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A Fuzzy EOQ Inventory Model with Advance Payment and Various Fuzzy Numbers

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Abstract

Objectives: To propose suitable inventory system with advance payment in a fuzzy situation by employing four types of fuzzy numbers that are triangular, trapezoidal, pentagonal and Hexagonal. **Methods:** We use the same numerical data as in Priyan et al.(1) to verify the results obtained by this research. To obtain the minimum total cost and the optimum solution, a comparison is made by employing four types of fuzzy numbers. Two inventory models are proposed here. Initially, crisp models are developed with fuzzy total inventory cost along crisp optimal replenishment cycle. Next, the fuzzy model is also formulated with fuzzy total inventory cost and fuzzy optimal replenishment cycle. Graded mean integration method is employed to defuzzify the total inventory cost and the extension of the Lagrangian method is used to determine the optimal replenishment cycle. **Findings:** Our results indicate that the optimal solutions of the fuzzy model slightly fluctuate from the solutions of the crisp model. Numerical examples have been given in order to show the applicability of the proposed model. We obtain minimum total cost when we defuzzify the trapezoidal fuzzy parameters using graded mean integration method. **Novelty:** In real-world circumstances, costs might be influenced by a foreign currency in which the expenses are frequently unknown. Instead of a stochastic environment, the decision-maker in this case is faced with a fuzzy one. This study is unique in itself as it undertakes to study inventory models with different fuzzy numbers. The research reveals that the fuzzy model, which has been defuzzified with the graded mean integration method, shows a proof of savings of 25% to 40% in the analytical solution compared to the previous model. Our model helps the decision maker to tackle the uncertainties in accounting flexibility in the input factors that always fit the real situation.

Keywords: Inventory costs; Advance payment; Fuzzy numbers; Graded mean integration method; Lagrangian method

1 Introduction

An advance payment, or simply an advance, is the part of a contractually due sum that is paid or received in advance for goods or services, while the balance included in the invoice will only follow the delivery. Such payments may be structured into a contract or offered to address a specific adversity situation. In some cases, the full amount due will be paid in advance, while in others, part of the money will be offered in advance and the other part will be paid later.

In inventory system, the decision maker should adopt a better trade of judgement for accounting flexibility in the characteristics of the model in order to tackle the uncertainty which fits to the real situations. Therefore, the practitioner should be more careful in accounting flexibility in the cost components. In real world, transactions between firms seldom complete instantaneously. When a seller is powerful and wants to control the risk of the cash flow, he would like the buyer to pay in a fixed period before the date of delivery. When the payment is paid in advance, the vendor benefits from the cash deposit since the advance payment from the customer can be taken as an interest-free loan. The buyer's inventory policies under delayed payment have been widely addressed. Priyan et al.[1] developed an EOQ inventory model with advance payment policy and fuzziness in the cost and demand parameters.

Due to the imperfect production plan, products that are damaged can be reworked after the completion of one cycle. The customers are then given the finest quality products. The enhanced goods entirely satisfy the shortages. Following a product replenishment and sale under the inventory's scarcity effect, the company looks at a finite horizon periodic combined rework and inventory management model. The improved items totally fulfil and allow for shortage. To meet demand, a multi-shipment policy is used. The researchers have often presumed that the demand is constant. The selling price, existing population, deterioration, and frequency of advertising all have an impact on demand. Throughout the years, a few researchers created inventory models that utilised into consideration things like deteriorating goods, items presently in short supply, various demand patterns and costs, production of specific goods, and combinations of these. Rajeswari et al.[2] assimilated a two-warehouse economic order quantity (EOQ) model with imperfect products under fuzzy environment. Pattnaik et al.[3] discussed an inventory model for imperfect items under different fuzzy scenario. Kuppulakshmi et al.[4] proposed a fuzzy inventory model for imperfect items with price discount and penalty maintenance cost.

Numerous researchers have used fuzzy concepts to address the issues with inventory control. Researchers in production and inventory management lures substantial consideration to Chen et al.'s[5] graded mean integration representation of generalized fuzzy numbers. Rahaman et al.[6] made an initiation to develop a fuzzy production inventory model with deterioration under Marxian approach of socio-political economy. Alsaedi et al.[7] proposed a sustainable green supply chain model with carbon emissions for defective items under learning in a fuzzy environment. Vasanthi et al.[8] developed an inventory model with ordering and holding cost as triangular fuzzy number and an imprecise total cost value is estimated by defuzzifying it by GMI method. Hemalatha and Annadurai[9,10] developed an optimal inventory model under fuzzy environment by considering fuzziness in the cost parameters. The article is designed as follows: Section 2 presents the methodology of the model. Results and discussion are described in Section 3. Finally, the conclusion of the study is summarized in Section 4.

1.1 Research gap and contribution of this model

The key to success in a production-distribution network is making the right decision. Regarding the replenishment decisions adopted under the advance payment, there have been many contributions. According to the aforementioned research results, taking system uncertainty into account is essential for ensuring the industrial sector's financial stability and advance payment. Most of the researchers applied the signed distance method for defuzzification. The present study fills the research gap by applying the Graded mean integration method for defuzzification.

The purpose of this problem is to accommodate the more practical aspects of real inventory systems and focuses on cost minimization. The present study proposed an inventory model with advance payment within the fuzzy framework, thereby adopting two methodologies in the fuzzy scenario. The Graded mean integration method is employed to defuzzify the total inventory cost and the extension of the Lagrangian method is used to find the optimal replenishment cycle. The novelty of this paper is proved by introducing various fuzzy numbers and demonstrating that the EOQ model achieves optimal value in the fuzzy model. The results suggest that operators should treat the input parameter's flexibility as a trapezoidal fuzzy number and Graded mean integration method for defuzzification.

2 Methodology

The methodology of the model includes the preliminary concepts for model building purposes, notations and assumptions that are used for the formulations of both models. In this research, an EOQ inventory model with advance payment policy is described by establishing fuzziness in the cost parameters. The fuzziness in the cost components are represented by the triangular, trapezoidal, pentagonal and Hexagonal fuzzy numbers. Our main objective is to study the impact of the impreciseness of cost components in the decision variables and the total cost. Two fuzzy models are developed. In the first one, crisp models are developed with fuzzy total inventory cost along crisp optimal replenishment cycle T. In the second one, the fuzzy model is formulated with fuzzy total inventory cost and fuzzy optimal replenishment cycle T. Graded mean integration method is employed to defuzzify the total inventory cost and the extension of the Lagrangian method is used to find the optimal replenishment cycle T. Numerical example is carried out to investigate the behaviour of our proposed models, and the results are compared with those obtained from the crisp model.

2.1 Preliminaries

Let us now describe the pertinent definitions of fuzzy sets as follows.

Definition 1: A fuzzy set $\widetilde{A}$ on the given universal set X is a set of ordered pairs on the real line $R, \tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$ is called a membership function. The membership function is also called as degree of compatibility or a degree of truth of X in $\widetilde{A}$ which is defined as$\mu_{\widetilde{A}} : X \to [0,1]$.

Definition 2: Triangular Fuzzy Numbers: Let $\widetilde{A} = (a_1, a_2, a_3), a_1 < a_2 < a_3$ be a fuzzy set on $R = (-\infty, \infty)$. It is called a triangular fuzzy number, if its membership function is

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & a_1 \le x \le a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 \le x \le a_3 \\ 0, & otherwise \end{cases}$$

Definition 3: Trapezoidal Fuzzy Numbers: Let $\widetilde{A} = (a,b,c,d), a < b < c < d$, be a fuzzy set on $R = (-\infty, \infty)$. It is called a trapezoidal fuzzy number, if its membership function is

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a}{b-a}, & a \le x \le b \\ 1, & b \le x \le c \\ \frac{d-x}{d-c}, & c \le x \le d \\ 0, & otherwise \end{cases}$$

Definition 4: Pentagonal Fuzzy Numbers: Let $\widetilde{A} = (a,b,c,d,e), a < b < c < d < e$, be a fuzzy set on $R = (-\infty, \infty)$. It is called a pentagonal fuzzy number, if its membership function is

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a}{b-a}, & a \le x \le b \\ \frac{x-b}{c-b}, & b \le x \le c \\ 1, & x = c \\ \frac{d-x}{d-c}, & c \le x \le d \\ \frac{d-x}{d-c}, & d \le x \le e \\ 0, & otherwise \end{cases}$$

Definition 5: Hexagonal Fuzzy Numbers: Let $\widetilde{A} = (a,b,c,d,e,f), a < b < c < d < e < f$, be a fuzzy set on $R = (-\infty, \infty)$. It is called a hexagonal fuzzy number, if its membership function is

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a}{2(b-a)}, & a \le x \le b \\ \frac{1}{2} + \frac{x-b}{c-b}, & b \le x \le c \\ 1, & c \le x \le d \\ 1 - \frac{x-d}{2(e-d)}, & d \le x \le e \\ \frac{f-x}{2(f-c)}, & e \le x \le f \end{cases}$$

2.2 Graded mean integration representation method

Chen and Hsieh [5] introduced Graded mean integration representation method based on the integral value of graded mean h-level of generalized fuzzy number for defuzzifying generalized fuzzy number. Here, we first describe generalized fuzzy number as follows:

Any fuzzy subset of the real line$R$, whose membership function satisfies the following conditions, is a generalized fuzzy number.

1. $\mu_{\tilde{A}}(x)$ is a continuous mapping from R to the closed interval $[0,1]$.
2. $\mu_{\tilde{A}}(x) = 0, -\infty \leq x \leq a_1$.
3. $\mu_{\tilde{A}}(x) = L(x)$ is strictly increasing on $[a_1, a_2]$.
4. $\mu_{\tilde{A}}(x) = w_A, a_2 \leq x \leq a_3$.
5. $\mu_{\widetilde{A}(x)=R(x)}$is strictly decreasing on$[a_3, a_4]$.
6. $\mu_{\tilde{A}}(x) = 0, a_4 \leq x \leq \infty$.

where $0 < w_A \leq 1$ and a_1, a_2, a_3 and a_4 are real numbers. Also, this type of generalized fuzzy number should be denoted as $\tilde{A} = (a_1, a_2, a_3, a_4; w_A)_{LR}$ When $w_A = 1$, it can be simplified as $\tilde{A} = (a_1, a_2, a_3, a_4)_{LR}$Secondly, by Graded mean integration representation method L^{-1} and $R^{-1}$are the inverse functions of $L$and R respectively, and the graded mean h - level value of generalized fuzzy number $\tilde{A} = (a_1, a_2, a_3, a_4; w_A)_{LR}$ is $\frac{h}{2}(L^{-1}(h) + R^{-1}(h))$. Then the Graded mean integration representation of $P(\widetilde{A})$ with grade w_A is $P(\tilde{A}) = \frac{\int_0^{W_A} \frac{h}{2}\left(L^{-1}(h)+R^{-1}(h)\right)\mathrm{dh}}{\int_0^{W_A} hdh}$, where $0 < h \leq w_A$and $0 < w_A \leq 1$. In our proposed fuzzy inventory models, we use four types of fuzzy number as the type of all fuzzy parameters. The Graded mean integration representation of all the four types of fuzzy numbers for defuzzifying is as follows.

(i) The Graded mean integration representation of a triangular fuzzy numbers $\widetilde{A} = (a_1, a_2, a_3)$ is defined as $d_F\widetilde{A} = \frac{a_1+4a_2+a_3}{6}$.

(ii) The Graded mean integration representation of a trapezoidal fuzzy numbers $\widetilde{A} = (a,b,c,d)$ is defined as $d_F\widetilde{A} = \frac{a_1+2a_2+2a_3+a_4}{6}$.

(iii) The Graded mean integration representation of a pentagonal fuzzy numbers $\widetilde{A} = (a,b,c,d,e)$ is defined as $d_F\widetilde{A} = \frac{a_1+3a_2+4a_3+3a_4+a_5}{12}$.

(iv) The Graded mean integration representation of a hexagonal fuzzy numbers $\widetilde{A} = (a,b,c,d,e,f)$ is defined as $d_F\widetilde{A} = \frac{a_1+3a_2+2a_3+2a_4+3a_5+a_6}{12}$.

2.3 Extension of the Lagrangian Method

Taha[11] discussed to solve the optimum solution of nonlinear programming problem with equality constraints by using Lagrangian Method, and showed the Lagrangian method may be extended to solve inequality constraints. The general idea of extending the Lagrangian procedure is that if the unconstrained optimum the problem does not satisfy all constraints; the constrained optimum must occur at a boundary point of the solution space. Suppose that the problem is given by Minimize $y = f(x)$, subject to $g_i(x) \geq 0$, $i = 1, 2, 3, \ldots..m$. The non-negativity constraints $x \geq 0$ if any are included in the m constraints. Then, the procedure of Extension of the Lagrangian method involves the following steps.

Step 1. Solve the unconstrained problem Minimize $y = f(x)$. If the resulting optimum satisfies all the constraints, stop because all constraints are redundant. Otherwise, set $k = 1$ and go to Step 2.

Step 2. Activate any k constraints (i.e., convert them into equality) and optimize $f(x)$ subject to the k active constraints by the Lagrangian method. If the resulting solution is feasible with respect to the remaining constraints, stop; it is a local optimum. Otherwise, activate another set of k constraint and repeat the step. If all sets of active constraints taken k at a time are considered without encountering a feasible solution, go to Step 3.

Step 3. If $k = m$, stop; no feasible solution exists. Otherwise, set $k = k + 1$ and go to Step 2.

2.4 Notations and Assumptions

We adopt the following notations and assumptions which are almost used in Priyan's[1] model to develop the mathematical model of the proposed inventory system.

2.4.1 Notations

T time interval between successive orders (decision variable)

D demand rate

A_C ordering cost per order

H_C unit stock-holding cost per item per unit time excluding interest charges

p_C unit purchase cost in $

β_d price discount factor for advance payment

t length of advance payment
I_c interest charges per \$ investment in stocks per year
ETC Expected total cost

2.4.2 Assumptions

1. The vendor offers price discount for the buyer if all the payment is paid in advance.
2. Replenishments are instantaneous and the shortages are not allowed
3. Time horizon is infinite

2.5 Crisp EOQ model with advance payment

In this paper the payment term paid by the vendor in advance payment is similar to Priyan et al.[1]. In the proposed scenario, the buyer's purchasing cost is $DTp_C\beta_d$ and incurs an ordering cost A at time zero. The inventory level before arrival of a procurement is zero. The purchase cost has to be financed at interest rate I_c, and the loan interest cost equals $\frac{DTp_CI_c\beta_dt}{T} = Dp_CI_c\beta_dt$ during this period. During the stock period, that is, from time t to $t+T$, the buyer makes payment to the interest- bearing account immediately after the selling of the goods. As the loan is being paid back, the interest payable is decreasing. On the last day of stock period, the buyer pays the remaining balance. Hence, the average outstanding amount of the loan is $DTp_C\beta_d$, and the interest cost is $\frac{DT^2p_CI_c\beta_d}{2}$ from time t to $t+T$ in one cycle. The physical holding cost is the same as that of the traditional economic order quantity inventory model and is not influenced by the payment terms. The behaviour of inventory for this model is depicted in Figure 1. Based on the assumptions described above with Figure 1, the buyer's expected total cost per unit time can be obtained as Priyan et al.[1] is

$$ETC = \frac{A_C}{T} + \frac{DTH_C}{2} + Dp_C\beta_dI_ct + \frac{Dp_C\beta_dI_cT}{2} \tag{1}$$

where $\frac{A_C}{T}$ is the ordering cost, $\frac{DTH_C}{2}$ is the holding cost(excluding interest charge), $Dp_C\beta_dI_ct$ is the cost of interest charges at the time of advance payment and $\frac{Dp_C\beta_dI_cT}{2}$ is the cost of interest charges when the goods are kept in stock.

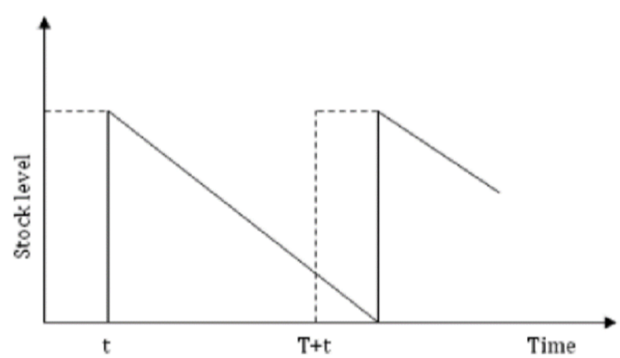

Fig 1. Time-weighted inventory when all the payment paid inadvance

2.5.1 Solution procedure

We proved the convexity of the buyer's expected total cost ETC in proposition 1 based on classical differential calculus optimization technique.

Proposition 1: The expected total cost ETC is a convex function of cycle time T.

The unique optimal replenishment cycle T (denoted by T^*) can be obtained by setting the first order partial derivatives of ETC with respect to T to zero, and simplifying, we get

$$T^* = \sqrt{\frac{2A_C}{D(H_C + p_C\beta_dI_C)}} \tag{2}$$

2.6 Fuzzy EOQ model with advance payment

There are two types of fuzzy models that explored in this section. In the first one, crisp models are developed with fuzzy total inventory cost along crisp optimal replenishment cycle T. In the second one, the fuzzy model is formulated with fuzzy total inventory cost and fuzzy optimal replenishment cycle T.

2.7 Fuzzy EOQ model for crisp optimal replenishment cycle T

We consider the crisp model in fuzzy environment. Considering that the cost components, ordering cost A_C , holding cost H_C and purchase cost p_C are all fuzzy.

2.7.1 Triangular fuzzy number

We represent cost components, A_C, H_C, and p_C by triangular fuzzy number as given below $\widetilde{A}_C = (A_{C1}, A_{C2}, A_{C3})$, $\widetilde{H}_C = (H_{C1}, H_{C2}, H_{C3})$ and $\widetilde{p}_C = (p_{C1}, p_{C2}, p_{C3})$. Accordingly, when the costs A_C, H_C, and $p_C$are fuzzified, we can obtain the total cost function in the fuzzy sense is given by

$$E\widetilde{T}C = (\widetilde{A}_C \oslash T) + ((D \otimes T \otimes \widetilde{H}_C) \oslash 2) + (D \otimes \widetilde{p}_C \otimes \beta_d \otimes I_c \otimes t) + ((D \otimes \widetilde{p}_C \otimes \beta_d \otimes I_c \otimes T) \oslash 2) \tag{3}$$

where $\oplus, \otimes, \oslash$ and $\ominus$ are the fuzzy arithmetical operations under the Function Principle, according to Chen[12]. Then we get the fuzzy total cost by using the Eq. (3) as

$$E\tilde{T}C = \left[\left(\frac{A_{C_1}}{T} + \frac{DTH_{C_1}}{2} + Dp_{C_1}\beta_d I_c t + \frac{Dp_{C_1}\beta_d I_c T}{2}\right)\right. \\ \left.\left(\frac{A_{C_2}}{T} + \frac{DTH_{C_2}}{2} + Dp_{C_2}\beta_d I_c t + \frac{Dp_{C_2}\beta_d I_c T}{2}\right), \left(\frac{A_{C_3}}{T} + \frac{DTH_{C_3}}{2} + Dp_{C_3}\beta_d I_c t + \frac{Dp_{C_3}\beta_d I_c T}{2}\right)\right] \tag{4}$$

Next, we defuzzify the fuzzy total cost by Graded mean integration representation method. The result is

$$\begin{aligned} d(E\tilde{T}C) = \\ \frac{1}{6}\left[\left(\frac{A_{C1}}{T} + \frac{DTH_{C1}}{2} + Dp_{C1}\beta_d I_c t + \frac{Dp_{C1}\beta_d I_c T}{2}\right)\right. \\ +4\left(\frac{A_{C2}}{T} + \frac{DTH_{C2}}{2} + Dp_{C2}\beta_d I_c t + \frac{Dp_{C2}\beta_d I_c T}{2}\right) \\ \left.+\left(\frac{A_{C3}}{T} + \frac{DTH_{C3}}{2} + Dp_{C3}\beta_d I_c t + \frac{Dp_{C3}\beta_d I_c T}{2}\right)\right]. \end{aligned} \tag{5}$$

Then a unique minimum of optimal replenishment cycle T^* is obtained by equating the first order partial derivates of $d(E\widetilde{T}C)$ with respect to T to zero, and simplifying further, we obtain

$$T^* = \sqrt{\frac{2\left(A_{C1} + 4A_{C2} + A_{C3}\right)}{D\left(\left(H_{C1} + 4H_{C2} + H_{C3}\right) + \left(\left(p_{C1} + 4p_{C2} + p_{C3}\right)\beta_d I_C\right)\right.}} \tag{6}$$

2.7.2 Trapezoidal fuzzy number

We represent cost components, A_C, H_C, and p_C by trapezoidal fuzzy number as given below $\widetilde{A}_C = (A_{C1}, A_{C2}, A_{C3}, A_{C4})$, $\widetilde{H}_C = (H_{C1}, H_{C2}, H_{C3}, H_{C4})$ and $\widetilde{p}_C = (p_{C1}, p_{C2}, p_{C3}, p_{C4})$. Accordingly, when the costs A_C, H_C, and p_C are fuzzified, we can obtain the total cost function in the fuzzy sense is given by

$$E\widetilde{T}C = \left(\left(\frac{A_{C1}}{T} + \frac{DTH_{C1}}{2} + Dp_{C1}\beta_d I_c t + \frac{Dp_{C1}\beta_d I_c T}{2}\right), \left(\frac{A_{C2}}{T} + \frac{DTH_{C2}}{2} + Dp_{C2}\beta_d I_c t + \frac{Dp_{C2}\beta_d I_c T}{2}\right),\right.$$

$$\left(\left(\frac{A_{C3}}{T} + \frac{DTH_{C3}}{2} + Dp_{C3}\beta_d I_c t + \frac{Dp_{C3}\beta_d I_c T}{2}\right), \left(\frac{A_{C4}}{T} + \frac{DTH_{C4}}{2} + Dp_{C4}\beta_d I_c t + \frac{Dp_{C4}\beta_d I_c T}{2}\right)\right]. \tag{7}$$

Next, we defuzzify the fuzzy total cost by Graded mean integration representation method. The result is

$$d(E\widetilde{T}C) = \frac{1}{6}\left(\left(\frac{A_{C1}}{T}+\frac{DTH_{C1}}{2}+Dp_{C1}\beta_d I_c t+\frac{Dp_{C1}\beta_d I_c T}{2}\right)+2\left(\frac{A_{C2}}{T}+\frac{DTH_{C2}}{2}+Dp_{C2}\beta_d I_c t+\frac{Dp_{C2}\beta_d I_c T}{2}\right)\right.$$
$$\left.+2\left(\left(\frac{A_{C3}}{T}+\frac{DTH_{C3}}{2}+Dp_{C3}\beta_d I_c t+\frac{Dp_{C3}\beta_d I_c T}{2}\right)+\left(\frac{A_{C4}}{T}+\frac{DTH_{C4}}{2}+Dp_{C4}\beta_d I_c t+\frac{Dp_{C4}\beta_d I_c T}{2}\right)\right]\right. . \quad (8)$$

Then a unique minimum of optimal replenishment cycle T^* is obtained by equating the first order partial derivates of $d(E\widetilde{T}C)$ with respect to T to zero.

We obtain

$$T^* = \sqrt{\frac{2\left(A_{C1}+2A_{C2}+2A_{C3}+A_{C4}\right)}{D\left(\left(H_{C1}+2H_{C2}+2H_{C3}+H_{C4}\right)+\left(\left(p_{C1}+2p_{C2}+2p_{C3}+p_{C4}\right)\beta_d I_C\right)\right.}} \quad (9)$$

2.7.3 Pentagonal fuzzy number

We represent cost components A_C, H_C, and p_C by pentagonal fuzzy number as given below$\widetilde{A_C} = (A_{C1},A_{C2},A_{C3},A_{C4},A_{C5})$, $\widetilde{H_C} = (H_{C1},H_{C2},H_{C3},H_{C4},H_{C5})$ and $\widetilde{p_C} = (p_{C1},p_{C2},p_{C3},p_{C4},p_{C5})$. Accordingly, when the costs A_C, H_C, and p_C are fuzzified, we can obtain the total cost function in the fuzzy sense is given by

$$\begin{aligned}
&E\tilde{T}C = \\
&\left[\left(\frac{A_{C1}}{T}+\frac{DTH_{C1}}{2}+Dp_{C1}\beta_d I_c t+\frac{Dp_{C1}\beta_d I_c T}{2}\right),\right.\\
&\left(\frac{A_{C2}}{T}+\frac{DTH_{C2}}{2}+Dp_{C2}\beta_d I_c t+\frac{Dp_{C2}\beta_d I_c T}{2}\right),\\
&\left(\frac{A_{C3}}{T}+\frac{DTH_{C3}}{2}+Dp_{C3}\beta_d I_c t+\frac{Dp_{C3}\beta_d I_c T}{2}\right),\\
&\left(\frac{A_{C4}}{T}+\frac{DTH_{C4}}{2}+Dp_{C4}\beta_d I_c t+\frac{Dp_{C4}\beta_d I_c T}{2}\right)\\
&\left.\left(\frac{A_{C5}}{T}+\frac{DTH_{C5}}{2}+Dp_{C5}\beta_d I_c t+\frac{Dp_{C5}\beta_d I_c T}{2}\right)\right].
\end{aligned} \quad (10)$$

Next, we defuzzify the fuzzy total cost by Graded mean integration representation method. The result is

$$\begin{aligned}
&d(E\tilde{T}C) = \\
&\frac{1}{12}\left[\left(\frac{A_{C1}}{T}+\frac{DTH_{C1}}{2}+Dp_{C1}\beta_d I_c t+\frac{Dp_{C1}\beta_d I_c T}{2}\right)\right.\\
&+3\left(\frac{A_{C2}}{T}+\frac{DTH_{C2}}{2}+Dp_{C2}\beta_d I_c t+\frac{Dp_{C2}\beta_d I_c T}{2}\right)\\
&+4\left(\frac{A_{C3}}{T}+\frac{DTH_{C3}}{2}+Dp_{C3}\beta_d I_c t+\frac{Dp_{C3}\beta_d I_c T}{2}\right)\\
&+3\left(\frac{A_{C4}}{T}+\frac{DTH_{C4}}{2}+Dp_{C4}\beta_d I_c t+\frac{Dp_{C4}\beta_d I_c T}{2}\right)\\
&\left.+\left(\frac{A_{C5}}{T}+\frac{DTH_{C5}}{2}+Dp_{C5}\beta_d I_c t+\frac{Dp_{C5}\beta_d I_c T}{2}\right)\right].
\end{aligned} \quad (11)$$

Then a unique minimum of optimal replenishment cycle T^* is obtained by equating the first order partial derivates of $d(E\widetilde{T}C)$ with respect to T to zero.

We obtain

$$T^* = \sqrt{\frac{2\left(A_{C1}+3A_{C2}+4A_{C3}+3A_{C4}+A_{C5}\right)}{D\left(\left(H_{C1}+3H_{C2}+4H_{C3}+3H_{C4}+H_{C5}\right)+\left(\left(p_{C1}+3p_{C2}+4p_{C3}+3p_{C4}+p_{C5}\right)\beta_d I_C\right)\right.}} \quad (12)$$

2.7.4 Hexagonal fuzzy number

We represent cost components, A_C, H_C, and p_C by hexagonal fuzzy number as given below. $\widetilde{A} = (A_1, A_2, A_3, A_4, A_5, A_6)$, $\widetilde{h} = (h_1, h_2, h_3, h_4, h_5, h_6)$ and $\widetilde{p} = (p_1, p_2, p_3, p_4, p_5, p_6)$. Accordingly, when the costs A_C, H_C, and p_C are fuzzified, we can obtain the total cost function in the fuzzy sense is given by

$$E\widetilde{T}C = \left(\left(\frac{A_{C1}}{T} + \frac{DTH_{C1}}{2} + Dp_{C1}\beta_d I_c t + \frac{Dp_{C1}\beta_d I_c T}{2}\right), \left(\frac{A_{C2}}{T} + \frac{DTH_{C2}}{2} + Dp_{C2}\beta_d I_c t + \frac{Dp_{C2}\beta_d I_c T}{2}\right),\right.$$

$$\left(\frac{A_{C3}}{T} + \frac{DTH_{C3}}{2} + Dp_{C3}\beta_d I_c t + \frac{Dp_{C3}\beta_d I_c T}{2}\right), \left(\frac{A_{C4}}{T} + \frac{DTH_{C4}}{2} + Dp_{C4}\beta_d I_c t + \frac{Dp_{C4}\beta_d I_c T}{2}\right)$$

$$\left(\left(\frac{A_{C5}}{T} + \frac{DTH_{C5}}{2} + Dp_{C5}\beta_d I_c t + \frac{Dp_{C5}\beta_d I_c T}{2}\right), \left(\frac{A_{C6}}{T} + \frac{DTH_{C6}}{2} + Dp_{C6}\beta_d I_c t + \frac{Dp_{C6}\beta_d I_c T}{2}\right)\right]. \quad (13)$$

Next, we defuzzify the fuzzy total cost by Graded mean integration representation method. The result is

$$\begin{aligned} &d(E\tilde{T}C) = \\ &\frac{1}{12}\left[\left(\frac{A_{C1}}{T} + \frac{DTH_{C1}}{2} + Dp_{C1}\beta_d I_c t + \frac{Dp_{C1}\beta_d I_c T}{2}\right)\right. \\ &+3\left(\frac{A_{C2}}{T} + \frac{DTH_{C2}}{2} + Dp_{C2}\beta_d I_c t + \frac{Dp_{C2}\beta_d I_c T}{2}\right) \\ &+2\left(\frac{A_{C3}}{T} + \frac{DTH_{C3}}{2} + Dp_{C3}\beta_d I_c t + \frac{Dp_{C3}\beta_d I_c T}{2}\right) \\ &+2\left(\frac{A_{C4}}{T} + \frac{DTH_{C4}}{2} + Dp_{C4}\beta_d I_c t + \frac{Dp_{C4}\beta_d I_c T}{2}\right) \\ &+3\left(\frac{A_{C5}}{T} + \frac{DTH_{C5}}{2} + Dp_{C5}\beta_d I_c t + \frac{Dp_{C5}\beta_d I_c T}{2}\right) \\ &\left.+\left(\frac{A_{C6}}{T} + \frac{DTH_{C6}}{2} + Dp_{C6}\beta_d I_c t + \frac{Dp_{C6}\beta_d I_c T}{2}\right)\right]. \end{aligned} \quad (14)$$

Then a unique minimum of optimal replenishment cycle T^* is obtained by equating the first order partial derivates of $d(E\widetilde{T}C)$ with respect to T to zero. We obtain

$$T^* = \sqrt{\frac{2(A_{C1} + 3A_{C2} + 2A_{C3} + 2A_{C4} + 3A_{C5} + A_{C6})}{D((H_{C1} + 3H_{C2} + 2H_{C3} + 2H_{C4} + 3H_{C5} + H_{C6}) + ((p_{C1} + 3p_{C2} + 2p_{C3} + 2p_{C4} + 3p_{C5} + p_{C6})\beta_d I_c)}}. \quad (15)$$

2.8 Fuzzy EOQ model for fuzzy optimal replenishment cycle T

In this case we consider that the fuzzy EOQ model by changing the crisp replenishment cycle T into fuzzy replenishment cycle T. We represent the cost components A_C, H_C, and p_C and the replenishment cycle T by the triangular, trapezoidal, pentagonal and Hexagonal fuzzy numbers.

2.8.1 Case 1: Triangular fuzzy number

We represent cost components A_C, H_C, and p_C and the replenishment cycle T by the triangular fuzzy number given below.

$\widetilde{A_C} = (A_{C1}, A_{C2}, A_{C3})$, $\widetilde{H_C} = (H_{C1}, H_{C2}, H_{C3})$ and $\widetilde{p_C} = (p_{C1}, p_{C2}, p_{C3})$ and $\widetilde{T} = (T_1, T_2, T_3)$. Accordingly, when the costs A_C, H_C, p_C and T are fuzzified, we obtain the total cost function in the fuzzy sense given by

$$E\widetilde{T}C = \left(\left(\frac{A_{C1}}{T_3} + \frac{DT_1H_{C1}}{2} + Dp_{C1}\beta_d I_c t + \frac{Dp_{C1}\beta_d I_c T_1}{2}\right), \left(\frac{A_{C2}}{T_2} + \frac{DT_2H_{C2}}{2} + Dp_{C2}\beta_d I_c t + \frac{Dp_{C2}\beta_d I_c T_2}{2}\right),\right.$$
$$\left(\frac{A_{C3}}{T_1} + \frac{DT_3H_{C3}}{2} + Dp_{C3}\beta_d I_c t + \frac{Dp_{C3}\beta_d I_c T_3}{2}\right) \quad (16)$$

Next, we defuzzify the fuzzy total cost by Graded mean integration representation method. Then we have

$$d(E\widetilde{T}C)=\frac{1}{6}\left(\left(\frac{A_{C1}}{T_3}+\frac{DT_1H_{C1}}{2}+Dp_{C1}\beta_dI_ct+\frac{Dp_{C1}\beta_dI_cT_1}{2}\right)+4\left(\frac{A_{C2}}{T_2}+\frac{DT_2H_{C2}}{2}+Dp_{C2}\beta_dI_ct+\frac{Dp_{C2}\beta_dI_cT_2}{2}\right)\right.$$

$$\left.+\left(\left(\frac{A_{C3}}{T_1}+\frac{DT_3H_{C3}}{2}+Dp_{C3}\beta_dI_ct+\frac{Dp_{C3}\beta_dI_cT_3}{2}\right)\right)\right] \tag{17}$$

with $0\le T_1\le T_2\le T_3$. It will not change the meaning of the Eq. (17), if we replace inequality conditions $0\le T_1\le T_2\le T_3$ into the inequality constraints $T_3-T_2\ge 0$, $T_2-T_1\ge 0$ and $T_1>0$. In the following steps, extension of the Lagrangian method is used to find the solutions of $T_1,T_2$and T_3 to minimize the defuzzified fuzzy total cost $d(E\widetilde{T}C)$ subject to $T_3-T_2\ge 0, T_2-T_1\ge 0$ and $T_1\ge 0$ in Eq. (17).

Step 1: Solve the unconstraint problem. Then a unique minimum of $d(E\widetilde{T}C)$ is obtained by equating the first order partial derivatives of $d(E\widetilde{T}C)$ with respect to $T_1,T_2$and T_3 to zero. We obtain $T_1=\sqrt{\frac{2A_{C3}}{D(H_{C1}+p_{C1}\beta_dI_c)}}$,$T_2=\sqrt{\frac{2(4A_{C2})}{D(4H_{C2}+4p_{C2}\beta_dI_c)}}$ and $T_3=\sqrt{\frac{2A_{C1}}{D(H_{C3}+p_{C3}\beta_dI_c)}}$.

The above results show that $T_1>T_2>T_3$and it does not satisfy the constraint $0\le T_1\le T_2\le T_3$.

Therefore, set k = 1 and go to step 2.

Step 2: Convert the inequality constraint $T_2-T_1\ge 0$ into equality constraint $T_2-T_1=0$. We optimize $d(E\widetilde{T}C)$ subject to $T_2-T_1=0$ by the Lagrangian method. The Lagrangian function is $L(T_1,T_2,T_3,\lambda)=d(E\widetilde{T}C)-\lambda(T_2-T_1)$.

Then a unique minimum of $L(T_1,T_2,T_3,\lambda)$ is obtained by equating the first order partial derivatives of $L(T_1,T_2,T_3,\lambda)$ with respect to T_1, T_2, T_3 and λ to zero. We obtain

$T_1=T_2=\sqrt{\frac{2(A_{C3}+4A_{C2})}{D((H_{C1}+4H_{C2})+(p_{C1}\beta_dI_c+4p_{C2}\beta_dI_c))}}$ and $T_3=\sqrt{\frac{2A_{C1}}{D(H_{C3}+p_{C3}\beta_dI_c)}}$.

Thus, the results show that $T_1>T_3$ does not satisfy the constraint $0\le T_1\le T_2\le T_3$. Therefore, set k = 2 and go to step 3.

Step 3: Convert the inequality constraint $T_2-T_1\ge 0$ and $T_3-T_2\ge 0$ into equality constraint $T_2-T_1=0$, $T_3-T_2=0$. We optimize $d(E\widetilde{T}C)$ subject to $T_2-T_1=0$ and $T_3-T_2=0$by the Lagrangian method. The Lagrangian function is $L(T_1,T_2,T_3,\lambda_1,\lambda_2)=d(E\widetilde{T}C)-\lambda_1(T_2-T_1)-\lambda_2(T_3-T_2)$.

Then a unique minimum of $L(T_1,T_2,T_3,\lambda_1,\lambda_2)$ is obtained by equating the first order partial derivatives of $L(T_1,T_2,T_3,\lambda_1,\lambda_2)$ with respect to T_1,T_2,T_3,λ_1 and λ_2 to zero.

We obtain $T^*=T_1=T_2=T_3=\sqrt{\frac{2(A_{C1}+4A_{C2}+A_{C3})}{D((H_{C1}+4H_{C2}+H_{C3})+(p_{C1}+4p_{C2}+p_{C3})\beta_dI_c)}}$.

Case 2: Trapezoidal fuzzy number

We represent cost components A_C, H_C, and p_C and the replenishment cycle T by the trapezoidal fuzzy number as given below $\widetilde{A}_C=(A_{C1},A_{C2},A_{C3},A_{C4})$, $\widetilde{H}_C=(H_{C1},H_{C2},H_{C3},H_{C4})$, $\widetilde{p}_C=(p_{C1},p_{C2},p_{C3},p_{C4})$ and $\widetilde{T}=(T_1,T_2,T_3,T_4)$. Accordingly, when the costs A_C, H_C, and p_C and $T$are fuzzified, we obtain the total cost function in the fuzzy sense given by

$$E\tilde{T}C=\left[\left(\frac{AC_1}{T_4}+\frac{DT_1HC_1}{2}+Dp_{C_1}\beta_dI_ct+\frac{Dp_{C_1}\beta_dI_CT_1}{2}\right),\left(\frac{AC_2}{T_3}+\frac{DT_2HC_2}{2}+Dp_{C_2}\beta_dI_ct+\frac{Dp_{C_2}\beta_dI_cT_2}{2}\right),\right.$$
$$\left.\left(\frac{AC_3}{T_2}+\frac{DT_3HC_3}{2}+Dp_{C_3}\beta_dI_ct+\frac{Dp_3\beta_dI_CT_3}{2}\right),\left(\frac{AC_4}{T_1}+\frac{DT_4HC_4}{2}+Dp_{C_4}\beta_dI_ct+\frac{Dp_{C_4}\beta_dI_cT_4}{2}\right)\right]. \tag{18}$$

Next, we defuzzify the fuzzy total cost by Graded mean integration representation method. Then, we have

$$d(E\tilde{T}C)=\frac{1}{6}\left[\left(\frac{A_{C_1}}{T_4}+\frac{DT_1H_{C_1}}{2}+Dp_{C_1}\beta_dI_ct+\frac{Dp_{C_1}\beta_dI_cT_1}{2}\right)+2\left(\frac{A_{C_2}}{T_3}+\frac{DT_2H_{C_2}}{2}+Dp_{C_2}\beta_dI_ct+\frac{Dp_{C_2}\beta_dI_CT_2}{2}\right)\right.$$
$$\left.+2\left(\frac{A_{C_3}}{T_2}+\frac{DT_3H_{C_3}}{2}+Dp_{C_3}\beta_dI_ct+\frac{Dp_{C_3}\beta_dI_cT_3}{2}\right)+\left(\frac{A_{C_4}}{T_1}+\frac{DT_4H_{C_4}}{2}+Dp_{C_4}\beta_dI_ct+\frac{Dp_{C_4}\beta_dI_CT_4}{2}\right)\right] \tag{19}$$

with $0\le T_1\le T_2\le T_3\le T_4$. It will not change the meaning of the Eq. (19), if we replace inequality conditions $0\le T_1\le T_2\le T_3\le T_4$ into the inequality constraints $T_4-T_3\ge 0, T_3-T_2\ge 0, T_2-T_1\ge 0$ and $T_1>0$. The extension of the Lagrangian method is used to find the solutions of $T_1,T_2,T_3$and T_4. By adopting the Lagrangian method to find the solutions of $T_1,T_2,T_3$and T_4as in the case 1. We have

$T^*=T_1=T_2=T_3=T_4=\sqrt{\frac{2(A_{C1}+2A_{C2}+2A_{C3}+A_{C4})}{D((H_{C1}+2H_{C2}+2H_{C3}+H_{C4})+(p_{C1}+2p_{C2}+2p_{C3}+p_{C4})\beta_dI_c)}}$

2.8.3 Case 3: Pentagonal fuzzy number

We represent cost components, A_C, H_C, p_C and the replenishment cycle T by the pentagonal fuzzy number as given below.

$\widetilde{A}_C = (A_{C1}, A_{C2}, A_{C3}, A_{C4}, A_{C5}), \widetilde{H}_C = (H_{C1}, H_{C2}, H_{C3}, H_{C4}, H_{C5}), \widetilde{p}_C = (p_{C1}, p_{C2}, p_{C3}, p_{C4}, p_{C5})$ and $\widetilde{T} = (T_1, T_2, T_3, T_4, T_5)$. Accordingly, when the costs A_C, H_C, p_C and T are fuzzified, we obtain the total cost function in the fuzzy sense given by

$$E\tilde{T}C = \left[\left(\frac{A_{C1}}{T_s} + \frac{DT_1H_{C1}}{2} + Dp_{C1}\beta_dI_ct + \frac{Dp_{C1}\beta_dI_cT_1}{2}\right), \left(\frac{A_{C2}}{T_4} + \frac{DT_2H_{C2}}{2} + Dp_{C2}\beta_dI_ct + \frac{Dp_{C2}\beta_dI_cT_2}{2}\right), \left(\frac{A_{C3}}{T_3} + \frac{DT_3H_{C3}}{2} + Dp_{C3}\beta_dI_ct + \frac{Dp_{C3}\beta_dI_cT_3}{2}\right), \left(\frac{A_{C4}}{T_2} + \frac{DT_4H_{C4}}{2} + Dp_{C4}\beta_dI_ct + \frac{Dp_{C4}\beta_dI_cT_4}{2}\right) \left(\frac{A_{CS}}{T_1} + \frac{DT_sH_{CS}}{2} + Dp_{CS}\beta_dI_ct + \frac{Dp_{CS}\beta_dI_cT_s}{2}\right)\right] \tag{20}$$

Next, we defuzzify the fuzzy total cost by Graded mean integration representation method. Then we have

$$d(E\tilde{T}C) = \frac{1}{12}\left[\left(\frac{A_c}{T_3} + \frac{DT_1H_C}{2} + Dp_a\beta_dI_ct + \frac{Dp_C\beta_dI_cT_1}{2}\right) + 3\left(\frac{A_c}{T_4} + \frac{DT_2H_{C2}}{2} + Dp_{c2}\beta_dI_ct + \frac{Dp_{C2}\beta_dI_cT_2}{2}\right) + 4\left(\frac{A_C}{T_3} + \frac{DT_3H_{C3}}{2} + Dp_{C3}\beta_dI_ct + \frac{Dp_{C3}\beta_dI_cT_3}{2}\right) + 3\left(\frac{A_{c4}}{T_2} + \frac{DT_4H_{C4}}{2} + Dp_{C4}\beta_dI_ct + \frac{Dp_{C4}\beta_{di}I_cT_4}{2}\right) + \left(\frac{A_S}{T_1} + \frac{DT_sH_C}{2} + Dp_{CS}\beta_dI_ct + \frac{Dp_{CS}\beta_iI_cT_s}{2}\right)\right] \tag{21}$$

with $0 \le T_1 \le T_2 \le T_3 \le T_4 \le T_5$. It will not change the meaning of the Eq. (21), if we replace inequality conditions $0 \le T_1 \le T_2 \le T_3 \le T_4 \le T_5$ into the inequality constraints $T_5 - T_4 \ge 0$, $T_4 - T_3 \ge 0$, $T_3 - T_2 \ge 0$, $T_2 - T_1 \ge 0$ and $T_1 > 0$. The extension of the Lagrangian method is used to find the solutions of T_1, T_2, T_3, T_4 and T_5as in the case 1. we obtain

$T^* = T_1 = T_2 = T_3 = T_4 = T_5 = \sqrt{\frac{2(A_{C5}+3A_{C4}+4A_{C3}+3A_{C2}+A_{C1})}{D((H_{C1}+3H_{C2}+4H_{C3}+3H_{C4}+H_{C5})+(p_{C1}+3p_{C2}+4p_{C3}+3p_{C4}+p_{C5})\beta_dI_c)}}$.

2.8.4 Case 4: Hexagonal fuzzy number

We represent cost components, A_C, H_C, p_C and the replenishment cycle T by the hexagonal fuzzy number as given below:

$\widetilde{A}_C = (A_{C1}, A_{C2}, A_{C3}, A_{C4}, A_{C5}, A_{C6})$, $\widetilde{H}_C = (H_{C1}, H_{C2}, H_{C3}, H_{C4}, H_{C5}, H_{C6})$, $\widetilde{p}_C = (p_{C1}, p_{C2}, p_{C3}, p_{C4}, p_{C5}, p_{C6})$ and $\widetilde{T} = (T_1, T_2, T_3, T_4, T_5, T_6)$. Accordingly, when the costs A_C, H_C, p_C and T are fuzzified, we obtain the total cost function in the fuzzy sense given by

$$E\tilde{T}C = \left[\left(\frac{A_{C1}}{T_6} + \frac{DT_1H_{C1}}{2} + Dp_{C1}\beta_dI_ct + \frac{Dp_{C1}\beta_dI_cT_1}{2}\right), \left(\frac{A_{C2}}{T_s} + \frac{DT_2H_{C2}}{2} + Dp_{C2}\beta_dI_ct + \frac{Dp_{C2}\beta_dI_cT_2}{2}\right) \left(\frac{A_{C3}}{T_4} + \frac{DT_3H_{C3}}{2} + Dp_{C3}\beta_dI_ct + \frac{Dp_{C3}\beta_dI_cT_3}{2}\right), \left(\frac{A_{c4}}{T_3} + \frac{DT_4H_{C4}}{2} + Dp_{C4}\beta_dI_ct + \frac{Dp_{C4}\beta_dI_cT_4}{2}\right) \left(\frac{A_{C3}}{T_2} + \frac{DT_sH_{CS}}{2} + Dp_{CS}\beta_dI_ct + \frac{Dp_{CS}\beta_dI_cT_3}{2}\right), \left(\frac{A_{C6}}{T_1} + \frac{DT_6H_{C6}}{2} + Dp_{C6}\beta_dI_ct + \frac{Dp_{C6}\beta_dI_cT_6}{2}\right)\right] \tag{22}$$

Next, we defuzzify the fuzzy total cost by Graded mean integration representation method. Then we have

$$d(E\tilde{T}C) = \frac{1}{12}\left[\left(\frac{A_{C_1}}{T_6} + \frac{DT_1H_{C_1}}{2} + Dp_{C_1}\beta_dI_ct + \frac{Dp_{C_1}\beta_dI_cT_1}{2}\right) + 3\left(\frac{A_{C_2}}{T_5} + \frac{DT_2H_{C_2}}{2} + Dp_{C_2}\beta_dI_ct + \frac{Dp_{C_2}\beta_dI_cT_2}{2}\right) + 2\left(\frac{A_{C_3}}{T_4} + \frac{DT_3H_{C_3}}{2} + Dp_{C_3}\beta_dICt + \frac{Dp_{C_3}\beta_dI_cT_3}{2}\right) + 2\left(\frac{A_{C_4}}{T_3} + \frac{DT_4H_{C_4}}{2} + Dp_{C_4}\beta_dI_ct + \frac{Dp_{C_4}\beta_dI_cT_4}{2}\right) + 3\left(\frac{A_{C_5}}{T_2} + \frac{DT_5H_{C_5}}{2} + Dp_{C_5}\beta_dI_ct + \frac{Dp_{C_5}\beta_dI_cT_5}{2}\right) + \left(\frac{A_{C_6}}{T_1} + \frac{DT_6H_{C_6}}{2} + Dp_{C_6}\beta_dI_ct + \frac{Dp_{C_6}\beta_dI_cT_6}{2}\right)\right] \tag{23}$$

with $0 \le T_1 \le T_2 \le T_3 \le T_4 \le T_5 \le T_6$. It will not change the meaning of the Eq. (23), if we replace inequality conditions $0 \le T_1 \le T_2 \le T_3 \le T_4 \le T_5 \le T_6$ into the inequality constraints $T_6 - T_5 \ge 0, T_5 - T_4 \ge 0, T_4 - T_3 \ge 0, T_3 - T_2 \ge 0, T_2 - T_1 \ge 0$ and $T_1 > 0$. The extension of the Lagrangian method is used to find the solutions of T_1, T_2, T_3, T_4, T_5 and T_6 as in the case 1. Then we have

$$T^* = T_1 = T_2 = T_3 = T_4 = T_5 = T_6 = \sqrt{\frac{2(A_{C6}+3A_{C5}+2A_{C4}+2A_{C3}+3A_{C2}+A_{C1})}{D((H_{C1}+3H_{C2}+2H_{C3}+2H_{C4}+3H_{C5}+H_{C6})+(p_{C1}+3p_{C2}+2p_{C3}+2p_{C4}+3p_{C5}+p_{C6})\beta_d I_c)}}.$$

3 Results and Discussion

3.1 Numerical analysis

Numerical analysis is given to illustrate the above solution procedure for both crisp and fuzzy model. We use the same numerical data as in Priyan et al.[1] to verify the results obtained by this paper. A_C=30, D=400, t=0.1, H_C=20, p_C=25, β_d=0.9, and I_C =0.2. Based on these values the the optimal replenishment cycle T^* and the expected total cost ETC for the crisp model are summarized in Table 1 . We set some triangular, trapezoidal, pentagonal and Hexagonal fuzzy numbers of the input parameters (A_C, H_C, and p_C) in Tables 2, 3, 4 and 5, to represent the components of fuzzy models. For each of these parameters, the variations in the values are arranged arbitrary and their defuzzified values are determined by applying the Graded mean integration method. Based on these values the the optimal replenishment cycle T^* and the expected total cost $E\widetilde{TC}$ for the fuzzy model along with crisp model are summarized in Table 6. The corresponding curves of the minimum expected total cost against T^* are plotted in Figure 2 as well.

Table 1. The input parameter for the Crisp model

A_C	D	t	H_C	p_C	β_d	I_c
52.5	700	0.1	35	43.75	0.9	0.2
45	600	0.1	30	37.5	0.9	0.2
37.5	500	0.1	25	31.25	0.9	0.2
30	400	0.1	20	25	0.9	0.2
22.5	300	0.1	15	18.75	0.9	0.2
15	200	0.1	10	12.5	0.9	0.2
7.5	100	0.1	5	6.25	0.9	0.2

Table 2. Fuzzy triangular values for the input parameter A_C, H_C and p_C

$\widetilde{A}_C$	$d_F\widetilde{A}_C$	$\widetilde{H}_C$	$d_F\widetilde{H}_C$	$\widetilde{p}_C$	$d_F\widetilde{p}_C$
(47.5, 52.5, 57.5)	52.5	(30,35, 40)	35	(38.75, 43.75, 48.75)	43.75
(40, 45, 50)	45	(25, 30, 35)	30	(32.5, 37.5, 42.5)	37.5
(32.5, 37.5, 42.5)	37.5	(20, 25, 30)	25	(26.25, 31.25, 36.25)	31.25
(25, 30, 35)	30	(15, 20, 25)	20	(20, 25, 30)	25
(17.5, 22.5, 27.5)	22.5	(10, 15, 20)	15	(13.75, 18.75, 23.75)	18.75
(10, 15, 20)	15	(5, 10, 15)	10	(7.5, 12.5, 17.5)	12.5
(2.5, 7.5, 12.5)	7.5	(1, 5, 9)	5	(1.25, 6.25, 11.25)	6.25

Table 3. Fuzzy trapezoidal values for the input parameter A_C, H_C and p_C

$\widetilde{A}_C$	$d_F\widetilde{A}_C$	$\widetilde{H}_C$	$d_F\widetilde{H}_C$	$\widetilde{p}_C$	$d_F\widetilde{p}_C$
(50.5, 52.5, 52.5, 54.5)	52.5	(30, 35, 35, 40)	35	(41.75, 43.75, 43.75, 45.75)	43.75
(40, 45, 45, 50)	45	(25, 30, 30,35)	30	(35.5, 37.5, 37.5, 39.5)	37.5
(35.5, 37.5, 37.5, 39.5)	37.5	(20, 25, 25, 30)	25	(30.25, 31.25, 31.25, 32.25)	31.25
(25, 30, 30, 35)	30	(15, 20, 20, 25)	20	(20, 25, 25, 30)	25
(20.5, 22.5, 22.5, 24.5)	22.5	(10, 15, 15, 20)	15	(15.75, 18.75, 18.75, 21.75)	18.75
(10, 15, 15, 20)	15	(5, 10, 10, 15)	10	(10.5, 12.5, 12.5, 14.5)	12.5
(5.5, 7.5, 7.5, 9.5)	7.5	(2, 5, 5, 8)	5	(4.25, 6.25, 6.25, 8.25)	6.25

Table 4. Fuzzy pentagonal values for the input parameter A_C, H_C and p_C

$\widetilde{A}_C$	$d_F\widetilde{A}_C$	$\widetilde{H}_C$	$d_F\widetilde{H}_C$	$\widetilde{p}_C$	$d_F\widetilde{p}_C$
(42.5, 47.5, 52.5, 57.5, 62.5)	52.5	(25, 30, 35, 40, 45)	35	(33.75, 38.75, 43.75, 48.75, 53.75)	43.75
(35, 40, 45, 50, 55)	45	(20, 25, 30, 35, 40)	30	(27.5, 32.5, 37.5, 42.5, 47.5)	37.5
(27.5, 32.5, 37,.5, 42.5, 47.5)	37.5	(15, 20, 25, 30, 35)	25	(21.25, 26.25, 31.25, 36.25, 41.25)	31.25
(20, 25, 30, 35, 40)	30	(10, 15, 20,25, 30)	20	(15, 20, 25, 30, 35)	25
(12.5, 17.5, 22.5, 27.5, 32.5)	22.5	(5, 10, 15, 20, 25)	15	(8.75, 13.75, 18.75, 23.75, 28.75)	18.75
(5, 10, 15, 20, 25)	15	(4, 7, 10, 13, 16)	10	(2.5, 7.5, 12.5, 17.5, 22.5)	12.5
(3.5, 5.5, 7.5, 9.5, 11.5)	7.5	(1, 3, 5, 7, 9)	5	(2.25, 4.25, 6.25, 8.25, 10.25)	6.25

Table 5. Fuzzy Hexagonal values for the input parameter A_C, H_C and p_C

$\widetilde{A}_C$	$d_F\widetilde{A}_C$	$\widetilde{H}_C$	$d_F\widetilde{H}_C$	$\widetilde{p}_C$	$d_F\widetilde{p}_C$
(27.5, 42.5, 52.5, 57.5, 62.5, 67.5)	52.5	(10, 25, 35, 40, 45, 50)	35	(18.75, 33.75, 43.75, 48.75, 53.75, 58.75)	43.75
(20, 35, 45, 50, 55, 60)	45	(5, 20, 30, 35, 40, 45)	30	(12.5, 27.5, 37.5, 42.5, 47.5, 52.5)	37.5
(12.5, 27.5, 37.5, 42.5, 47.5, 52.5)	37.5	(2, 15, 25, 30, 35, 38)	25	(6.25, 21.25, 31.25, 36.25, 41.25, 46.25)	31.25
(5, 20, 30, 35, 40, 45)	30	(6, 10, 20, 24, 28, 32)	20	(6, 13, 25, 30, 35, 40)	25
(2.5, 12.5, 22.5, 27.5, 31.5, 35.5)	22.5	(3, 8, 15, 18, 21, 24)	15	(3.75, 10.75, 18.75, 21.75, 25.75, 30.75)	18.75
(2, 5, 15, 18, 23, 28)	15	(1, 7, 10, 12, 13, 15)	10	(2.5, 4.5, 12.5, 16.5, 18.5, 20.5)	12.5
(1.5, 3.5, 7.5, 9.5, 10.5, 12.5)	7.5	(2, 2, 5, 6, 7, 9)	5	(1.25, 4.25, 6.25, 7.25, 8.25, 9.25)	6.25

Table 6. Comparison of crisp and fuzzy model

Crisp		**Triangular**		**Trapezoidal**		**Pentagonal**		**Hexagonal**	
T^*	ETC	T^*	$E\widetilde{T}C$	T^*	$E\widetilde{T}C$	T^*	$E\widetilde{T}C$	T^*	$E\widetilde{T}C$
0.0591	2326.44	0.0591	2322.42	0.0591	2053.58	0.0591	2312.26	0.0591	2260.55
0.0639	1813.72	0.0639	1809.37	0.0639	1595.81	0.0639	1798.34	0.0639	1738.69
0.0700	1352.90	0.0700	1348.12	0.0700	1188.83	0.0700	1335.93	0.0700	1270.05
0.0782	946.81	0.0782	941.44	0.0782	826.61	0.0782	927.59	0.0782	875.41
0.0904	599.31	0.0904	593.05	0.0904	522.49	0.0904	576.39	0.0904	542.48
0.1107	316.11	0.1107	308.21	0.1107	270.04	0.1107	297.79	0.1107	277.45
0.1565	107.10	0.1565	96.75	0.1565	91.36	0.1565	100.50	0.1565	95.92

Table 7. Comparison study on trapezoidal fuzzy parameters

Trapezoidal input parameters			**Priyan et al. [1] model**		**Our model**		
$d_F\tilde{A}_C$	$d_F\widetilde{H}_C$	$d_F\tilde{p}_C$	T^*	$E\widetilde{T}C$	T^*	$E\widetilde{T}C$	% savings
18	12	16	0.0778	578.1	0.0786	347.55	40
24	16	21	0.0779	767.5	0.0779	517.21	33
36	24	29	0.0785	1126.2	0.0797	842.22	25
42	28	34	0.0785	1315.5	0.0781	939.73	29
60	40	45	0.0790	1843.5	0.0788	1329.40	28

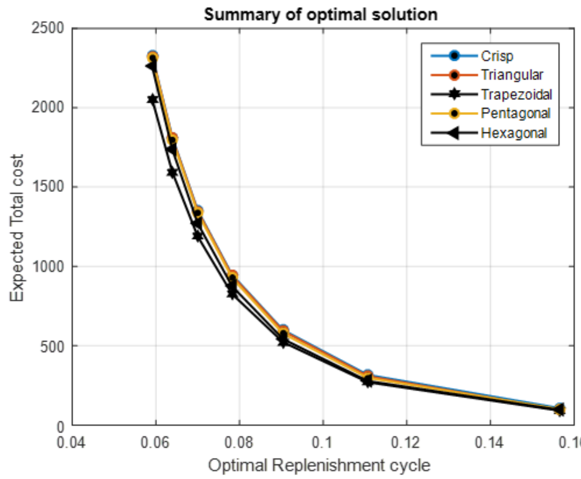

Fig 2. Summary of optimal solution

3.2 Comparison

A comparative study of the proposed model with that of optimal replenishment cycle T^* and the expected total cost $E\widetilde{TC}$ for the fuzzy model are shown in Table 6. Our results indicate that the optimal solutions of the fuzzy model slightly fluctuate from the solutions of the crisp model (see Table 6). Hence, the research reveals that in all the models, the decision variables and the expected total cost are sensitive to the level of fuzziness in the cost components. From the Table 6 we observe that out of all the fuzzy numbers, trapezoidal fuzzy number (case 2) gives the optimum solution.

The procedures followed in the fuzzy inventory model (case 2) for trapezoidal fuzzy numbers by taking the same data as in Priyan et al.[(1)] model for the input parameters is considered. Also, compared with the previous model and the results are tabulated in Table 7. In Priyan et al.[(1)] model, they found the optimal replenishment cycle by classical differential calculus optimization technique and total cost by using signed distance method for defuzzification. In the proposed fuzzy model (by case 2), we use Graded mean integration method to defuzzify the fuzzy total cost and obtain an estimate of the total cost in the fuzzy sense. Extension of the Lagrangian method is used to find the optimal replenishment cycle T of the model. From this an efficient result for the proposed fuzzy model is attained.

The results in the numerical examples indicate that savings of the total cost are realized through trapezoidal fuzzy number. Furthermore, we optimize replenishment cycle T of the model by adopting Extension of the Lagrangian method and Graded mean integration method to defuzzify the fuzzy total cost and obtain an estimate of the minimum total cost in the fuzzy sense. It is observed that uncertain cost parameters give 25% to 40% of the savings in the total cost respectively. One of the repercussions of this convergence is that if the fuzziness in the cost components is trapezoidal fuzzy number, then the total relevant cost could be automatically improved. Among the four cases, trapezoidal fuzzy number is more matched to real life supply chains that could be seen by made a comparison with the previous model. By computing the proposed models, specifically, from the results of numerical examples, we observe that a significant amount of savings can be easily achieved. Also, it shows that Graded mean integration method to defuzzify the fuzzy total cost when there is an option of improving the system, it is advisable to apply the method for defuzzification.

4 Conclusion

In China's steel industry, large steel factories request advance payments, specifically from small clients. In real world applications, the input cost and other parameters in the EOQ inventory problem may not be known precisely or it may be uncertain due to some uncontrollable factors. Hence, approximate solution methodologies have been illustrated for the solution of a class of realistic inventory problems. The selection decisions are complex, as decision making is more challenging nowadays. If the uncertainty is insignificant, it may be possible to use some classical inventory formula. Fuzzy methodologies provide a useful way to model vagueness in human recognition and judgment. Moreover, fuzzy numbers are largely applied on data analysis, artificial intelligence, and decision making.

This work approaches the Graded mean integration method for defuzzification with various fuzzy numbers. To the best of our knowledge, none of the previous researchers considered inventory model with advance payment and various fuzzy numbers. An overarching theme of the proposed model is to enable the estimation of the minimum total cost in the fuzzy sense.

The purpose of this model is to provide an EOQ inventory model with advance payment to achieve minimum total cost under fuzzy environment. The fuzziness in the cost components are represented by the triangular, trapezoidal, pentagonal and Hexagonal fuzzy numbers. we use Graded mean integration method to defuzzify the fuzzy total cost and obtain an estimate of the total cost in the fuzzy sense. Extension of the Lagrangian method is used to find the optimal replenishment cycle of the model. Numerical example is provided to ascertain the sensitiveness of fuzziness in the components. The analytical solution of the fuzzy model that has been defuzzified with the graded mean integration method shows a proof of savings 25% to 40% in the analytical solution compared to the solution of the Priyan et al.[1] model.

There are several extensions of this work that could constitute future research related to this field. One immediate probable extension could be to discuss the effect of shortage. Another possible extension of this work may be conducted by considering the supplier's provision of a permissible delay in payments in this integrated inventory model.

References

1) Priyan S, Palanivel M, Uthayakumar R. Mathematical modelling for EOQ inventory system with advance payment and fuzzy parameters. *International Journal of Supply and Operations Management*. 2014;1(3):260–278. Available from: https://dx.doi.org/10.22034/2014.3.01.
2) Rajeswari C, Sugapriya D, Nagarajan D. An analysis of uncertain situation and advance payment system on a double-storage fuzzy inventory model. *OPSEARCH*. 2022;59:20–40. Available from: https://doi.org/10.1016/j.matpr.2020.10.769.
3) Pattnaik S, Nayak MM, Acharya M. Linearly Deteriorating EOQ Model for Imperfect Items with Price Dependent Demand under Different Fuzzy Environments. *Turkish Journal of Computer and Mathematics Education*. 2021;12(13):5328–5349. Available from: https://turcomat.org/index.php/turkbilmat/article/download/9726/7430/17307.
4) Kuppulakshmi V, Sugapriya C, Kavikumar J, Nagarajan D. Fuzzy Inventory Model for Imperfect Items with Price Discount and Penalty Maintenance Cost. *Mathematical Problems in Engineering*. 2023;p. 1–15. Available from: https://doi.org/10.1155/2023/1246257.
5) Chen SH, Hsieh CH. Graded mean integration representation of generalized fuzzy number. *Journal of Chinese Fuzzy Systems*. 1999;5(2):1–7. Available from: https://www.scinapse.io/papers/2286090524.
6) Rahaman M, Mondal SP, Alam S, De SK, Ahmadian A. Study of a Fuzzy Production Inventory Model with Deterioration Under Marxian Principle. *International Journal of Fuzzy Systems*. 2022;24(4):2092–2106. Available from: https://doi.org/10.1007/s40815-021-01245-0.
7) Alsaedi BSO, Alamri OA, Jayaswal MK, Mittal M. A Sustainable Green Supply Chain Model with Carbon Emissions for Defective Items under Learning in a Fuzzy Environment. *Mathematics*. 2023;11(2):301. Available from: https://doi.org/10.3390/math11020301.
8) Vasanthi P, Ranganayaki S, Kasthuri R. Fuzzy inventory model without shortages using GMI approach. *Journal of Physics: Conference Series*. 2022;(1). Available from: https://doi.org/10.1088/1742-6596/2332/1/012002.
9) Hemalatha S, Annadurai K. Optimization of a fuzzy inventory model with pentagonal fuzzy numbers. *International Journal of Mathematics and Computer Research*. 2023;11(03):3277–3287. Available from: https://doi.org/10.47191/ijmcr/v11i3.01.
10) Hemalatha S, Annadurai K. Optimal Policy for an inventory model using pentagonal fuzzy numbers. *International Journal of Mathematics Trends and Technology*. 2023;69(3):7–15. Available from: https://doi.org/10.14445/22315373/IJMTT-V69I3P502.
11) Taha HA. Operations research. Eaglewood Cliffs, NJ, USA. Prentice-Hall. 1997. Available from: http://zalamsyah.staff.unja.ac.id/wp-content/uploads/sites/286/2019/11/9-Operations-Research-An-Introduction-10th-Ed.-Hamdy-A-Taha.pdf.
12) Chen SH. Operations on fuzzy numbers with function principle. *Tamkang Journal of Management Sciences*. 1985;6:13–26. Available from: https://doi.org/10.1016/S0020-0255(97)10070-6.