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REVIEW ARTICLE

Mathematical Perspectives of Leverage Centrality: A Review

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Abstract

Objectives: In complex networks, the identification of key regions is determined through centrality measures. There are various centrality measures in which leverage centrality is specially designed for neural networks. In this article, we review all the recent research works on leverage centrality, focusing mainly on its mathematical perspective. Further, our present research work and the future scope of leverage centrality are given. **Methods:** For this systematic review, we referred all the relevant articles in this area from 2019 to till present. **Findings:** Leverage centrality analysis of some infrastructure networks and group leverage centrality are recently investigated. At the application level, leverage centrality has been used in the analysis of functional magnetic resonance imaging (fMRI) data, and real-world networks including airline connections, electrical power grids, co-authorship collaborations, molecular interaction networks, and sparse complex networks. **Novelty:** Brain networks have demonstrated hierarchical structure and may be decomposed into modules or neighborhoods of nodes that perform similar processes. A novel centrality metric called leverage centrality proposed by Joyce et al. may be of particular use in such hierarchical networks as an aid in identifying hubs, nodes that are important to maintaining local topological structure.

Keywords: Centrality Measure; Neural Network; Fmri; Hubs; Group Leverage Centrality

1 Introduction

Node centrality measures are among the most commonly used analytical techniques for networks. They have long helped analysts to identify important nodes that hold power in a complex network. The integration of computational bio-modeling approaches with different hybrid network-based techniques provides additional information about the behavior of complex systems [1,2]. Many different centrality measures have been proposed, but the degree to which they offer unique information, and whether it is advantageous to use multiple centrality measures to define node roles, is unclear [3]. The identification of critical nodes can be divided into three categories. The first is based on local information, such as degree centrality. Here a large amount of information in the network is ignored, resulting in inaccurate results. The second is from the perspective of global structure, such as betweenness centrality, closeness centrality, and eigenvector centrality. It is time-consuming and is not suitable for large-scale networks. The third is a kind of method between globality and locality. Leverage centrality belongs to this category. Leverage centrality captures nodes in the network which are connected to more nodes than their neighbors and, therefore, control the content and quality of the information received by their neighbors[4].

Leverage centrality can allocate a negative value to a node in contrast with the measurement of other centrality measures. A node of positive value can have more connections than its neighbors[5,6]. Leverage centrality considers the degree of a node relative to its neighbors and operates under the principle that a node in a network is central if its immediate neighbors rely on that node for information. A node with negative leverage centrality is considered to be influenced by its neighbors, as the neighbors connect and interact with farther nodes. A node with positive leverage centrality, on the other hand, influences its neighbors since the neighbors tend to have fewer connections[7].

Now the major recent works on leverage centrality include the leverage centrality analysis of some infrastructure networks like wheel, cycle, path, and their related networks[8] and the notion of group leverage centrality[9]. In addition to the findings above, leverage centrality is used in the determination of regional development priorities in the graph representation of Kalimantan island[10] and the correlation between leverage centrality and critical nodes is investigated for the first time in[11]. Leverage centrality is strongly correlated with page rank centrality and degree centrality in the context of complex networks[12]. In spite of its great application in diverse fields, leverage centrality has yet to be explored more from a mathematical standpoint. In this article, we also highlight the future scope of research in this emerging field.

2 Methodology

We have reviewed the recent articles that highlight the mathematical perspectives of leverage centrality as well as those research that utilize leverage centrality as a tool for hub identification. In this review article, to include all the major results with figures, we excluded the proofs. The structure of the paper is as follows: There are four major sections in which the first describes some basic properties and recent major works on leverage centrality. The second section outlines some of the basic propositions on leverage centrality. In the third section, we present results on leverage centrality analysis of some infrastructure networks and group leverage centrality. The last section includes a conclusion with our present research on leverage centrality.

2.1 Some basic propositions on leverage centrality

Definition 2.1 The degree of a vertex v is the number of edges incident to v and is denoted by $deg(v)$.

The formal definition of leverage centrality is as follows:

Definition 2.2[8] Leverage centrality is a measure of the relationship between the degree of a given node v and the degree of each of its neighbors v_i, averaged over all neighbors Nv, and is defined as:

$$l(v)=\frac{1}{\deg(v)}\sum_{v_i\in N_v}\frac{\deg(v)-\deg(v_i)}{\deg(v)+\deg(v_i)}$$

It is seen from the definition that this measure is unique from existing measures and counts not only the degree of a given node but also the degree of neighbors.

Proposition 2.1 [9] For any graph $G, \sum_{v\in G} l(v) \leq 0$.

A vertex of the lowest degree (highest degree) cannot have a positive (negative) leverage centrality. In the star graph $K_{1,n-1}$, there are $n-1$ vertices with negative leverage centrality. Therefore, it is possible to have all the vertices in a graph except for one to have negative leverage centrality, similarly all but one has positive leverage centrality[9].

Theorem 2.1[9] In a graph G of order n, the maximum number of vertices with positive leverage centrality is $n-1$.

In regular graphs, the leverage centrality of all the vertices is zero. In fact, $l(v) = 0$ for every vertex v if and only if G is a regular graph[9].

3 Results and Discussion

This section presents the formulae for the leverage centrality of nodes in some infrastructure networks including wheels and related networks, cycles and related networks and also paths and related networks.

3.1 Leverage Centrality of Some Infrastructure Networks[8]

Wheels and related networks

3.1.1 Wheel networks

The wheel W_n , with n spokes is a graph that contains an $n-$cycle and one additional central node c that is adjacent to all nodes of the cycle. Label the nodes of the $n-$cycle sequentially as $\{v_0, v_1, \ldots, v_{n-1.}\}$

Theorem 3.1 [8] Let $G = W_n$, of order $n+1$. Then, for $v \in V(G)$,

$$l(v) = \begin{Bmatrix} \frac{n-3}{n+3}, & \text{if } v = c \\ \frac{-(n-3)}{3(n+3)}, & \text{otherwise} \end{Bmatrix}$$

Here the central node has the maximum node degree and is the best-connected node within the network.

3.1.2 Gear networks

Gear network is a wheel graph with a node added between each pair adjacent graph nodes of the outer cycle. The central node c of G_n has degree of n. Label the major (degree three)and minor nodes(degree two), respectively, as $\{v_0, v_1, \ldots, v_{n-1}\}$and $\{w_0, w_1, \ldots, w_{n-1}\}$ and let w_i be adjacent to the nodes v_i and v_{i+1} for $0 \le i \le n-1$, where $i+1$ is taken modulo n .

Theorem 3.2 Let $G = G_n$ of order $2n + 1$. Then, for $v \in V(G)$

$$l(v) = \begin{Bmatrix} \frac{n-3}{n+3} & \text{if } v = c \\ \frac{7-n}{5(n+3)} & \text{if } v = v_i \\ \frac{-1}{5} & \text{if } v = w_i \end{Bmatrix}$$

Minor nodes of the underlying network are of the lowest degree and so cannot have a positive leverage centrality.

Friendship networks

Friendship graph f_n is collection of n triangles with a common point. The central node c of f_n has a node degree of $2n$.

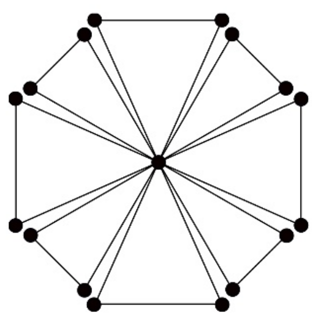

Fig 1. Friendship network f_n for $n = 8$.

Theorem 3.3 Let $G = f_n$ of order $2n+1$. Then, for $v \in V(G)$,

$$l(v) = \begin{Bmatrix} \frac{n-1}{n+1} & \text{if } v = c \\ \frac{-(n-1)}{2(n+1)} & \text{otherwise} \end{Bmatrix}$$

For $n > 1$, the central node has a positive leverage centrality whereas other nodes of the triangles have negative leverage centralities.

3.1.4 Helm networks

Helm H_n is a graph of order $2n+1$ obtained from a wheel W_n with cycle C_n having a pendant link attached to each node of the cycle. H_n consists of the node set $V(H_n)$= $(v_i : 0 \le i \le n-1\} \cup (u_i : 0 \le i \le n-1\} \cup (c\}$ and link set $E(H_n)$ $\{v_i v_{i+1} : 0 \le i \le n-1\}$

$\cup \{v_i u_i : 0 \le i \le n-1\}$, where $i+1$ is taken modulo n. The central node c of H_n has a node degree of n.

$\cup \{v_i c : 0 \le i \le n-1\}$

Theorem 3.4 Let $G = H_n$ of order $2n + 1$. Then, for $v \in V(G)$

$$l(v)=\left\{\begin{array}{lc} \frac{n-4}{n+4} & \text{if} \quad v=c \\ \frac{-(n-16)}{10(n+4)} & \text{if } v=v_i (0\le i\ \le n-1) \\ \frac{-3}{5} & \text{if } v=u_i (0\le i\ \le n-1) \end{array}\right\}$$

For $n>5$, the central node has the highest leverage centrality and so the best-connected node within the network.

3.1.5 Sunflower networks

Sunflower network SF_n consists of a wheel with central node c and an $n-$cycle $\{v_0, v_1, \ldots, v_{n-1}\}$ and additional n nodes $\{u_0, u_1, \ldots, u_{n-1}\}$ where u_i is joined by links to v_i, v_{i+1} for $0\le i\le n-1$, where $i+1$ is taken modulo n. The central node c of SF_n has a node degree of n.

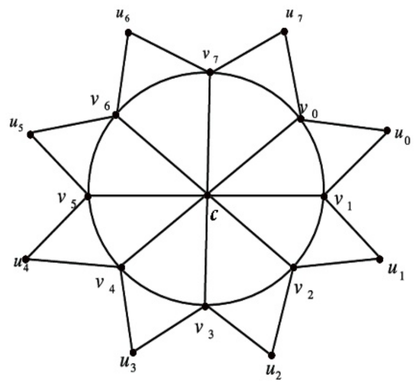

Fig 2. Sunflower network SF_n for $n=8$.

Theorem 3.5 Let $G=SF_n$ of order $2n+1$. Then, for $v\in V(G)$

$$l(v)=\left\{\begin{array}{lc} \frac{n-5}{n+5} & \text{if} \quad v=c \\ \frac{-(n-65)}{35(n+5)} & \text{if } v=v_i (0\le i\ \le n-1) \\ \frac{-3}{7} & \text{if } v=u_i (0\le i\ \le n-1) \end{array}\right\}$$

For $n>6$, the central node has the highest degree, highest leverage centrality, therefore is the best-connected node and influences its neighbors.

Cycles and related networks

3.1.6 Cycle networks

Theorem 3.6 Let $G=C_n$ of order n. Then, $\forall v\in V(G)$, $l(v)=0$.

3.1.7 Fans

If we join a node of C_n to all other nodes, then the resulting graph is called a fan and is denoted by F_n. Let $(c, v_0, v_1, \ldots, v_{n-2}\}$ be the nodes of F_n, where v_0 and v_{n-2} are the nodes of degree two and let c be the node that is connected to all other nodes. Then c is the central node of $F_n$with degree $n-1$. The nodes of degree two are referred to as minor nodes and nodes of degree three to as major nodes.

Theorem 3.7 Let $G=F_n, n>5$ of order n. Then, for $v\in V(G)$

$$l(v)=\left\{\begin{array}{lc} \frac{n(n-3)}{(n+1)(n+2)} & \text{if } v=c \\ \frac{2(11-2n)}{15(n+2)} & \text{if } v \text{ is a major node (adjacent to a minor node)} \\ \frac{4-n}{3(n+2)} & \text{if } v \text{ is a major node (not adjacent to a minor node)} \\ \frac{7-3n}{5(n+1)} & \text{if } v \text{ is a minor node} \end{array}\right\}$$

For $n\ge 6$, the central node has positive leverage centrality and influences its neighbors since the neighbors tend to have far fewer connections.

3.1.8 k-pyramids

The join graph $C_n \vee N_k (n \geq 3,\ k \geq 1)$ where N_k is the null graph of order k is called $k-$pyramid and is denoted by $kP(n)$. The 2-pyramid graph $C_n \vee N_2$ is called bipyramid graph and is denoted by $BP(n)$. Let $\{u_1, u_1, \ldots, u_n\}$ be the nodes of C_n and $\{w_1, w_2, \ldots, w_k\}$ be the nodes of N_k.

Theorem 3.8 Let $G = kP(n)$ of order $n + k$. Then, for $v \in V(G)$

$$l(v) = \left\{ \begin{array}{ll} \frac{k(k-n+2)}{(k+2)(k+n+2)} & \text{if } v = u_i (1 \leq i \leq n) \\ \frac{n-k-2}{n+k+2} & \text{if } v = w_j (1 \leq j \leq k) \end{array} \right\}$$

Corollary 3.1 Let $G = BP(n)$ of order $n+2$. Then, for $v \in V(G)$

$$l(v) = \left\{ \begin{array}{ll} \frac{-(n-4)}{2(n+4)} & \text{if } v = u_i (1 \leq i \leq n) \\ \frac{n-4}{n+4} & \text{if } v = w_j (1 \leq j \leq 2) \end{array} \right\}$$

If $n = k+2$, then the underlying network is regular yielding $l(v) = 0$ for $\forall v \in V(kP(n))$.

If $n < k+2$, then $l(u_i) > 0 (1 \leq i \leq n)$ whereas $l(w_j) < 0 (1 \leq j \leq k)$ meaning that the nodes of C_n influence their neighbors.

If $n > k+2$, then $l(w_j) > 0 (1 \leq j \leq k)$ and $l(u_i) < 0 (1 \leq i \leq n)$.

3.1.9 n-gon books

When k copies of $C_n (n \geq 3)$ share a common link, it will form an $n-$gon book of k pages and is denoted by $B(n,k)$. Here the nodes of degree 2 are referred to as minor nodes and nodes of degree $k+1$ to as major nodes.

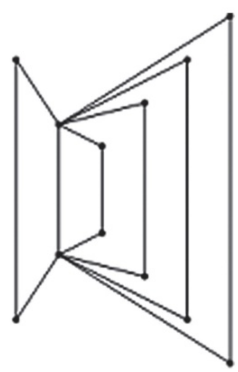

Fig 3. $n-$gon book of k pages $B(n,k)$ for $n = 4,\ k = 5$.

Theorem 3.9 Let $G = B(n,k), k > 1$ of order $(n-2)k+2$. Then, for $v \in V(G)$

$$l(v) = \left\{ \begin{array}{l} \frac{k(k-1)}{(k+1)(k+3)} \text{ if } v \text{ is a major node} \\ \frac{-(k-1)}{(k+3)} \text{ if } n = 3 \text{ and } v \text{ is a minor node} \\ \frac{-(k-1)}{2(k+3)} \text{ if } n > 3 \text{ and } v \text{ is a minor node adjacent to a major node} \\ 0 \text{ if } n > 4 \text{ and } v \text{ is a minor node adjacent to only minor nodes} \end{array} \right\}$$

Major nodes have the highest leverage centralities since their degrees are the highest with respect to other nodes in the neighborhood and are the best-connected nodes within the network.

Paths and related networks

3.1.10 Paths

There are three types of nodes in $P_n, n \geq 5$.

- Type I: v is an end node with $deg(v) = 1$.
- Type II: v is a node adjacent to an end node.
- Type III: v is a node adjacent only to the nodes of degree 2.

Theorem 3.10 Let $G = P_n, n \geq 5$ of order n. Then, for $v \in V(G)$

$$l(v)=\left\{\begin{array}{cc}\frac{-1}{3} & \text{if } v \text{ is a node of Type I}\\ \frac{1}{6} & \text{if } v \text{ is a node of Type II}\\ 0 & \text{if } v \text{ is a node of Type III}\end{array}\right\}$$

The nodes of Type II with positive leverage centralities influence its neighbors and are the best-connected nodes within the network.

3.1.11 Regular Caterpillars

A tree T is called a caterpillar, if removal of all its pendant nodes results in a path called the spine of T. If all nodes of the spine have equal number of pendant nodes, then the resulting graph is called regular caterpillar and is denoted by $T_{n,m}$ where n is the number of nodes of the spine and m is the number of pendant nodes attached to each node of the spine.

There are six types of nodes in $T_{n,m}$ as follows:

- Type 1: v is an end node of P_n.
- Type 2: v is a node of P_n adjacent to an end node.
- Type 3: v is a node of P_n adjacent only to the nodes of degree 2 in P_n.
- Type 4: $u_{ij}(1 \leq j \leq m)$ is a pendant node attached to node v_i, where v_i is of Type 1
- Type 5: $u_{ij}(1 \leq j \leq m)$ is a pendant node attached to node v_i, where v_i is of Type 2.
- Type 6: $u_{ij}(1 \leq j \leq m)$ is a pendant node attached to node v_i, where v_i is of Type 3.

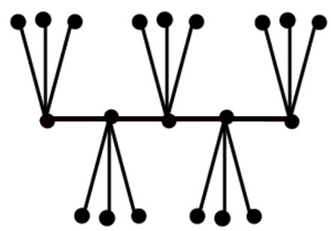

Fig 4. Regular caterpillar $T_{n,m}$ for $n=5,\ m=3$.

Theorem 3.11 Let $G=T_{n,m}, n>3$ of order $n(m+1)$. Then, for $v \in V(G)$

$$l(v)=\left\{\begin{array}{cc}\frac{m\left(2m^2+3m-1\right)-2}{(m+1)(2m+3)(m+2)} & \text{if } v \text{ is a node of Type 1}\\ \frac{m\left(2m^2+5m+4\right)+3}{(m+2)(2m+3)(m+3)} & \text{if } v \text{ is a node of Type 2}\\ \frac{m(m+1)}{(m+2)(m+3)} & \text{if } v \text{ is a node of Type 3}\\ \frac{-m}{(m+2)} & \text{if } v \text{ is a node of Type 4}\\ \frac{-(m+1)}{(m+3)} & \text{if } v \text{ is a node of Type 5 or Type 6}\end{array}\right\}$$

For $m \geq 3$, the nodes of Type 1 with the highest leverage centralities are the best-connected nodes within the network.

Another work in this field is due to [9] is the concept of group leverage centrality which is an extension of leverage centrality.

3.2 Group Leverage Centrality

As the brain network is highly complex and vast, it can be clustered into working regions, or lobes, to effectively model it. Since every lobe of the brain is a set of nodes, [9] introduced a new centrality measure for a subset of nodes- group leverage centrality. It is a measure of how important a subset of nodes is in the network. The effects of meditation on different lobes of the brain were quantified using this new centrality measure.

The two different ways in which group leverage centrality be defined as:

3.2.1 Total Group Leverage Centrality

Total group leverage centrality (TG) is defined as the average of leverage centralities of all the nodes in a given set of nodes S, where S is the subset of V, the vertices of the graph in consideration.

$$TG(S)=\frac{1}{|S|}\sum_{v\in S} l(v) \quad , \quad S \subseteq V$$

Total group leverage centrality is found by finding the average of all the leverage centralities of the vertices (all neighbors) present in the specified subset.

3.2.2 Complement Group Leverage Centrality

Complement group leverage centrality (CG) is defined as the average of leverage centralities of all the nodes in a given set of nodes S, where S is the subset of V, (the vertices of the graph in consideration) and only the vertices in the set $V \backslash S$ are considered for computing the leverage centralities.

$$CG(S) = \frac{1}{|S|} \sum_{v \in S} \left(\frac{1}{k_v} \sum_{i \in SN_v} \left(\frac{k_v - k_i}{k_v + k_i} \right) \right) \text{ where } S' = V \backslash S$$

Here, k_v and N_v represent the degree and set of neighbors of node v respectively.

Complement group leverage centrality is found by finding the average of all the leverage centralities of the vertices while considering only the neighbors outside the specified subset. This centrality has been used to found that meditation has profound effects on the human mind, and can cause increased cognitive processing and perception, decreased stress, and general well-being.

As it is a novel metric, the research in this field is going on tremendously and we expect a rapid solution of the existing challenges in this field, from a mathematical perspective.

4 Conclusion

In this review article, we included all the relevant and recent research works on leverage centrality. This includes leverage centrality analysis of some infrastructure networks like wheels, cycles, paths and their related networks. Also, a new centrality measure derived from leverage centrality is also mentioned along with its application. For the further development of the theory, our present research work is on certain graph products for the leverage centrality analysis of nodes and finding how it is related to the centralities of the component graphs. In the future, the study can be extended to various graph operations.

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