INDIAN JOURNAL OF SCIENCE AND TECHNOLOGY

RESEARCH ARTICLE

OPEN ACCESS
Received: 26-05-2023
Accepted: 01-06-2023
Published: 18-07-2023

Citation: Chinthamani S, Lokesh P (2023) The Extension of Chebyshev Polynomial Bounds Involving Bazilevic Function. Indian Journal of Science and Technology 16(27): 2040-2046. https://doi.org/10.17485/IJST/v16i27.icrms-207

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Funding: None

Competing Interests: None

Published By Indian Society for Education and Environment (iSee)

ISSN
Print: 0974-6846
Electronic: 0974-5645

The Extension of Chebyshev Polynomial Bounds Involving Bazilevic Function

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Abstract

Objectives: To propose a new class of bi-univalent function based on Bazilevic Sakaguchi function using the trigonometric polynomials $T_n\left(q,e^{i\theta}\right)$and to find the Taylor – Maclaurin coefficient inequalities and Fekete – Szego inequality for upper bounds. **Methods:** The Chebychev's polynomial has vast applications in GFT. The powerful tool called convolution (Or Hadamard product), subordination techniques are used in designing the new class. In establishing the core results, derivative tests, triangle inequality and appropriate results that are existing are used. **Findings:**The trigonometric polynomials are applied and a class of Bi-univalent functions $P_{\Sigma}^{a,b,c}(\lambda,\tau,q,\theta)$ involving Bazilevic Sakaguchi function is derived. More over, the maximum bounds for initial coefficients and Fekete-Szego functional for the underlying class are computed. This finding opens the door to young researchers to move further with successive coefficient estimates and related research. **Novelty:**In recent days, several studies on Chebyshev's polynomial are revolving around univalent function classes among researchers. But in this article a significant amount of interplay between Chebyshev's polynomial and Bazilevic Sakaguchi function associated with Bi-univalent functions is clearly established.

Keywords: Bistarlike functions; Bi-Starlike Functions; Bi-Univalent Functions; Sakaguchi Type Functions; Subordination; Trigonometric Polynomials

1 Introduction

Let A represent the family of functions f that are analytic in the open unit disk $\Delta = (z \in C : (z| < 1\}$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} \rho_k z^k \tag{1}$$

For $h(z) \in A$, given by

$$h(z) = z + \sum_{k=2}^{\infty} h_k z^k$$

Let S mean the subclass of A consisting of univalent functions inΔ. It is well known (refer[1,2]) that every function of $f \in S$ virtually possesses an inverse of f, defined by $f^{-1}[f(z)] = z, (z \in \Delta)$ and $f[f^{-1}(w)] = w, ((w| < r_0(f); r_0(f) \geq \frac{1}{4})$, where

$$f^{-1}(w) = w - \rho_2 w^2 + (2\rho_2^2 - \rho_3)w^3 - (5\rho_2^3 - 5\rho_2\rho_3 + \rho_4)w^4 + \ldots\ldots \tag{2}$$

When the function $f \in A$ is bi-univalent, both $f$and $f^{-1}$are univalent inΔ. Let $\sum$ be the class of bi-univalent functions in Δ given by (1). In fact, Feras Yousef et al.[2] have revived the study of analytic and bi-univalent functions in recent years. Many researchers investigated and propounded various subclasses of bi-univalent functions and fixed the initial coefficients $(\rho_2|$and $(\rho_3|$[3–6]

For analytic functions f and g, f is said to be subordinate to g, denoted$f(z) \prec g(z)$, if there is an analytic function w such that $w(0) = 0, (w(z)| < 1$ and $f(z) = g(w(z))$.

A function $f \in S$ is said to be Bazilevic function if it satisfies (see[7]):

$\Re\left(\frac{z^{1-\lambda} f'(z)}{(f(z))^{1-\lambda}}\right) > 0, \quad (z \in \Delta, \lambda \geq 0)$

This class of the function was denoted by B_λ. Consequently when$\lambda = 0$, the class of starlike function is obtained.

In recently, P.Lokesh et al[8] investigated the inequalities of coefficient for certain classes of Sakaguchi type functions that satisfy geometrical condition as

$$R\left(\frac{(s-t)z(f'(z)}{f(sz) - f(tz)}\right) > \alpha \tag{3}$$

for complex numbers s, t but s $\neq$ t and α $(0 \leq \alpha < 1)$.

The convolution or Hadamard product of two functions $f, g \in A$is defined by $f * g$and is defined by

$(f * g)(z) = z + \sum_{n=2}^{\infty} \rho_n \delta_n z^n$.

where$f$is given by (1) and $g(z) = z + \sum_{n=2}^{\infty} \delta_n z^n$.

Let $R = (-\infty, \infty)$be the set of real numbers. C be the complex numbers and

$$N := \{1, 2, 3, \ldots\} = N_0 / (0\}$$

be the set of positive integers. Let $\Delta = (z \in C : (z| < 1\}$ be open unit disc in C. A well known, the trigonometric polynomials $T_n\left(q, e^{i\theta}\right)$ are expressed by the generating function

$$\xi_q\left(e^{i\theta}, z\right) = \frac{1}{(1 - ze^{i\theta})(1 - qze^{-i\theta})}$$

$= \sum_{n=0}^{\infty} T_n\left(q, e^{i\theta}\right) z^n, (q \in (-1, 1), \theta \in (-\pi, \pi], z \in \Delta)$.

where

$$T_n\left(q, e^{i\theta}\right) = \frac{e^{i(n+1)\theta} - q^{n+1} e^{-i(n+1)\theta}}{e^{i\theta} - qe^{-i\theta}} \, (n \geq 2)$$

with

$$T_0\left(q, e^{i\theta}\right) = 1, T_1\left(q, e^{i\theta}\right) = e^{i\theta} + qe^{-i\theta}, T_2\left(q, e^{i\theta}\right) = e^{2i\theta} + q^2 e^{-2i\theta} + q \ldots$$

The obtained results for $q = 1$give the corresponding ones for Chebyshev polynomials of the second kind. The classical Chebyshev polynomials which are used in this paper, have been in the late eighteenth century, when was defined using de Moivre's formula by Chebyshev(refer[9]). Such polynomials as (for example) the Fibonacci polynomials, the Lucas polynomials, the Pell polynomials and the families of orthogonal polynomials and other special polynomials as well as their generalizations are potentially important in the fields of probability, statistics, mechanics and number theory[10–14]

2 Methodology

In the present work, the convolution operator $I_{a,b,c}$due to Hohlov (refer[15,16], which is special case of the Rajavadivelu Themangani et al (refer[17]) is recalled.

For the complex parameters a, b & c $(c \neq 0, -1, -2, \ldots)$ the Gaussian hyper geometric function ${}_2F_1(a, b, c : z)$ is defined as

$${}_2F_1(a, b, c : z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = 1 + \sum_{n=2}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1}} \frac{z^{n-1}}{(n-1)!} \, (z \in \Delta).$$

where $(a)_n$is the Pochhammer symbol (or the shifted factorial) given by

$$(a)_n := \frac{\Gamma(a+k)}{\Gamma(a)} = \left(\begin{array}{c} 1, n=0 \\ a(a+1)(a+2)\ldots(a+n-1), n \in N := \{1,2,\ldots\} \end{array} \right.$$

Now, let us consider a linear operator introduced by Isra Al-shbeil et al (refer[18]) and

$$I_{a,b,c} : A \to A.$$

defined by the Hadamard product$I_{a,b,c}f(z) = (z_2F_1(a,b,c:z)) * f(z)$.

It is observed that, for a function $f$of the form (1),

$$I_{a,b,c}f(z) = z + \sum_{n=2}^{\infty} \Psi_n \rho_n z^n, (z \in \Delta).$$

where $\Psi_n = \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}$.

In this paper, a new class of bi-univalent function based on Bazilevic Sakaguchi function using the trigonometric polynomials $T_n\left(q, e^{i\theta}\right)$ is established. Furthermore, the coefficient bounds and Fekete – Szego inequalities are also derived for this class.

Definition 1 : For $0 \le \lambda < 1, (\tau| \le 1, \tau \neq 1, q \in (-1,1), \theta \in (-\pi, \pi]$, a function $f \in \Sigma$ given by (1) is said to be in the class $P_{\Sigma}^{a,b,c}(\lambda, \tau, q, \theta)$if it satisfies the following conditions,

$$\frac{((1-\tau)z)^{1-\lambda}\left(I_{a,b,c}f(z)\right)'}{\left(I_{a,b,c}f(z) - I_{a,b,c}f(\tau z)\right)^{1-\lambda}} \prec \varphi_q\left(e^{i\theta}, z\right), (z \in \Delta) \tag{4}$$

$$\frac{((1-\tau)z)^{1-\lambda}\left(I_{a,b,c}g(w)\right)'}{\left(I_{a,b,c}g(w) - I_{a,b,c}g(\tau w)\right)^{1-\lambda}} \prec \varphi_q\left(e^{i\theta}, w\right), (w \in \Delta) \tag{5}$$

where the function $g = f^{-1}$.

By taking the parameters $\lambda = 0$ and $\tau = 0$, which was introduced by Sahsene Altinkaya et al (refer[19]).

3 Result and Discussion

In this section, we obtain the extension of Chebyshev polynomial bounds$(\rho_2|$ and $(\rho_3|$for the function of the class $P_{\Sigma}^{a,b,c}(\lambda, \tau, q, \theta)$.

Theorem 1 Let the function $f$given by (1) be in the class $P_{\Sigma}^{a,b,c}(\lambda, \tau, q, \theta)$.

Then

$$|\rho_2| \le \frac{\left|e^{i\theta} + qe^{-i\theta}\right| \sqrt{2\left|e^{i\theta} + qe^{-i\theta}\right|}}{\sqrt{2\left(\begin{array}{c} \left(e^{2i\theta} + q^2e^{-2i\theta} + 2q\right)\left(3-(1-\lambda)\left(1+\tau+\tau^2\right)\right)\Psi_3 \\ -\left(\begin{array}{c} \left(e^{2i\theta} + q^2e^{-2i\theta} + 2q\right)(1-\lambda)(1+\tau)(2(1-\tau)+\lambda(1+\tau)) \\ +2\left(e^{2i\theta} + q^2e^{-2i\theta} + q\right)(2-(1-\lambda)(1+\tau))^2 \end{array} \right)\Psi_2^2 \end{array} \right|}} \tag{6}$$

and

$$(\rho_3| \le \frac{\left(e^{i\theta} + qe^{-i\theta}\right|}{\left(3-(1-\lambda)\left(1+\tau+\tau^2\right)\right)\Psi_3} + \frac{\left(e^{2i\theta} + q^2e^{-2i\theta} + 2q\right|}{(2-(1-\lambda)(1+\tau))^2\Psi_2^2}, \tag{7}$$

Proof. Since $f \in P_{\Sigma}^{a,b,c}(\lambda, \tau, q, \theta)$, there is two analytic functions ϕ, χsuch that

$$\phi(0) = 0, (\phi(z)| = \left(r_1z + r_2z^2 + r_3z^3 + \ldots\right| < 1, (z \in \Delta)$$

$$\chi(0) = 0, (\chi(w)| = \left(s_1w + s_2w^2 + s_3w^3 + \ldots\right| < 1, (w \in \Delta)$$

We can express as

$$\frac{\left(\left(1-\tau\right)z\right)^{1-\lambda}\left(I_{a,b,c}f\left(z\right)\right)'}{\left(I_{a,b,c}f\left(z\right)-I_{a,b,c}f\left(\tau z\right)\right)^{1-\lambda}}=1+T_1\left(q,e^{i\theta}\right)\phi\left(z\right)+T_2\left(q,e^{i\theta}\right)\phi^2\left(z\right)+\ldots$$

and

$$\frac{\left(\left(1-\tau\right)w\right)^{1-\lambda}\left(I_{a,b,c}g\left(w\right)\right)'}{\left(I_{a,b,c}g\left(w\right)-I_{a,b,c}g\left(\tau w\right)\right)^{1-\lambda}}=1+T_1\left(q,e^{i\theta}\right)\chi\left(w\right)+T_2\left(q,e^{i\theta}\right)\chi^2\left(w\right)+\ldots$$

or, equivalently,

$$\frac{((1-\tau)z)^{1-\lambda}\left(\Im_{a,b,c}f(z)\right)'}{\left(\Im_{a,b,c}f(z)-\Im_{a,b,c}f(\tau z)\right)^{1-\lambda}}=1+T_1\left(q,e^{i\theta}\right)r_1z+\left(T_1\left(q,e^{i\theta}\right)r_2+T_2\left(q,e^{i\theta}\right)r_1^2\right)z^2+\ldots \tag{8}$$

and

$$\frac{\left(\left(1-\tau\right)w\right)^{1-\lambda}\left(I_{a,b,c}g\left(w\right)\right)'}{\left(I_{a,b,c}g\left(w\right)-I_{a,b,c}g\left(\tau w\right)\right)^{1-\lambda}}=1+T_1\left(q,e^{i\theta}\right)s_1w+\left(T_1\left(q,e^{i\theta}\right)s_2+T_2\left(q,e^{i\theta}\right)s_1^2\right)w^2+\ldots \tag{9}$$

It is well known that

$$|r_i|\leq 1 \text{ and } |s_i|\leq 1, (\forall i\ \in N) \tag{10}$$

From the equations (8) and (9), we obtain

$$\left(2-\left(1-\lambda\right)\left(1+\tau\right)\right)\Psi_2\rho_2=T_1\left(q,e^{i\theta}\right)r_1 \tag{11}$$

$$\begin{aligned}\left(3-\left(1-\lambda\right)\left(1+\tau+\tau^2\right)\right)\Psi_3\rho_3-\frac{(1-\lambda)(1+\tau)}{2}(2(1-\tau)+\lambda(1+\tau))\Psi_2^2\rho_2^2\\=T_1\left(q,e^{i\theta}\right)r_2+T_2\left(q,e^{i\theta}\right)r_1^2\end{aligned} \tag{12}$$

$$-\left(2-\left(1-\lambda\right)\left(1+\tau\right)\right)\Psi_2\rho_2=T_1\left(q,e^{i\theta}\right)s_1 \tag{13}$$

$$\begin{aligned}&\left(\begin{array}{l}2\left(3-\left(1-\lambda\right)\left(1+\tau+\tau^2\right)\right)\Psi_3\\-\frac{(1-\lambda)(1+\tau)}{2}(2(1-\tau)+\lambda(1+\tau))\Psi_2^2\end{array}\right)\rho_2^2-\left(3-\left(1-\lambda\right)\left(1+\tau+\tau^2\right)\right)\Psi_3\rho_3\\&=T_1\left(q,e^{i\theta}\right)s_2+T_2\left(q,e^{i\theta}\right)s_1^2\end{aligned} \tag{14}$$

From the equations (11) and (13), we easily find

$$r_1=-s_1 \tag{15}$$

$$2(2-\left(1-\lambda\right)\left(1+\tau\right))^2\Psi_2^2\rho_2^2=T_1^2\left(q,e^{i\theta}\right)\left(r_1^2+s_1^2\right), \tag{16}$$

Summing the equations (12) and (14), we get

$$\begin{aligned}\left(2\left(3-\left(1-\lambda\right)\left(1+\tau+\tau^2\right)\right)\Psi_3-(1-\lambda)(1+\tau)(2(1-\tau)+\lambda(1+\tau))\Psi_2^2\right)\rho_2^2\\=T_1\left(q,e^{i\theta}\right)\left(r_2+s_2\right)+T_2\left(q,e^{i\theta}\right)\left(r_1^2+s_1^2\right)\end{aligned} \tag{17}$$

By substituting the values of $\left(r_1^2+s_1^2\right)$ from (16) in the right side of (17), we get

$$\left(\begin{array}{c} 2\left(3-(1-\lambda)\left(1+\tau+\tau^2\right)\right)\Psi_3-(1-\lambda)(1+\tau)(2(1-\tau)+\lambda(1+\tau))\Psi_2^2 \\ -\frac{2(2-(1-\lambda)(1+\tau))^2\Psi_2^2 T_2\left(q,e^{i\theta}\right)}{T_1^2\left(q,e^{i\theta}\right)} \\ =T_1\left(q,e^{i\theta}\right)(r_2+s_2), \end{array}\right)\rho_2^2$$

which yields

$$\rho_2^2=\frac{T_1^3\left(q,e^{i\theta}\right)(r_2+s_2)}{\left[\begin{array}{l} T_1^2\left(q,e^{i\theta}\right)\left(\begin{array}{c} 2\left(3-(1-\lambda)\left(1+\tau+\tau^2\right)\right)\Psi_3 \\ -(1-\lambda)(1+\tau)(2(1-\tau)+\lambda(1+\tau))\Psi_2^2 \end{array}\right) \\ -2(2-(1-\lambda)(1+\tau))^2\Psi_2^2 T_2\left(q,e^{i\theta}\right) \end{array}\right]} \tag{18}$$

By subtracting the equation (14) from equation (12), we find

$$\begin{aligned} &2\left(3-(1-\lambda)\left(1+\tau+\tau^2\right)\right)\Psi_3\rho_3-2\left(3-(1-\lambda)\left(1+\tau+\tau^2\right)\right)\Psi_3\rho_2^2 \\ &=T_1\left(q,e^{i\theta}\right)(r_2-s_2) \end{aligned} \tag{19}$$

In view of equation (16), the equation (19) becomes

$$\rho_3=\frac{T_1\left(q,e^{i\theta}\right)(r_2-s_2)}{2\left(3-(1-\lambda)\left(1+\tau+\tau^2\right)\right)\Psi_3}+\frac{T_1^2\left(q,e^{i\theta}\right)\left(r_1^2+s_1^2\right)}{2(2-(1-\lambda)(1+\tau))^2\Psi_2^2}$$

By applying equation (10), we can easily obtain the desired inequalities in Theorem 1.

Taking the parameters $\lambda=0$and $\tau=0$in the Theorem 1, we get the following remark.

Remark 1 If $f\in P_\Sigma^{a,b,c}(q,\theta)$, then

$$(\rho_2|\leq\frac{\left(e^{i\theta}+qe^{-i\theta}\right|\sqrt{\left(e^{i\theta}+qe^{-i\theta}\right|}}{\sqrt{\left(2\left(e^{2i\theta}+q^2e^{-2i\theta}+2q\right)\Psi_3-\left(2e^{2i\theta}+2q^2e^{-2i\theta}+3q\right)\Psi_2^2\right|}},$$

and

$$(\rho_3|\leq\frac{\left(e^{i\theta}+qe^{-i\theta}\right|}{2\Psi_3}+\frac{\left(e^{2i\theta}+q^2e^{-2i\theta}+2q\right|}{\Psi_2^2}.$$

which was investigated by Sahsene Altinkaya et al[19].

Fekete – Szego inequality for the function class$P_\Sigma^{a,b,c}(\lambda,\tau,q,\theta)$

In this section, we provide Fekete – Szego inequalities for function in the class $P_\Sigma^{a,b,c}(\lambda,\tau,q,\theta)$. This inequality is given in the following theorem.

Theorem 2 For $\mu\in R$the function $f\in P_\Sigma^{a,b,c}(\lambda,\tau,\mu,q,\theta)$.

$$\left|\rho_3-\mu\rho_2^2\right|\leq\left\{\begin{array}{l} \frac{\left|e^{i\theta}+qe^{-i\theta}\right|}{\left(3-(1-\lambda)\left(1+\tau+\tau^2\right)\right)\Psi_3},|\mu-1|\leq\gamma \\ \frac{2|1-\mu|\left|e^{2i\theta}+q^2e^{-2i\theta}+2q\right|\left|e^{i\theta}+qe^{-i\theta}\right|}{2\left(e^{2i\theta}+q^2e^{-2i\theta}+2q\right)\left(3-(1-\lambda)\left(1+\tau+\tau^2\right)\right)\Psi_3} \qquad |,|\mu-1|\geq\gamma \\ -\left(\begin{array}{c} \left(e^{2i\theta}+q^2e^{-2i\theta}+2q\right)(1-\lambda)(1+\tau)(2(1-\tau)+\lambda(1+\tau)) \\ +2\left(e^{2i\theta}+q^2e^{-2i\theta}+q\right)(2-(1-\lambda)(1+\tau))^2 \end{array}\right)\Psi_2^2 \end{array}\right.$$

$$where\gamma=\frac{\left\{\begin{array}{c} 2\left(e^{2i\theta}+q^2e^{-2i\theta}+2q\right)\left(3-(1-\lambda)\left(1+\tau+\tau^2\right)\right)\Psi_3 \\ -\left(\begin{array}{c} \left(e^{2i\theta}+q^2e^{-2i\theta}+2q\right)(1-\lambda)(1+\tau)(2(1-\tau)+\lambda(1+\tau)) \\ +2\left(e^{2i\theta}+q^2e^{-2i\theta}+q\right)(2-(1-\lambda)(1+\tau))^2 \end{array}\right)\Psi_2^2 \end{array}\right\}}{2\left(e^{2i\theta}+q^2e^{-2i\theta}+2q\right)\left(3-(1-\lambda)\left(1+\tau+\tau^2\right)\right)\Psi_3}.$$

Proof. From the equation (18) and the equation (19), we observe that

$$\rho_3-\mu\rho_2^2=\frac{T_1\left(q,e^{i\theta}\right)(r_2-s_2)}{2\left(3-(1-\lambda)\left(1+\tau+\tau^2\right)\right)\Psi_3}+\frac{(1-\mu)T_1^3\left(q,e^{i\theta}\right)(r_2+s_2)}{\left[\begin{array}{l} T_1^2\left(q,e^{i\theta}\right)\left(\begin{array}{c} 2\left(3-(1-\lambda)\left(1+\tau+\tau^2\right)\right)\Psi_3 \\ -(1-\lambda)(1+\tau)(2(1-\tau)+\lambda(1+\tau))\Psi_2^2 \end{array}\right) \\ -2(2-(1-\lambda)(1+\tau))^2\Psi_2^2 T_2\left(q,e^{i\theta}\right) \end{array}\right]}$$

$$= T_1\left(q,e^{i\theta}\right)\left[\begin{array}{c}\left(\zeta(\mu)+\frac{1}{2\left(3-(1-\lambda)\left(1+\tau+\tau^2\right)\right)\Psi_3}\right)r_2\\+\left(\zeta(\mu)-\frac{1}{2\left(3-(1-\lambda)\left(1+\tau+\tau^2\right)\right)\Psi_3}\right)s_2\end{array}\right]$$

$$where\zeta(\mu)=\frac{(1-\mu)T_1^2\left(q,e^{i\theta}\right)}{\left[\begin{array}{c}T_1^2\left(q,e^{i\theta}\right)\left(\begin{array}{c}2\left(3-(1-\lambda)\left(1+\tau+\tau^2\right)\right)\Psi_3\\-(1-\lambda)(1+\tau)(2(1-\tau)+\lambda(1+\tau))\Psi_2^2\end{array}\right)\\-2(2-(1-\lambda)(1+\tau))^2\Psi_2^2T_2\left(q,e^{i\theta}\right)\end{array}\right]}$$

Then, in view of equation (10), we get

$$\left|\rho_3-\mu\rho_2^2\right|\leq\begin{cases}\frac{\left|T_1\left(q,e^{i\theta}\right)\right|}{\left(3-(1-\lambda)\left(1+\tau+\tau^2\right)\right)\Psi_3},0\leq|\zeta(\mu)|\leq\frac{1}{\left(3-(1-\lambda)\left(1+\tau+\tau^2\right)\right)\Psi_3}\\2|\zeta(\mu)|\left|T_1\left(q,e^{i\theta}\right)\right|,|\zeta(\mu)|\geq\frac{1}{\left(3-(1-\lambda)\left(1+\tau+\tau^2\right)\right)\Psi_3}\end{cases}$$

This evidently completes the proof of Theorem 2.

Taking the parameters $\lambda=0$and $\tau=0$, in Theorem 2,

Remark 2 For $\mu\in R$, let the function $f\in P_{\Sigma}^{a,b,c}(\mu,q,\theta)$. Then

$$\left|\rho_3-\mu\rho_2^2\right|$$
$$\leq\begin{cases}\frac{\left|e^{i\theta}+qe^{-i\theta}\right|}{2\Psi_3},|\mu-1|\leq\frac{2\left(e^{2i\ \theta}+q^2e^{-2i\ \theta}+2q\right)\Psi_3-\left(2e^{2i\ \theta}+2q^2e^{-2i\ \theta}+3q\right)\Psi_2^2}{2\left(e^{2i\ \theta}+q^2e^{-2i\ \theta}+2q\right)\Psi_3}\\\frac{|1-\mu|\left|e^{2i\ \theta}+q^2e^{-2i\ \theta}+2q\right|\left|e^{i\theta}+qe^{-i\theta}\right|}{\left|2\left(e^{2i\ \theta}+q^2e^{-2i\ \theta}+2q\right)\Psi_3-\left(2e^{2i\ \theta}+2q^2e^{-2i\ \theta}+3q\right)\Psi_2^2\right|},\\|\mu-1|\geq\frac{2\left(e^{2i\ \theta}+q^2e^{-2i\ \theta}+2q\right)\Psi_3-\left(2e^{2i\ \theta}+2q^2e^{-2i\ \theta}+3q\right)\Psi_2^2}{2\left(e^{2i\ \theta}+q^2e^{-2i\ \theta}+2q\right)\Psi_3}\end{cases}$$

4 Conclusion

In the present investigation, a new class of bi univalent function based on Bazilevic Sakaguchi function using the trigonometric polynomials $T_n\left(q,e^{i\theta}\right)$ is obtained in the open unit disc. Furthermore, belonging to this class, the Taylor – Maclaurin coefficient inequalities and the well knownFekete – Szego inequalities are also derived. These findings can further be improved by finding sharpness. Moreover, Hankel Determinants and Toeplitz determinants for various integral orders can be computed in the future.

5 Declaration

This work has been presented in "International conference on Recent Strategies in Mathematics and Statistics (ICRSMS2022), Organized by the Department of Mathematics of Stella Maris College and of IIT Madras during 19 to 21 May, 2022 at Chennai, India. The Organizer claims the peer review responsibility.

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