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Establishment of Order of 4-Tuples Concerning Familiar Polynomials with Captivating Condition

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Abstract

Objectives: Finding the feature of a number pattern is a fascinating area of current research. There are numerous features, one of which is that the Area by Perimeter of a Pythagorean triangle is a nasty number, figurate number, and so on. This manuscript aims to identify the precise sorts of 4-tuples combining Legendre polynomials and Probabilist's Hermite polynomials in which the product of any two polynomials is added by one square to another polynomial. **Method:** By using specific transformation, the assumption can be turned into a universal second-degree equation with two variables known as the Pell equation. The solution to this equation yields 3-tuples and their extension into 4-tuples from 2-tuples in a precise manner, as explained. **Findings:** This paper describes some types of 4-tuples involving Legendre polynomials and Probabilist's Hermite polynomials in which the product of any two polynomials is added by the number one in any 4-tuple results in the square of an alternate polynomial. **Novelty:** A Python program is achieved for displaying all 4-tuples with the numerical values of the specified unknown combination with the special property.

Keywords: Legendre polynomials; Probabilist's Hermite polynomials; Pell equation; Integer solutions; Diophantine m-tuples

1 Introduction

A set of m positive integers $(a_1, a_2, a_3, \ldots a_m)$ is called a Diophantine m-tuple if $(a_i a_j + 1)$ is a perfect square for all $1 \le i \le j \le m$. For a widespread assessment of numerous articles one can study[1–10]. In this manuscript, the particular forms of 4-tuples $\{(\alpha_1, \beta_1, \gamma_1, p_1), (\beta_1, \gamma_1, \gamma_2, q_1), (\gamma_1, \gamma_2, \gamma_3, s_1), (\gamma_2, \gamma_3, \gamma_4, t_1) \ldots.\}$ involving Legendre polynomial and Probabilist's Hermite polynomial in which the product of any two polynomials added by the number one in the whole 4-tuples grade square of another polynomial is recognized. Also, a Python Program for showing all the 4-tuples by arithmetic values of the variable together with the chosen statement is obtained.

2 Methodology

2.1 Elementary Definition

2.1.1 Definition of Legendre polynomial

The explicit depiction of Legendre's polynomial is stated by

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)^2 (x-1)^{n-k}(x+1)^k$$

The first few Legendre polynomials are listed below

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \tfrac{1}{2}\left(3x^2-1\right), P_3(x) = \tfrac{1}{2}\left(5x^3-3x\right),$$
$$P_4(x) = \tfrac{1}{8}\left(35x^4-30x^2+3\right), P_5(x) = \tfrac{1}{8}\left(63x^5-70x^3+15x\right)$$

2.1.2 Definition of Probabilist's Hermite polynomial

The probabilist's Hermite polynomial is specified by

$$H_{e_n}(x) = (-1)e^{\frac{x^2}{2}}\frac{d^n}{dx^n}e^{-\left(\frac{x^2}{2}\right)}$$

The leading probabilist's Hermite polynomials are designated by

$$H_{(e_0)}(x) = 1, H_{(e_1)}(x) = x, H_{(e_2)}(x) = x^2-1, H_{(e_3)}(x) = x^3-3x, H_{(e_4)}(x) = x^4-6x^2+3$$

2.1.3 Technique of Examinations

The precise order of attaining 3-tuples and their extension into 4-tuples from 2-tuples by utilizing the universal solutions of second - degree equation with two variables so-called Pell equation is clarified in Sections 3.1 and 3.2.

2.1.4 Extension of 2-tuples to 3-tuples

Let $\alpha_1 = 2P_2(x)$ and $\beta_1 = 3H_{(e_2)}(x)$ be such that $\alpha_1\beta_1+1$ is a perfect square.
This means that the couple (α_1, β_1) signifies Diophantine 2-tuples with the property $D(1)$.
Let γ_1 be the non - zero polynomial in x with supplementary conditions that

$$\alpha_1\gamma_1+1 = a^2 \tag{1}$$

$$\beta_1\gamma_1+1 = b^2 \tag{2}$$

Segregation of γ_1 from Equation (1) and Equation (2) illustrates that

$$\beta_1 a^2-\alpha_1 b^2 = (\beta_1-\alpha_1) \tag{3}$$

Now, introduce the suitable linear conversions

$$a = U+\alpha_1 V \ and \ b = U+\beta_1 V \tag{4}$$

Replacing Equation (4) in Equation (3), this equation turned out to be the succeeding familiar quadratic equation named as Pel equation

$$U^2 = 1+SV^2 \ where \ S = \alpha_1\beta_1 \tag{5}$$

The preliminary solution to Equation (5) be distinguished by

$$U_0 = 3x^2-2 \ and \ V_0 = 1 \tag{6}$$

Application of such solutions to Equation (5) indicated in Equation (6) and the selected choices of α_1 and β_1 in Equation (4) provides the values of a and b as shown below

$$a = 6x^2 - 3 \text{ and } b = 6x^2 - 5 \tag{7}$$

In vision of Equation (1), the value of γ_1 is calculated by

$$\gamma_1 = 12x^2 - 8$$

Thus, the collection of polynomials $(\alpha_1, \beta_1, \gamma_1) = \left(3x^2 - 1, 3x^2 - 3, 12x^2 - 8\right)$ characterizes.
Diophantine 3-tuples sustaining the property $D(1)$.
Next, begin with the pair (β_1, γ_1) let us select γ_2 to be any other non -zero polynomial together with the following double equations

$$\gamma_2 \beta_1 + 1 = c^2 \tag{8}$$

$$\gamma_2 \gamma_1 + 1 = d^2 \tag{9}$$

Now, let us resolve Equation (8) and Equation (9), by inputting appropriate transformations

$$c = M + \beta_1 N \text{ and } d = M + \gamma_1 N \tag{10}$$

which straight the Pell equation

$$M^2 = 1 + PN^2 \tag{11}$$

where $P = \beta_1 \gamma_1$
Securing the corresponding initial solution of Equation (11) as

$$M_0 = 6x^2 - 5 \text{ and } N_0 = 1 \tag{12}$$

By applying this solution Equation (12) and the choices of β_1 and γ_1 in Equation (10), the values of c and d are determined as $c = 9x^2 - 8$ and $d = 18x^2 - 13$.
In view of (8), the value of γ_2 is acquired by $\gamma_2 = 27x^2 - 21$
Hence, $(\beta_1, \gamma_1, \gamma_2) = \left(3x^2 - 3, 12x^2 - 8, 27x^2 - 21\right)$ registers Diophantine 3-tuples satisfying the condition $D(1)$.
Similarly, if (γ_1, γ_2) is a couple of 2 -tuples with $\gamma_1 \gamma_2 + 1$ is a square and γ_3 is the third element for extending this 2 -tuples into 3 -tuples, then by reiterating the corresponding method as approved above the chance of γ_3 is determined as $\gamma_3 = 75x^2 - 55$
In the same way, perform the same technique to the 2-tuples consisting quadratic polynomials $(\gamma_2, \gamma_3) = \left(27x^2 - 21, 75x^2 - 55\right)$, it is possible to find the triple $(\gamma_2, \gamma_3, \gamma_4) = \left(12x^2 - 8, 27x^2 - 21, 75x^2 - 55\right)$.
Thus, it is scrutinized that $(\alpha_1, \beta_1, \gamma_1), (\beta_1, \gamma_1, \gamma_2)(\gamma_1, \gamma_2, \gamma_3)(\gamma_2, \gamma_3, \gamma_4)$ etc. represents a specific order of 3 -tuples nourishing the property $D(1)$.

2.1.5 Extension of order of 3-tuples into order of 4-tuples

Stating with the 3 -tuples $(\alpha_1, \beta_1, \gamma_1)$, let p_1 to be the fourth tuple together with the following declarations that

$$p_1 \alpha_1 + 1 = m_1^2 \tag{13}$$

$$p_1 \beta_1 + 1 = m_2^2 \tag{14}$$

$$p_1 \gamma_1 + 1 = m_3^2 \tag{15}$$

The sequence of non-zero solutions in integers to Equation (5) by using its elementary solutions is specified by

$$U_n = \frac{1}{2}\left\{\left(U_0+\sqrt{S}V_0\right)^{(n+1)} + \left(U_0-\sqrt{S}V_0\right)^{(n+1)}\right\}$$
$$V_n = \frac{1}{2\sqrt{S}}\left\{\left(U_0+\sqrt{S}V_0\right)^{n+1} - \left(U_0-\sqrt{S}V_0\right)^{n+1}\right\} n = 0,1,2,3,\ldots$$

Therefore, the first solution (U_1, V_1) to Equation (5) is attained by

$$U_1 = S + \left(3x^2-2\right)^2 \; and \; V_1 = 6x^2 - 4$$

To treasure the 4^{th}-tuple p_1, let us introduce the first-degree transformations as

$$m_1 = U_1 + \alpha_1 V_1 = 36x^4 - 42x^2 + 11 \tag{16}$$

$$m_2 = U_1 + \beta_1 V_1 = 36x^4 - 54x^2 + 19 \tag{17}$$

$$m_3 = U_1 + \gamma_1 V_1 = 90x^4 - 120x^2 + 39 \tag{18}$$

By using any one of the suitable transformations listed from Equations (16), (17) and (18) in any one of the equations from Equations (13), (14) and (15) the possibility of the fourth tuple is acknowledged by $p_1 = 432x^6 - 864x^4 + 564x^2 - 120$.

Therefore, $(\alpha_1, \beta_1, \gamma_1, p_1) = \left(3x^2-1, 3x^2-3, 12x^2-8, 432x^6-864x^4+564x^2-120\right)$ is Diophantine 4-tuples supporting the characteristics $D(1)$.

Next commence with precise forms of 3 -tuples $(\beta_1, \gamma_1, \gamma_2), (\gamma_1, \gamma_2, \gamma_3)(\gamma_2, \gamma_3, \gamma_4)$ etc.

Let q_1, s_1, t_1 etc to be the fourth tuple in the successive 3 -tuples and by picking appropriate transformations, each 3 -tuples are prolonged into 4 -tuples as prearranged below.

$$\{(\beta_1,\gamma_1,\gamma_2,q_1)(\gamma_1,\gamma_2,\gamma_3,s_1)(\gamma_2,\gamma_3,\gamma_4,t_1)\ldots\} =$$
$$\left(\left(3x^2-3, 12x^2-8, 27x^2-21, 3888x^6-9504x^4+7716x^2-2080\right),\right.$$
$$\left(12x^2-8, 27x^2-21, 75x^2-55, 97200x^6-211680x^4+153588x^2-37128\right),$$
$$\left\{\left(27x^2-21, 75x^2-55, 194x^2-144, 1555200x^6-3516480x^4+2650188x^2-665720\right),\ldots\right\}$$

Note that, in each of the 4 -tuples the product of any two polynomials enlarged by the number one is the square of some polynomial.

Table 1. The subsequent table displays numerical illustrations of order of 4-tuples for few values of x

x	$(\alpha_1,\beta_1,\gamma_1,p_1)$	$(\beta_1,\gamma_1,\gamma_2,q_1)$	$(\gamma_1,\gamma_2,\gamma_3,s_1)$	$(\gamma_2,\gamma_3,\gamma_4,t_1)$
2	(11, 9, 40, 15960)	(9, 40, 87, 125552)	(40, 87, 245, 3411144)	(87, 245, 624, 53204152)
3	(26, 24, 100, 249900)	(24, 100, 222, 2131892)	(100, 222, 620, 55057884)	(222, 620, 1584, 872091892)
4	(47, 45, 184, 1557192)	(45, 184, 411, 13613600)	(184, 411, 1145, 346361400)	(411, 1145, 2928, 5511617608)
5	(74, 72, 292, 6223980)	(72, 292, 654, 55000820)	(292, 654, 1820, 1390252572)	(654, 1820, 4656, 22167788980)

2.1.6 The Python Program for substantiating all such order of 4-tuples gratifying the declaration is presented below.

```
import math
  x = 2
  while (x <= 10) :
  print (' x = ', x)
  T = 1
  a = 3 * x ** 2 - 1
  b = 3 * x ** 2 - 3
```

$r = 12 * x^{**2} - 8$
$s = 27 * x^{**2} - 21$
$t = 75 * x * 2 - 55$
$n = 1$
$D = a^* b$
$U1 = D + (3 * x * * 2 - 2) * * 2$
$V1 = 6 * x^{**2-4}$
$m1 = U1 + a * V1$
$p1 = (m1 * * 2 - 1)/a$
$z = \text{math.sqrt}\ (a * b + n)$
$p = z + a^* T$
$c = (p * * 2 - n)/a$
$print('(a, b, c, d, p1)', a, b, c, p1)$
$D1 = b^* r$
$U2 = D1 + (6 * x * * 2 - 5) * * 2$
$V2 = 12 * x^{**2} - 10$
$m2 = U2 + b * V2$
$p2 = (m2 * * 2 - 1)/b$
$z = \text{math} \cdot \text{sqrt}(b^* r + n)$
$p = z + b^* T$
$c1 = (p * * 2 - n)/b$
$\text{print}\ ('(b, c, d, p_2)', b, c, c1, p_2)$
$D_3 = r^* s$
$U_3 = D_3 + (18 * x * *_2 - 13) * *_2$
$V_3 = 36 * x * *_2 - 26$
$m_3 = U_3 + r^* V_3$
$p_3 = (m_3 3^{**} 2 - 1)/r$
$z = \text{math .sqr}(r * s + n)$
$p = z + r * T$
$c_2 = (p * * 2 - n)/r$
$\text{print}\ ('(c, d, e, p_3)', c, c1, c2, p_3)$
$D_4 = s^* t$
$U_4 = D_4 + (45 * x * 2 - 34) * *_2$
$V_4 = 90 * x^{**2} - 68$
$m_4 = U_4 + s^* V_4$
$p_4 = (m4 *^* 2 - 1)/s$
$z = \text{math .sqrt}(s^* t + n)$
$p = z + S^* T$
$c_3 = (p * * 2 - n)/s$
$\text{print}\ ('(d, e, f, p_4)', c_1, c_2, c_3, p_4)$
$x = x + 1$

3 Remark

If one can choose other combination of these two polynomials, few of them can be extended to quadruple and remaining cannot be extended to quadruple.

- **Example 1**

Consider the Legendre polynomial $\alpha_1 = P_0 = 1$and Probabilist's Hermite polynomial $\beta_1 = H_{(e_2)}(x) = x^2 - 1$
Then it can be extended into quadruple like $\left(1, x^2-1, x^2+2x, 4x^4+8x^3-4x\right)$

$$\left(x^2-1, x^2+2x, 4x^2+4x-3, 16x^6+48x^5+4x^4-72x^3-8x^2+36x-8\right), \ldots$$

The next table provides numerical examples of 4-tuples in order for a few numerals.

x	$(\alpha_1, \beta_1, \gamma_1, p_1)$	$(\beta_1, \gamma_1, \gamma_2, q_1)$	$(\gamma_1, \gamma_2, \gamma_3, s_1)$	$(\gamma_2, \gamma_3, \gamma_4, t_1)$
2	$(1,3,8,120)$	$(3,8,21,2080)$	$(8,21,55,37128)$	$(21,55,144,596390)$
3	$(1,8,15,528)$	$(8,15,45,21736)$	$(8,21,55,37128)$	$(45,112,299,5423892)$
4	$(1,15,24,1520)$	$(15,24,77,111112)$	$(24,77,187,1382880)$	$(77,187,504,26438010)$
5	$(1,24,35,3480)$	$(24,35,117,393472)$	$(35,117,280,4587264)$	$(117,280,759,91599152)$

- **Example 2**

Consider the Legendre polynomial $\alpha = 2P_3 = 5x^3 - 3x$ and Probabilist's Hermite polynomial $\beta = H_{(e_2)}(x) = x^2 - 1$. Then it cannot be extended into quadruple.

4 Results and Discussion

Several authors researched numerous $D(n)$ triples employing polygonal and centered polygonal numbers and discovered that some tuples are extended to $D(n)$ quadruple while few triples are not.

This manuscript specially focused on polynomials, which identify various kinds of 4tuples involving Legendre polynomials and Probabilist's Hermite polynomials, in which the product of any two polynomials is added by the number one in any 4-tuple results in the square of an alternate polynomial. In addition, a Python program has developed for presenting all 4-tuples with numerical values of the specified unknown combination with the special property.

5 Conclusion

In this manuscipt, certain forms of 4-tuples $\{(\alpha_1, \beta_1, \gamma_1, p_1), (\beta_1, \gamma_1, \gamma_2, q_1), (\gamma_1, \gamma_2, \gamma_3, s_1), \ldots\}$ concerning Legendre polynomials and Probabilist's Hermite polynomials in which the product of any two polynomials added by the number one in all the 4-tuples results in the square of an alternative polynomial has documented. Also, a Python Program for showing all the 4-tuples with numerical values of the selected unknown combined with the special property has attained. Similarly, a researcher can analyze 5-tuples and 6-tuples having familiar polynomials or figurate numbers with some other property.

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