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RESEARCH ARTICLE

Numerical Solution of Non-linear Integro-differential Equations using Operational Matrix based on the Hosoya Polynomial of a Path Graph

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Abstract

Objectives: Introduction to new numerical techniques to solve differential, difference, and integro-differential equations (IDEs) are always remaining the thrust area of research for many scientists over the centuries. The prime objective of this work is to contribute a new numerical technique to solve IDEs. **Method**: To address non-linear integro-differential equations, we computed an operational matrix of derivatives based on the Hosoya polynomial of the path graph in this work. **Findings**: Using the derived operational matrix, we have solved both Volterra and Fredholm integro-differential equations. Taking suitable examples accuracy of the projected method is demonstrated in this paper in terms of a graphical representation of the absolute error. The results of the examples reveal that the projected method is a suitable method to solve IDEs. **Novelty**: The application of the Hosoya polynomial of path graph to solve integro-differential equations is a novel approach in the field of numerical analysis.

Keywords: Volterra Integrodifferential equations; Fredhlom Integrodifferential equations; Graph theorypolynomials

1 Introduction

In this paper, we have introduced a novel numerical technique to solve integro-differential equations by computing an operational matrix using the Hosoya polynomial of the path graph. Integrals and derivatives are fundamental calculus methods that have a wide range of uses in science and engineering. Many scholars are focusing on designing computational schemes for discovering solutions to different problems including derivatives and integrals. Many systems in science and engineering are governed by differential, integral, and integro-differential equations. In the fields of dispersive waves, ocean circulations, and electromagnetic theory, integro-differential equations play a major role. Integro-differential equations also play a crucial role in characterizing physical, biological, and social problems. Polymer rheology, a variety of models of population growth, compartmental systems, mathematical modeling of discrete particle diffusion in a turbulent fluid, aeroelastic phenomena, unsteady aerodynamics, and nuclear reactors are only a few of the applications. For example, Volterra looked at population development, concentrating his studies on genetic influences, and as a result, the analysis of integro-differential equations that appear naturally in the simulation of advanced scientific problems has gotten a lot of interest.

Integro-differential equations can be used in Ecology as well. Indeed, optimal search theory suggests that predators can use long leaps to locate prey that is scattered and dispersed randomly, with Brownian motion being more effective only when prey is abundant. Because of its numerous applications in domains like biochemistry, electrical engineering (Communications networks and coding theory), computer science (algorithms and computations), and operations research, graph theory is quickly becoming a mainstream topic in mathematics (scheduling). The adoption of the Hosoya polynomial method is the most recent method which provides a mathematical formulation in the field of science and engineering. Jalilian[(1)] adopted the exponential spline function to solve Fredholm integro-differential equations of the second kind. To acquire the approximate solutions of a mixed Volterra-Fredholm integro-differential equation[(2)], Hamoud and colleagues discussed the Adomian decomposition approach and modified decomposition method. To determine the analytical solution to the linear and nonlinear FIDE and VIDE, A. S. Khan[(3)] has used a variation of parameter techniques. Numerous academics have recently worked on integro-differential equations and achieved superior results[(4–11)]. The study of integral and IDEs, which contain two different types of integral operators, was the main emphasis of Hamoud and Ghadle[(12)]. They compared and contrasted various approaches to numerically solve the integral and integro-differential equations of Volterra and Fredholm using various kernels. As a result, none of the methods provide a solution with 100% accuracy. To resolve FIDEs, Hamoud et al.[(13)] suggest a modified variational iteration method. This study offers an analytical approximation to ascertain how the solution will behave. To resolve the fuzzy integro-differential equations, Hamoud and Ghadle[(14)] used the Adomian decomposition method, modified Adomian decomposition method, variational iteration approach, and Homotopy perturbation method. The Hosoya polynomial of a graph was introduced in Hosoya's seminal paperback in 1988 and received a lot of attention afterward. The polynomial was later independently introduced and considered by Sagan et al. under the name Wiener polynomial of a graph. The Hosoya polynomial's key benefit is its depth of knowledge on distance-based graph invariants. For instance, it is simple to determine the renowned Wiener index of a graph by knowing the Hosoya polynomial of the graph, which is the first derivative of the polynomial at point 1. Benzenoid graphs[(15)], tori, armchair open-ended nanotubes, zigzag polyhexnanotorus, and Fibonacci and Lucas cubes are some of the graphs theoretical concepts used by the researchers to solve differential, integral, and integro-differential equations.

1.1 Properties of Hosoya Polynomial

A simple graph is a pair $G=(\upsilon,E)$, where υ is a set of element s called vertices, and E is a set of elements called edges. Let $u,v\in\upsilon$. The vertices U and v are said to be adjacent if there is an edge between U and v,i.e.,$(U,v)\in E$. A subset x of vertices is called independent if the vertices in X are pairwise non-adjacent. Let υ be the vertices of G. The path p_n is a graph with n vertices where υ_i is adjacent to υ_{i+1}, $i=1,2,\ldots,n-1$. The length of a path is the number of edges in it. A graph G is said to be connected if every pair of points of G is joined by some path. The distance between the vertices υ_i and υ_j in G is equal to the length of the shortest path joining them and is denoted by $d(\upsilon_i,\upsilon_j)$. For more details about the graph theory one can refer the book[(16)]. The Hosoya Polynomial of a graph G is given by

$H(G,x)=\sum_{k\geq 0} d(G,x^k)$,

where $d(G,x^k)$ is the number of pairs of vertices in the graph G that are distance k apart.

The Hosoya polynomial of a path graph P_n is:

$H(P_n,x)=n+(n-1)x+(n-2)x^2+\cdots+[n-(n-2)]x^{n-2}+[n-(n-1)]x^{n-1}$.

In particular

$H(P_1,x)=1$,

$H(P_2,x)=x+2$,

$H(P_3,x)=x^2+2x+3$.

The prime objective of this paper is strategy to use the Hosoya polynomial method for the numerical solution of non-linear IDEs with Volterra and Fredholm type equations of the form

$$\sum_{i=0}^{n} p_i u^{(i)}(x)=f(x)+\lambda\int_{\varphi(x)}^{\iota(x)} k(x,t)\,g((u(t]))\,dt u^{(p)}(a)=u_p,\ p=0,1,2,\ldots,(n-1), \quad (1)$$

where λ is constant, $g([u(t)])$, $p(x)$ and $K(x,t)$ are given functions, whereas $u(x)$ is to be determined.

2 Function Approximation

If $u(x)$ is a square integrable function in R, then $w(x)$ may be approximated in terms of Hosoya polynomial graph as

$u(x) = \sum_{j=0}^{\infty} e_j H(K_{jj};x)$

. If we use only first $(m+1)$ terms of the Hosoya polynomial of path graph, then we have

$u(x) = \sum_{j=0}^{m} e_j H(K_{jj};x) = E^T \psi(x)$

where the Hosoya polynomial co-efficient vector E and Hosoya polynomial vector $\psi(x)$ are given by

$E^T = [e_0, e_1, ..., e_m]$

$$\psi(x) = (H(K_{00};x), H(K_{11};x), \ldots H(K_{mm};x))^T \tag{2}$$

3 Hosoya polynomial method for IDEs

We Know that

$H(P_n,x) = (x+1)^{n-1} + 1,$
$H(P_1,x) = 1,$
$H(P_2,x) = x+2,$
$x = -2 + H(P_2,x),$
$x = -2H(P_1,x) + H(P_2,x),$
$H(P_3,x) = x^2 + 2x + 3,$
$x^2 = H(P_3,x) - 2x - 3,$
$x^2 = H(P_1,x) - 2H(P_2,x) + H(P_3,x),$
$H(P_4,x) = x^3 + 2x^2 + 3x + 4,$
$x^3 = H(P_4,x) - (2x^2 + 3x + 4),$
$x^3 = H(P_2,x) - 2H(P_3,x) + H(P_4,x).$

Similarly,

$x^4 = H(P_3,x) - 2H(P_4,x) + H(P_5,x),$
$x^5 = H(P_4,x) - 2H(P_5,x) + H(P_6,x),$
$x^6 = H(P_5,x) - 2H(P_6,x) + H(P_7,x).$

In general

$$x^n = H(P_{n-1},x) - 2H(P_n,x) + H(P_{n+1},x), \qquad n = 2,3,4,\ldots \tag{3}$$

The derivative of Hosoya polynomial of path graph is

$D(H(P_1,x)) = 0,$

$$D(H(P_2,x)) = 1,$$

$$= H(P_1,x),$$

$= (1,0,0,\ldots 0]\,\psi(x),$

$D(H(P_3,x)) = 2x+2,$

$$= 2(H(P_1,x) - 2) + 2,$$

$$= -2(H(P_1,x)) + 2(H(P_2,x)),$$

$$= [-2,2,0,\ldots,0]\,\psi(x),$$

$D(H(P_4,x)) = 3x^2 + 4x + 3,$

$$= -2H(P_1,x) - 2H(P_2,x) + 3H(P_3,x),$$

$$= [-2, -2, 3, 0, \ldots, 0]\psi(x).$$

Similarly,

$D(H(P_{n.n}, x)) = 2n(1+x)^{n-1}$,

$= [n, 0, 0, \ldots\ldots n, 0]\psi(x)$.

The first derivative of the vector $\psi(x)$ for $m = 6$ can be written as:

$$\frac{d}{dx}\psi(x) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 & 0 & 0 \\ -2 & -2 & 3 & 0 & 0 & 0 & 0 \\ -2 & -2 & -2 & 4 & 0 & 0 & 0 \\ -2 & -2 & -2 & -2 & 5 & 0 & 0 \\ -2 & -2 & -2 & -2 & -2 & 6 & 0 \end{pmatrix} \psi(x)$$

In general, we may write first derivative of the vector $\psi(x)$ as

$$\frac{d}{dx}\psi(x) = D^{(1)}\psi(x), \tag{4}$$

where $D^{(1)} = (d_{ij})$ is the Hosoya polynomial operational matrix of derivatives of order $(m+1) \times (m+1)$ and

$$d_{ij} = \begin{cases} 0, & for\ 1 \le j \le i-2, \\ 0, & for\ i \le 2, \\ -2, & for\ i \le n-j. \end{cases}$$

By using relation (7) it is clear that $\frac{d^n\psi(x)}{dx^n} = (D^{(1)})^n\psi(x)$. Where $n \in N$.

4 Use of Hosoya polynomial method for handling IDEs

Consider the non-linear IDEs (1) subject to the suitable initial conditions. We approximate $u(x)$ as in the section (4). Also, from the initial condition we get

$$E^T D^{(p)}\Psi(x_0) = u_p. \tag{5}$$

Therefore, the residual $R(x)$ of equation (1) is given by:

$R(x) = \sum p_i E^T D^{(i)}\psi(x) - f(x) - \lambda \int_a^b k(x,t) g(E^T\psi(t))dt.$

The application of the Hosoya polynomial method requires that $R(x)$ must vanish at certain collocation points. We select the collocation points as: $\frac{2i}{m+1}$, $i = 1, 2, ..m-n$. Therefore

$$R(\frac{2i}{m+1}) = 0, \quad i = 1, 2, \ldots, m-n. \tag{6}$$

Equations (6) with Equation (5) generate a system of linear or nonlinear equations in the unknown expansion coefficients e_i of dimension $(m+1)$. Applying Newton's iterative method, we can solve this system of equations.

5 Results and Discussions

The analysis of the Hosoya polynomial technique is demonstrated in this part by using the method to solve the non-linear integro-differential problems.

Example 1 First we consider the integro-differential equation

$$u'(x) = -u(x) + \frac{1}{2}(e^{-2} - 1) + \int_0^1 [u(t)]^2 dt,, \qquad u(0) = 0. \tag{7}$$

The exact solution is $u(x) = e^{-x}$.

Solution: Using the proposed method on equation (1), we get

$$E^T D\psi(x) = -E^T \psi(x) + \frac{1}{2}(e^{-2}-1) + \int_0^1 [E^T \psi(t)]^2 dt, \tag{8}$$

with $E^T \psi(0) = 1$. For $m = 3$, collocating the points $\frac{2i}{m+1}$, for $i = 1,2,3$ we obtain the system of equations as

$$-\left(a_1 + \frac{11a_2}{5} + \frac{86a_3}{25} + \frac{586a_4}{125}\right)^2 + a_1 + \frac{16a_2}{5} + \frac{36a_3}{25} + \frac{1626a_4}{125} + \frac{11}{5}(2a_3 - 2a_4) + \frac{1}{2}\left(1 - \frac{1}{e^2}\right),$$

$$-\left(a_1 + \frac{12a_2}{5} + \frac{99a_3}{25} + \frac{698a_4}{125}\right)^2 + a_1 + \frac{17a_2}{5} + \frac{49a_3}{25} + \frac{1933a_4}{125} + \frac{12}{5}(2a_3 - 2a_4) + \frac{1}{2}\left(1 - \frac{1}{e^2}\right),$$

$$-\left(a_1 + \frac{13a_2}{5} + \frac{114a_3}{25} + \frac{842a_4}{125}\right)^2 + a_1 + \frac{18a_2}{5} + \frac{64a_3}{25} + \frac{2302a_4}{125} + \frac{13}{5}(2a_3 - 2a_4) + \frac{1}{2}\left(1 - \frac{1}{e^2}\right).$$

and from initial condition

$$a_1 + 2a_2 + 3a_3 + 4a_4 = 1. \tag{9}$$

Solving the above system of equations and equation (9), we get
$a_0 = 3.46135,$
$a_1 = -2.05187,$
$a_2 = 0.697457,$
$a_3 = -0.112493.$
This yields the numerical solution as
$u_1(x) = -0.112493x^3 + 0.472471x^2 - 0.994439x + 1.$
For $m = 6$, we get the solution

$$u_2(x) = 0.000847432x^6 - 0.00762689x^5 + 0.041161x^4 - 0.16646x^3 + 0.499953x^2 - 0.999994x + 1 \quad .$$

The results of the proposed method for Example (1) are exhibited in Table 1 with 2 choices of m.The authors of[14] computed the solutions to Example 1 and found absolute errors of $8.12 \times 10^{-4}$and $2.11 \times 10^{-4}$for $m = 5$ and $m = 9$ respectively. The current technique, however, yields a maximum absolute error of 3.2×10^{-7} for $m = 6$ (Figure 1). This shows that the current strategy is effective for solving IDEs.

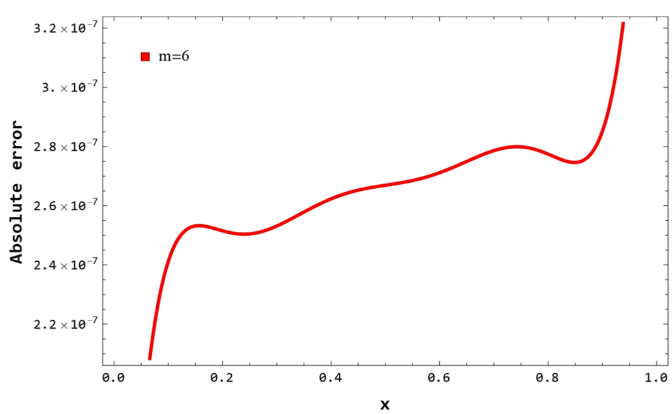

Fig 1. The absolute error for Example 1 for m = 6

Example 2 Consider the non-linear Fredhlom integro-differential equation, as follows

$u'(x) = 1 - \frac{1}{3}x^3 + \int_0^1 x^3 (u(t))^2 dt, \qquad u(0) = 0.$
The exact solution is $u(x) = x$.
The numerical solution is given as

$$u_1(x) = -1.324206093356754x^3 \times 10^{-15} + 1.3600418403042467x^2 \times 10^{-15} + x$$

$+6.719955987530877 \times 10^{-17}$.
For $m = 6$, we get the solution

$$u_2(x) = -1.0609361884974525x^6 \times 10^{-14} + 4.248372556965464x^5 \times 10^{-14}$$

$-6.029568457781779x^4 \times 10^{-14} + 3.92990039050865x^3 \times 10^{-14}$
$-1.2034181439836931x^2 \times 10^{-14} + x - 1.2553598836768822 \times 10^{-16}$.

To measure the accuracy of the studied approach, the absolute errors are presented Figure 1 for m=6.

In [14], the authors have computed the solution of Example 2 and obtained absolute error 3.26×10^{-6} and 2.44×10^{-4} for m=5 and m=9 respectively. Whereas the present method gives maximum absolute error as 3×10^{-16} for m=6 (Figure 2). This indicates that present method is reliable to solve IDE.

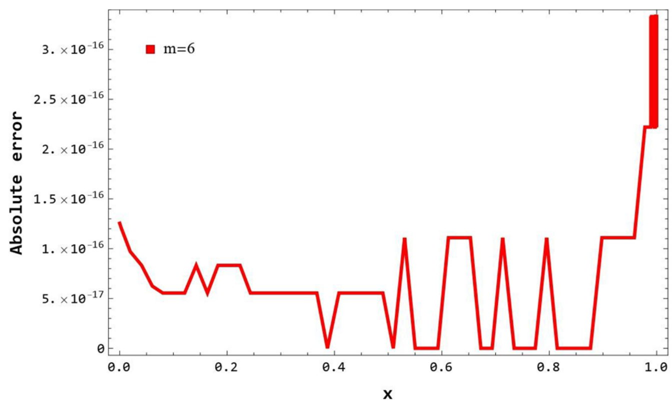

Fig 2. The absolute error for Example 2 for m = 6

Example 3 Consider the nonlinear Volterra integro-differential equation
$\int_0^x (x-t)u^2(t)dt + \int_0^x (x-t)u''(t)dt = \frac{3}{4}x^2 + \frac{1}{2}cos2x - \frac{1}{32}cos4x - \frac{15}{32} \quad u(0) = 2, \quad u'(0) = 0.$
The exact solution is
$u(x) = 1 + cos2x$.
For $m = 3$, The numerical solution is given as
$u_1(x) = 0.388509x^3 - 2.05098x^2 + 2$.
For $m = 6$, we get the solution
$u_2(x) = 2 - 1.33227 \times 10^{-15}x - 1.99991x^2 - 0.00141409x^3 + 0.674391x^4 - 0.0184099x^5 - 0.0715553x^6$.

The results in Table 1 shows the numerical solutions are in a very good agreement with the exact solution. It is obvious that in order to get the same accuracy, our method is easy implementation.

Table 1. Approximate and exact solutions for Example 3

x	Exact	HPM with m=3	HPM with m=6
0	2	2	2
0.1	1.9800	1.9798	1.9800
0.2	1.9210	1.9210	1.9210
0.3	1.8253	1.8259	1.8253
0.4	1.6967	1.6967	1.6967
0.5	1.5403	1.5358	1.5403
0.6	1.3623	1.3455	1.3623
0.7	1.1699	1.1282	1.1699

Continued on next page

Table 1 continued

0.8	0.9708	0.8862	0.9707
0.9	0.7727	0.6219	0.7726
1	0.5838	0.3375	0.5831

Example 4 Consider the non-linear Volterra integro-differential equation, as follows
$\int_0^x (x-t)u^2(t)dt + \int_0^x e^{(x-t)}u'(t)dt = -\frac{9}{4} - \frac{5}{2}x + \frac{1}{2}x^2 + 2e^x + \frac{1}{4}e^{2x} + xe^x, \quad u(0) = 2.$
The exact solution is $u(x) = 1 + e^x$.
The numerical solution for $m = 3$ is given as

$$u_1(x) = 0.225321x^3 + 0.477784x^2 + 1.00237x + 2.$$

For $m = 6$, we get the solution

$$u_2(x) = 0.00213916x^6 + 0.00737859x^5 + 0.0422561x^4 + 0.166478x^3 + 0.500029x^2 + 0.999998x + 2.$$

To measure the accuracy of the studied approach, the absolute errors are presented in Figure 3 for $m = 6$.

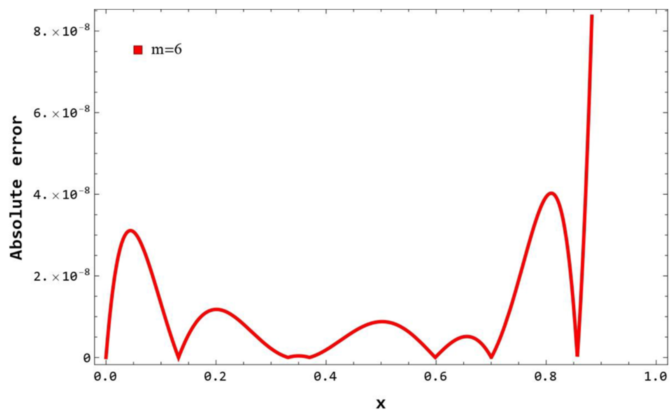

Fig 3. The absolute error for Example 4 for m = 6

Example 5 Consider the nonlinear Volterra integro-differential equation

$$u'(x) - u(x) + 2\int_0^x sinx(u(t))^2 dt = cosx + (1-x)\,sinx + cosx\,sin^2x, \qquad u(0) = 0.$$

The exact solution is $u(x) = sinx$.
The numerical solution is given as
$u_1(x) = 2.22045 \times 10^{-16} + 1.00211x - 0.00950606x^2 - 0.153177x^3.$
For $m = 6$, we get the solution
$u_2(x) = 2.77556 \times 10^{-17} + 1.00001x - 0.0000646591x^2 - 0.166385x^3 - 0.000677485x^4$
$+0.00925267x^5 - 0.000660979x^6.$
From Figure 4 it is observed that Hosoya polynomial method gives satisfactory accuracy even for lower value of $m(m = 6)$.

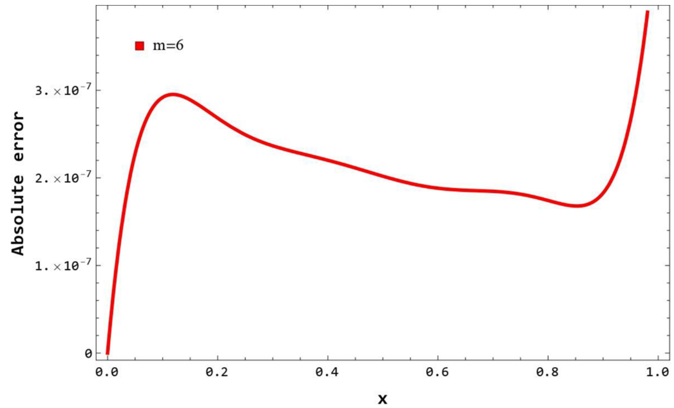

Fig 4. The absolute error for Example 5 for m = 6

6 Conclusion

Using the Hosoya polynomial of the path graph an operational matrix of the derivative has been derived in this paper. This technique is a novel way to deal with IDEs. Applying the operational matrix, we have obtained the numerical solution of IDEs in both Volterra and Fredhlom sense by considering five examples. The results obtained for the solutions of these examples are presented either in tabular form or with graphical representation. The projected absolute errors reveal that the current method gives higher accuracy even for smaller values of m as compared to the available literature. Solutions obtained in this paper suggest that Hosoya polynomial-based operational matrix method is easy to implement and can be used as an efficient method to solve IDEs. As a future direction of study, we can consider other graph theory polynomials to solve IDEs.

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