-2$
$f^*(v_2 v_{2n-1})$	$\lvert(n+1)d\rvert$
$f^*(v_2 v_{2n})$	$\lvert(n+2)d\rvert$
$f^*(v_3 v_{3j})$	$\lvert(n+3+j)d\rvert, 1 \leq j \leq n$
$f^*(v_1 u_1)$	$\lvert(3n+6)d\rvert$
$f^*(v_2 u_2)$	$\lvert(n-1)d\rvert$
$f^*(v_2 u_3)$	$\lvert nd\rvert$
$f^*(v_3 u_4)$	$\lvert(2n+4)d\rvert$
$f^*(u_1 u_2)$	$\lvert(2n+5)d\rvert$
$f^*(u_3 u_4)$	$\lvert(n+3)d\rvert$

It is clear that the function f is injective and also Table 5 shows that $f^*: E \to \{d, 2d, 3d, 4d, \ldots, qd\}$ is bijective. Hence, f is arithmetic sequential graceful labeling and the $W-$graph is arithmetic sequential graceful graph.

Example 3.9: The $W-$graph and its arithmetic sequential graceful labeling are shown in Figure 5.

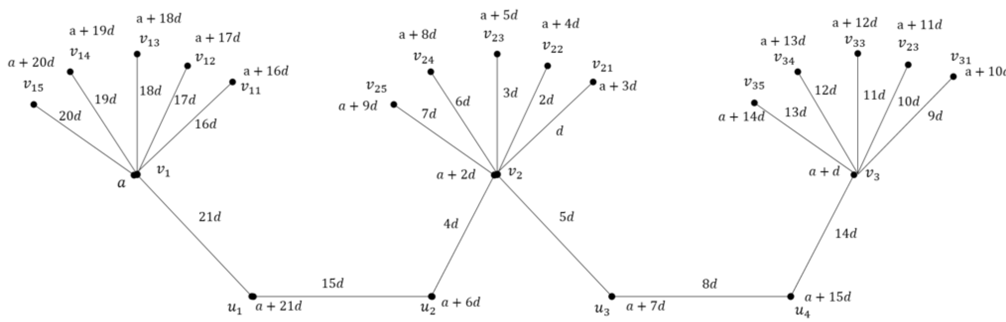

Fig 5. The $W-$graph and its arithmetic sequential graceful labeling

Theorem 3.10:

The graph $B(t,\ n)$ is an arithmetic sequential graceful graph.

Proof:

Let $V(G) = \{v_0\} \cup \{u_i : 1 \leq i \leq t\} \cup \{v_{i\,j} : 1 \leq i \leq t\,;\, 1 \leq j \leq n_i\}$ and
$E(G) = \{u_i v_{i\,j} : 1 \leq i \leq t\,;\, 1 \leq j \leq n_i\} \cup \{v_0\, v_{i\,i} : 1 \leq i \leq t\}$
Here $|V| = 1 + t + \sum_{i=1}^{t} n_i$ and $|E| = t + \sum_{i=1}^{t} n_i$
We define a function $f : V(G) \to \{a, a+d, a+2d, a+3d, \ldots, 2(a+qd)\}$
The vertex labeling is as follows:
$f(v_0) = a + [\sum_{k=1}^{t} n_k]\, d$
$f(v_{1\,j}) = a + (j-1)d \quad 1 \leq j \leq n_1$
$f(v_{(i+1)j}) = a + [\sum_{k=1}^{i} n_k + j - 1]\, d, \quad 1 \leq i \leq t-1; 1 \leq j \leq n_{i+1}$
$f(u_i) = a + [\sum_{k=1}^{i} n_k + t - i]\, d, \quad 1 \leq i \leq t$

Table 6. Edge labels of the graph $B(t,\ n)$, $t\ \geq 1,\ n = \sum_{k=1}^{t} n_k$

$f^*(\mathbf{u}\,\mathbf{v}),\forall\, uv\ \in E(G)$	**Edge labels**
$f^*\left(u_1 v_{1\ j}\right)$	$\left\lvert\ \left[\sum_{k=1}^{t} n_k + t - j\right] d\right\rvert,\ \ 1 \leq j \leq n_1$
$f^*\left(u_{(i+1)} v_{(i+1)\ j}\right)$	$\left\lvert\left[\sum_{k=1}^{t} n_k - \sum_{k=1}^{i} n_k + t - i - j\right] d\right\rvert,\quad 1 \leq i \leq t-1; 1 \leq j \leq n_{i+1}$
$f^*\left(v_0 v_{1\ 1}\right)$	$\left\lvert\left[\sum_{k=1}^{t} n_k\right] d\right\rvert$
$f^*\left(v_0\ v_{i\ i}\right)$	$\left\lvert\left[\ \sum_{k=1}^{t} n_k - \sum_{k=1}^{i-1} n_k - i + 1\right] d\right\rvert,\ \ 2 \leq i \leq t$

It is clear that the function f is injective and also Table 6 shows that $f^* : E \rightarrow \{d, 2d, 3d, 4d, \ldots, qd\}$ is bijective. Hence, f is arithmetic sequential graceful labeling and the graph$B(t,\ n)$ is arithmetic sequential graceful graph.

Example 3.11: The Banana tree and its arithmetic sequential graceful labeling are shown in Figure 6.

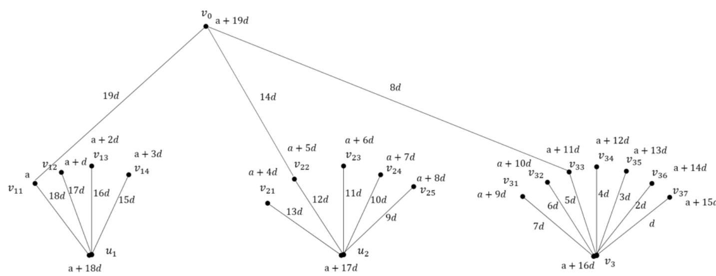

Fig 6. The Banana tree and its arithmetic sequential graceful labeling

4 Conclusion

We showed here arithmetic sequential graceful labeling of some graphs obtained by star graph. Here we proved four new results. We discussed graceful labeling of W-star, Square of graph G, Splitting graph, Banana tree. Labeling pattern is demonstrated by means of illustrations, which provide better understanding of derived results. Analyzing arithmetic sequential graceful labeling on other families of graph are our future work.

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