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* **Corresponding author**.

srsaranya1995@gmail.com

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Derivation of Finite Number of Integer Solutions to Particular Form of Mordell's Equation $a^2 = b^3 + r^2, r = 8, 9, 10$

V Pandichelvi[1], S Saranya[2]*

1 Assistant Professor, PG & Research Department of Mathematics, Urumu Dhanalakshmi College (Affiliated to Bharathidasan University), Trichy, India
2 Research Scholar, PG & Research Department of Mathematics, Urumu Dhanalakshmi College (Affiliated to Bharathidasan University), Trichy, India

Abstract

Objectives: The perception of solving Diophantine equations is a massive area of research. Mordell equation is a type of Diophantine equation such that the difference of square and cube of numbers remains constant. Various authors analyses Mordell kind equations for existence of solutions by applying several methods. The objective of this manuscript is whether an equation of the form such that the difference of square and cube of numbers provides square of a particular integer. The possibility of visualization of the surface of the considered equation by using the solutions with the support of MATLAB algorithm is also investigated. **Method:** Diophantine equations may have infinite number of solutions or finite number of solutions or no solutions in integers. There is no universal method for solving Diophantine equations. The particular type of Mordell's equation is scrutinized by the concept of divisibility. **Findings:** A special type of Mordell's equation such that the difference of square and cube of numbers offers square of a certain integer is inspected by daubing the notions of divisibility. Furthermore, the integer solutions and the corresponding surface of the equation are recognized with the support of MATLAB programs. **Novelty:** In the analysis, there is no solutions for few cases. The results are confirmed by MATLAB programs. If solutions to the equation exist, the surface of an equation is envisioned with the assistance of MATLAB program.

Keywords: Mordell's Diophantine Equation; Integer Solutions; Divisibility

1 Introduction

In [1–3], the author's determined integer solutions of several types of Mordell's equations. Various methods of evaluating integer solutions to the Mordell's equation $a^2 = b^3 + r$ is analysed in [4–7]. In [8,9], Gauthier, Sebastien, François Le , Mohanty, S. P. analysed Mordell's equation $y^2 = x^3 + k$. In [10], the authors investigated finite number of integer solutions of the Mordell's equation $Y^2 = X^3 + C$, where $C = \pm 9, 36, -16$.

In this text, a definite form of Mordell's equation $a^2 = b^3 + r^2,\ r = 8,\ 9,\ 10$ such that the difference of square of two integers grades a cube of an integer is studied by using the properties of divisibility. Also, an integer solutions and the relevant diagram of the equation
$a^2 = b^3 + r^2,\ r \in Z - \{0\}$ are envisaged by consuming the MATLAB program.

2 Method of exploration of integer solutions to the Mordell's form equations

The motive of this segment is to treasure complete integer solutions for Mordell's type Diophantine equation of the form $a^2 = b^3 + r^2$ where $r = 8$, $9,\ 10$ such that the difference of two square numbers is a cubical integer.

Theorem 2.1

The finite number of integer solutions to the Mordell's equation $a^2 = b^3 + 64$ are $(a,b) = ((\pm 8,0),\ (0,-4),\ (\pm 24,8)\}$.

Proof:

Amend the contemplate equation as given below

$$b^3 = a^2 - 64 = (a+8)(a-8) \tag{1}$$

Here both a and b are either even or odd. If m is the common divisor of $(a+8)$ and $(a-8)$, then m also divides their difference $(a+8)-(a-8) = 16$.

This implies that m essentially be any of the values $1,2,4,8,16$.

Case 1:

Suppose a and b are even. Then $a+8$ and $a-8$ are both even. So $HCF(a+8, a-8)$ is any of the values $2,4,8,16$.

Subcase 1 (i):

If $HCF\ (a+8, a-8) = 2$, then by dividing (1) by 8, it is emblazoned by
$\left(\frac{b}{2}\right)^3 = \left(\frac{a+8}{2}\right)\left(\frac{a-8}{4}\right)$ or $\left(\frac{b}{2}\right)^3 = \left(\frac{a+8}{4}\right)\left(\frac{a-8}{2}\right)$

Division of each component by a multiple of $2 = HCF(a+8, a-8)$, both factors are co-prime and hence is a cube of an integer. Presume that
$\left(\frac{a+8}{2}\right) = x^3$ and $\left(\frac{a-8}{4}\right) = y^3$ or $\left(\frac{a+8}{4}\right) = x^3$ and $\left(\frac{a-8}{2}\right) = y^3$
$\Longrightarrow 2x^3 - 8 = 4y^3 + 8$ or $4x^3 - 8 = 2y^3 + 8$
$\Longrightarrow x^3 - 2y^3 = 8$ or $2x^3 - y^3 = 8$

The opportunity of the above equations are distinguished by
$x = -2,\ y = -2$ or $x = 2, y = 2$

These two choices of $(x.y)$ grants two integer solutions $(a,b) = (\pm 24, 8)$.

Subcase 1 (ii):

Take $\mathrm{HCF}(a+8, a-8) = 4$

Dividing equation (1) by 4^3 and following the analogous method as in subcase 1(i), it is expressed by
$\left(\frac{a+8}{4}\right) = x^3$ and $\left(\frac{a-8}{16}\right) = y^3$ or $\left(\frac{a+8}{16}\right) = x^3$ and $\left(\frac{a-8}{4}\right) = y^3$
$\Longrightarrow x^3 - 4y^3 = 4$ or $4x^3 - y^3 = 4$

These two equations are fulfilled only when $x = 0,\ y = -1$ or $x = 1, y = 0$

The choice $x = 0,\ y = -1$ provides the integer solution as $(a,b) = (-8,0)$.

The options $x = 1, y = 0$ exemplifies the solution as $(a,b) = (8,0)$.

Consequently, $(\pm 8, 0)$ are the only solutions of the equation (1).

Subcase 1 (iii):

Select $HCF(a+8, a-8) = 8$.

Dividing each side of (1) by 8^3 and proceeding the similar technique as in subcase 1(i), it is revealed that
$\left(\frac{a+8}{8}\right) = x^3$ and $\left(\frac{a-8}{64}\right) = y^3$ or $\left(\frac{a+8}{64}\right) = x^3$ and $\left(\frac{a-8}{8}\right) = y^3$
$\Longrightarrow x^3 - 8y^3 = 2$ or $8x^3 - y^3 = 2$

It is scrutinized that both equations do not hold for any integer choices of x and y which is confirmed by the succeeding MATLAB program (Figure 1).

Subcase 1 (iv):

Select $HCF(a+8, a-8) = 16$. By applying the same performance as in subcase 1(i) by dividing both sides of equation (1) by 16^3, it is demonstrated that
$x^3 - 16y^3 = 2$ or $16x^3 - y^3 = 2$

The overhead resultant equations are satisfied by $x = 1,\ y = 0$ and $x = 0,\ y = -1$ respectively.

These two selections of x and y affords that $a = \pm 8$ and $b = 0$.

```
clc, clear
for y = −15000: 15000
for x = −1000: 1000
if((x.^3 − 8 ∗ y.^2 − y) == 2)
fprintf('x = %d, y = %d', x, y)
else
fprintf('no solution,\n')
if((x.^3 − 8 ∗ y.^2 − y) == −2)
fprintf('x = %d, y = %d', x, y)
else
fprintf('no solution,\n')
end
end
end
end
Subsequently a and b do not belong to Z.
```

Fig 1. MATLAB program

Case 2:

Suppose a and b are odd. Then $a+8$ and $a-8$ must be odd. It is confirmed that $HCF\ (a+8, a-8) = 1$which means that $a+8,\ a-8$ are relatively prime.

Choose $a+8 = x^3$ and $a-8 = y^3$

These two equations together give $x^3 - y^3 = 16$ and it is sustained by $x = 2$ and $y = -2$.

Successively, the solutions to the original equation is $(a,b) = (0,-4)$.

Case 3:

In case 1 and 2, the fact about the unique factorization domain in the ring of integer is measured. But one of the factors $\frac{a+8}{m^2}$ or $\frac{a-8}{m^2}$ in the right-hand side of (1) when divided by m^2 may be a factional number.

The product of an integer and a fractional number is a cube of an integer justifies the following conditions.

$\frac{a+8}{m^2} = \frac{x}{y}$ and $\frac{a-8}{m} = x^2y$ or $\frac{a+8}{m^2} = \frac{x^2}{y}$ and $\frac{a-8}{m} = xy$

These adoptions afford the succeeding equations

$mx^2y^2 + 16y - m^2x = 0$ or $mxy^2 + 16y - m^2x^2 = 0$

Solving them for y, it is acquired that

$y = \frac{-16 \pm \sqrt{16^2 + 4m^3x^3}}{2mx^2}$ or $y = \frac{-16 \pm \sqrt{16^2 + 4m^3x^3}}{2mx}$

In these two equations, the discriminant is a positive integer if and only if $x = 1$ and $m = 8$.

Thus, $\frac{a+8}{64} = \frac{1}{y}$ or $\frac{a-8}{8} = y$

$\Longrightarrow y^2 + 2y - 8 = 0$

$\Longrightarrow y = 2$ or $y = -4$

$\Longrightarrow a = \pm 24$ and $b = 8$.

Finally, it is concluded that, $(a,b) = ((\pm 8, 0),\ (0,-4),\ (\pm 24, 8)\}$ are the only integer solutions to the Mordell's kind equation $a^2 = b^3 + 64$.

Theorem 2.2

The only one solution of the Mordell's form of an equation $a^2 = b^3 + 81$ is $(a,b) = (\pm 9, 0)$.

Proof:

The Proof is analogous to Theorem 2.1

Theorem 2.3

The possible integer solutions to the particular Mordell's type equation $a^2 = b^3 + 100$ are $(a,b) = ((\pm6,4), (\pm9,20), (\pm10,0), (\pm15,5)\}$.

Proof:

The Proof is analogous to Theorem 2.1

Computation of integer solutions of $a^2 = b^3 + r^2$ where $r \in Z - \{0\}$ is unveiled with the succeeding MATLAP program (Figure 2)

```
clc,
clear
for a = -5000: 5000
for b = -5000: 5000
for r = 612
if((a.^2 - b.^3) == r.^2)
fprintf('a = %d, b = %d\n', a, b)
end
end
end
end
z1 = [a; b; r];
fsurf(z1)
colormap(cool)
 title('a^2 = b^3 + 612^2')
Output of this MATLAB program is
listed below
a = -1989, b = 153
a = -1700, b = 136
a = -612, b = 0
a = -36, b = -72
a = 36, b = -72
a = 612, b = 0
a = 1700, b = 136
a = 1989 , b = 153
```

Fig 2. MATLAB program

The spectacle of $a^2 = b^3 + 612^2$

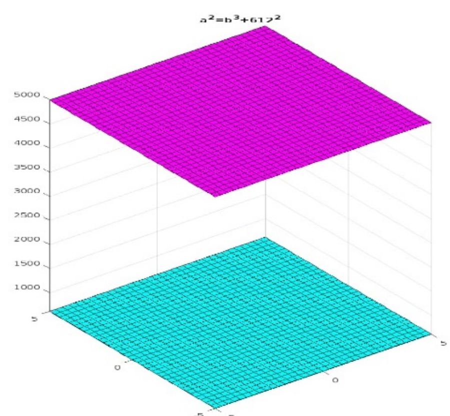

Fig 3. The spectacle of $a^2 = b^3 + 612^2$

3 Conclusion

In this paper, Mordell's equation of the form $a^2 = b^3 + r^2,\ r = 8,\ 9,\ 10$ is inspected for integer solutions. Also, the surface of the same equation for its integer solutions evaluated by MATLAB program is envisioned. In this way, one can investigate integer solutions to the equation such the sum of two cubical integers is a square number.

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