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RESEARCH ARTICLE

Jacobsthal Matrices and their Properties

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Abstract

Objectives: Matrices with Jacobsthal numbers are used in the medical image processing applications. The Cholesky factorization of the matrix with the Jacobsthal number is anlayzed. We also investigate the upper and lower bounds of the eigenvalues of the symmetric Jacobsthal and Jacobsthal-Lucas matrices. **Methods:** In this paper, we define a factor matrix and use the factorization techniques to get Cholesky decomposition of the Jacobsthal, Jacobsthal-Lucas matrix and inverses of these matrices. The bounds for eigenvalues are obtained using majorization techniques. **Findings:** The Cholesky factorization has been obtained using the factor matrix technique for any matrix of order n with entries from the Jacobsthal and Jacobsthal Lucas sequences. **Novelty:** Factorization of Lucas and symmetric Lucas matrix has already been obtained using the factorization technique. In this paper we give the factorization of the matrices with entries from the Jacobsthal and Jacobsthal Lucas sequences. Mathematics Subject Classification (2020). 15A23, 11B39, 15A18,15A42

Keywords: Jacobsthal matrix; Jacobsthal-Lucas matrix; symmetric; eigenvalues

1 Introduction

Many recurrence sequences of integers play a vital role in science and engineering. The zeros of characteristic polynomial of Fibonacci sequence matrix is always a fixed number $\frac{1\pm\sqrt{5}}{2}$. A.H. Ganie[(1)], introduced a new sequence of Jacobsthal type having a generalized order j and he established the generalized Binet formula. Fugen Torunbalci Aydin[(2)] , investigated the generalized form of these sequences in complex and dual forms using these numbers. N.Irmak[(3)], gave the factorization of the Lucas, inverse Lucas matrix and Cholesky factorization of the symmetric Lucas matrix. E.Andrade and D.C Olivera, C. Manzaneda introduced the concept on circulant like matrices properties involving Horadam, Fibonacci, Jacobsthal and Pell numbers[(4)]. E.E Polatli and Y Soykan introduced the concept of third –order Jacobsthal sequence and third- order Jacobsthal-Lucas sequence[(5)]. D Brod, A S Lina and Iwona Wloch introduced the concept of two generalizations of dual-hyperbolic balancing numbers[(6)].

Several sequences of integers are described in the book A Handbook of integer sequences N.Sloane. Y üksel Soykan(7), investigated a study on generalized Jacobsthal- Padovan numbers(7). Alaa Al-Kateeb investigated a generalization of Jacobsthal and Jacobsthal-Lucas numbers(8). Also Nayil Kilic studied On k-Jacobsthal and k-Jacobsthal-Lucas hybrid numbers(9).

D. Fathima, M. m Albaidani, A. H. Ganie and A. Akhter introduced about New structure of Fibonacci numbers using concept of $\triangle$–operator(10). A. H. Ganie and Afroza, studied about New type of difference sequence space of Fibonacci numbers(11). T. A. Tarray, P. A. Naik and R. A. Najar, investigated the Matrix representation of an all inclusive Fibonacci sequences(12). S. Uygun, introduced the concept of On the Jacobsthal and Jacobsthal Lucas Sequences at Negative Indices(13). Engin Özkan ·Mine Uysal and A. D. Godase, studied about Hyperbolic k-Jacobsthal and k-Jacobsthal-Lucas Quaternions(14). A. H. Ganie also introduced the concept Nature of Phyllotaxy and Topology of H-matrix(15). Hakan AKKUÅž, Rabia ÜREGEN and Engin ÖZKANA introduced the concept of New Approach to k −Jacobsthal Lucas Sequences(16).

In this paper, In Section 2, we recall the definition of Jacobsthal numbers and we investigate some properties of these matrix formed by Jacobsthal numbers. In Section 3, we define Jacobsthal-Lucas numbers and we investigate some properties of these matrices formed by these numbers. In section 4, we investigate the upper and lower bounds of the eigenvalues of these matrix by using the concept of majorization techniques.

2 Methodology

2.1 Jacobsthal matrix

For $n \geq 0$, the Jacobsthal number is defined by the following recurrence relation(17)

$$J_n = J_{n-1} + 2J_{n-2} \text{ for } n > 1, J_0 = 0, J_1 = 1 \tag{1}$$

We can extend Jacobsthal sequences through negative values of n by means of the recurrence (1)

$$J_{-n} = (-1)^{n+1} \frac{J_n}{2^n} \tag{2}$$

From (1) and (2), we have listed the few values of Jacobsthal number in the table

Table 1. Jacobsthal numbers

n	0	1	2	3	4	5	6	7	8	9	10	…
J_n	0	1	1	3	5	11	21	43	85	171	341	…
J_{-n}	0	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{3}{8}$	$\frac{-5}{16}$	$\frac{11}{32}$	$\frac{-21}{64}$	$\frac{43}{128}$	$\frac{-85}{256}$	$\frac{171}{512}$	$\frac{-341}{1024}$	…

Based on Jacobsthal number we define Jacobsthal matrix J_n and symmetric Jacobsthal matrix $\widehat{J_n}$as follows:

$$\mathrm{J}_n = [J_{ij}] = \begin{cases} J_{i-j+1}, & i-j+1 \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$\hat{J_n} = [a_{ij}] = [a_{ji}] = \begin{cases} \sum_{k=1}^{i} J_k^2, & i = j \\ 2a_{i,j-2} + a_{i,j-1}, & i+1 \leq j \end{cases} \tag{4}$$

where $a_{1,0} = 0$. For example,

$$\mathrm{J}_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 \\ 5 & 3 & 1 & 1 & 0 \\ 11 & 5 & 3 & 1 & 1 \end{bmatrix} \text{ and } \hat{J_5} = \begin{bmatrix} 1 & 1 & 3 & 5 & 11 \\ 1 & 2 & 4 & 8 & 16 \\ 3 & 4 & 11 & 19 & 41 \\ 5 & 8 & 19 & 36 & 74 \\ 11 & 16 & 41 & 74 & 157 \end{bmatrix} \tag{5}$$

Consider the identity matrix I_n of order $n \times n$. We define the matrices M_n, $\overline{J_n}$ and N_k as follows:

$$M_0 = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} \quad M_{-1} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} \tag{6}$$

and $M_k = M_0 + I_k$, $k = 1, 2, \ldots \bar{J_n} = [1] + J_{k-1}$, $N_1 = I_n$, $N_2 = I_{n-3} \oplus M_{-1}$ and $N_k = I_{n-k} \oplus M_{k-3}$ for $k \geq 3$.

Now we define a factor matrix as $P_n = [p_{ij}] = \begin{cases} 1 & i = j \\ 2 & i = j+1 \\ 0 & \text{otherwise} \end{cases}$

By using the matrices N_k and P_n ,we have the following theorem.

2.1.1 Observation

Generalized norm of Jacobsthal vector is defined by $\left\|\overrightarrow{J_n}\right\|^k = \left\langle \overrightarrow{J_n}, \overrightarrow{J_{n+1}}, \ldots, \overrightarrow{J_{n+k}} \right\rangle = {J_n}^2 + {J_{n+1}}^2 + \cdots + {J_{n+k}}^2$

Theorem 2.1.2

The Jacobsthal matrix J_n can be factored by the $N_k$and P_n as follows:

$J_n = N_1 N_2 \ldots N_n P_n$

$= P_n N_1 N_2 \ldots N_n$

For example,

$$J_5 = P_5 N_1 N_2 N_3 N_4 N_5$$

$$J_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 \\ 5 & 3 & 1 & 1 & 0 \\ 11 & 5 & 3 & 1 & 1 \end{bmatrix}$$

The other factorization of J_n for $n \times n$ matrix is as follows:

We define

$$D_n = [d_{ij}] \text{ by } d_{ij} = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ J_2 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ J_n & 0 & \cdots & 1 \end{bmatrix} \tag{7}$$

Theorem 2.1.3

For $n \geq 2$, the Jacobsthal matrix $J_n$can be factored by the D_n 's as

$$J_n = D_n (I_1 \oplus D_{n-1}) (I_2 \oplus D_{n-2}) \ldots (I_{n-2} \oplus D_2) \tag{8}$$

Lemma 2.1.4

Let k be the non-negative integer and $P_n^{-1} = \left[p'_{ij}\right]$ be the inverse of the matrix P_n. Then

$$p'_{ij} = \begin{cases} 0 & i < j \\ (-2)^k & i = j+k \end{cases} \tag{9}$$

holds.

Proof. Let $r_{ij} = \sum_{k=1}^{n} p_{ik}\, p_{kj}'$. Clearly, $r_{ii} = 1$ and $r_{ij} = 0$ for $i < j$.

Then $r_{ij} = 2(-2)^k + 1(-2)^{k+1} = 0$ for $i > j$ follows. This completes the lemma.

For example,

$$P_6^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ 4 & -2 & 1 & 0 & 0 & 0 \\ -8 & 4 & -2 & 1 & 0 & 0 \\ 16 & -8 & 4 & -2 & 1 & 0 \\ 32 & 16 & -8 & 4 & -2 & 1 \end{bmatrix}$$

The inverses of the matrices M_0 and M_{-1} are given below:

$M_0 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ $M_{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

We know that ${M_k}^{-1} = {M_0}^{-1} + I_k$

Define $T_k = {M_k}^{-1}$

Then $T_1 = {N_1}^{-1} = I_n$, $T_2 = {N_2}^{-1} = I_{n-3} \oplus {M_{-1}}^{-1}$ and $T_n = {M_{n-3}}^{-1}$ and we have

$${D_n}^{-1} = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ -J_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -J_n & 0 & \cdots & 1 \end{bmatrix}$$

and $(I_k \oplus D_{n-k})^{-1} = I_k \oplus {D_{n-k}}^{-1}$

Now inverses of the Jacobsthal matrix is given by

$$J_n^{-1} = \left[a'_{ij}\right] = \begin{cases} 1 \text{ if } i = j \\ -3 \text{ if } i = j+1 \\ 4(-1)^{i-j} 2^{i-j-2} \quad \text{if} \quad i \geq j+2 \\ 0 \quad \text{otherwise} \end{cases}$$

Here we find the inverses of the Jacobsthal matrix by using the matrices ${N_k}^{-1}$ and ${P_n}^{-1}$. Thus the following theorem explains the factorization of the inverse Jacobsthal matrix.

Theorem 2.1.5

The inverses of the Jacobsthal matrix ${J_n}^{-1}$ can be factored by the ${N_k}^{-1}$ and ${P_n}^{-1}$ as

$$\begin{aligned} J_n^{-1} &= N_n^{-1} N_{n-1}^{-1} \ldots N_2^{-1} N_1^{-1} P_n^{-1} \\ &= T_n T_{n-1} \ldots T_2 T_1 P_n^{-1} \\ &= (I_{n-2} \oplus D_2)^{-1} \ldots (I_1 \oplus C_{n-1})^{-1} C_n^{-1} \end{aligned}$$

For example,

$$J_6^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 \\ 4 & -2 & 1 & 0 & 0 & 0 \\ -8 & 4 & -2 & 1 & 0 & 0 \\ 16 & -8 & 4 & -2 & 1 & 0 \\ 32 & 16 & -8 & 4 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 0 & 0 \\ 4 & -3 & 1 & 0 & 0 & 0 \\ -8 & 4 & -3 & 1 & 0 & 0 \\ 16 & -8 & 4 & -3 & 1 & 0 \\ 32 & 16 & -8 & 4 & -3 & 1 \end{bmatrix} \quad (10)$$

2.2 Jacobsthal-Lucas matrix

For $n \geq 0$, the Jacobsthal-Lucas number is defined by the following recurrence relation [6]

$$j_n = j_{n-1} + 2j_{n-2} \text{ for } n > 1, j_0 = 2, j_1 = 1 \quad (11)$$

We can extend Jacobsthal sequences through negative values of n by means of the recurrence (1)

$$J_{-n} = (-1)^{n+1} \frac{J_n}{2^n} \quad (12)$$

From (1) and (2), we have listed the few values of Jacobsthal-Lucas number in the Table 2

Table 2. Jacobsthal-Lucas numbers

n	0	1	2	3	4	5	6	7	8	9	10	…
J_n	2	1	5	7	17	31	65	127	257	511	1025	…
J_{-n}	-2	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{7}{8}$	$\frac{-17}{16}$	$\frac{31}{32}$	$\frac{-65}{64}$	$\frac{125}{128}$	$\frac{-257}{256}$	$\frac{511}{512}$	$\frac{-1025}{1024}$	…

Based on Jacobsthal-Lucas number we define Jacobsthal-Lucas matrix j_n and symmetric Jacobsthal matrix $\widehat{j_n}$as follows:

$$\mathrm{j}_n = [j_{rs}] = \begin{cases} j_{r-s+1}, & r-s+1 \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (13)$$

$$\widehat{j_n} = [a_{ij}] = [a_{ji}] = \begin{cases} \sum_{k=1}^{i} j_k^2, & i = j \\ 2a_{i,j-2} + a_{i,j-1} + 4, & i+1 \leq j \\ 2a_{i,j-2} + a_{i,j-1}, & i+2 \leq j \end{cases} \quad (14)$$

where $a_{1,0} = 0$. For example,

Consider the identity matrix $I_n$of order $n \times n$. We define the matrices M_n , $\bar{j_n}$ and N_k as follows:

$$\mathrm{j}_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 5 & 1 & 1 & 0 & 0 \\ 7 & 5 & 1 & 1 & 0 \\ 11 & 7 & 5 & 1 & 1 \end{bmatrix} \text{ and } \hat{j_5} = \begin{bmatrix} 1 & 5 & 7 & 5 & 31 \\ 5 & 26 & 40 & 92 & 172 \\ 7 & 40 & 75 & 159 & 309 \\ 11 & 92 & 159 & 364 & 686 \\ 31 & 172 & 309 & 686 & 1325 \end{bmatrix} \quad (15)$$

$$M_0 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad M_{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad (16)$$

and $M_k = M_0 + I_k$, $k = 1, 2, \ldots \bar{j_n} = [1] + j_{k-1}$, $N_1 = I_n$, $N_2 = I_{n-3} \oplus M_{-1}$ and $N_k = I_{n-k} \oplus M_{k-3}$ for $k \geq 3$.

Now we define a factor matrix as $\mathrm{Q}_n = [q_{ij}] = \begin{cases} 1 & i = j \\ 2 & i = j+1 \\ 0 & \text{otherwise} \end{cases}$

By using the matrices N_k and Q_n , we have the following theorem.

Theorem 2.2.1

The Jacobsthal matrix j_n can be factored by the $N_k$and Q_n as follows:

$$\mathrm{j}_n = N_1 N_2 \ldots N_n \mathrm{Q}_n$$
$$= \mathrm{Q}_n N_1 N_2 \ldots N_n$$

For example, $\mathrm{j}_5 = \mathrm{Q}_5 N_1 N_2 N_3 N_4 N_5$

$$\mathrm{j}_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 & 0 \\ 7 & 5 & 1 & 0 & 0 \\ 17 & 7 & 5 & 1 & 0 \\ 31 & 17 & 7 & 5 & 1 \end{bmatrix}$$

The other factorization of j_n for $n \times n$ matrix is as follows:

We define

$$\mathrm{D}_n = [d_{ij}] \text{ by } d_{ij} = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ J_2 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ J_n & 0 & \cdots & 1 \end{bmatrix} \tag{17}$$

3 Results and Discussion

Theorem 3.1

For $n \geq 2$, the Jacobsthal-Lucas matrix $j_n$can be factored by the $D_n\,'s$ as

$$j_n = D_n (I_1 \oplus D_{n-1}) (I_2 \oplus D_{n-2}) \ldots (I_{n-2} \oplus D_2) \tag{18}$$

Lemma 3.2

Let k be the non-negative integer and $Q_n^{-1} = \left(q_{ij}^{`} \right]$be the inverse of the matrix P_n. Then

$$q'_{ij} = \begin{cases} 0 & i < j \\ (-4)^k & i = j + k \end{cases} \tag{19}$$

holds.

Proof. Let $w_{ij} = \sum_{k=1}^{n} q_{ik}\, q_{kj}^{`}$. Clearly, $q_{ii} = 1$ and $q_{ij} = 0$ for $i < j$.

Then q_{ij} =4$(-4)^k + 1(-4)^{k+1} = 0$ for $i > j$ follows. This completes the lemma.

For example,

$$Q_6^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 & 0 & 0 \\ 16 & -4 & 1 & 0 & 0 & 0 \\ -64 & 16 & -4 & 1 & 0 & 0 \\ 256 & -64 & 16 & -4 & 1 & 0 \\ -1024 & 256 & -64 & 16 & -4 & 1 \end{bmatrix}$$

The inverses of the matrices M_0 and M_{-1} are given below:

$$M_0 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad M_{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

We know that $M_k^{-1} = M_0^{-1} + I_k$

Define $T_k = M_k^{-1}$

Then $T_1 = N_1^{-1} = I_n$, $T_2 = N_2^{-1} = I_{n-3} \oplus M_{-1}^{-1}$ and $T_n = M_{n-3}^{-1}$ and we have

$$D_n^{-1} = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ -J_2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -J_n & 0 & \cdots & 1 \end{bmatrix}$$

and $(I_k \oplus D_{n-k})^{-1} = I_k \oplus D_{n-k}^{-1}$

Now inverses of the Jacobsthal-Lucas matrix is given by

$$J_n^{-1} = [a'_{ij}] = \begin{cases} 1 & \text{if } i = j \\ -5 & \text{if } i = j+1 \\ 18(-1)^{i-j}4^{i-j-2} & \text{if } \quad i \geq j+2 \\ 0 & \text{otherwise} \end{cases}$$

Theorem 3.3

For $n \geq 1$, positive integer, $T_n T_{n-1} \dots T_1 Q_n^{-1} \widehat{j_n} = j_n^T$ and the Cholesky factorization is given by $\widehat{j_n} = j_n j_n^T$.

Proof. Together with the facts $T_n T_{n-1} \dots T_1 Q_n^{-1} = j_n^{-1}$ and $j_n^{-1} \widehat{j_n} = j_n^T$, we have $\widehat{j_n} = j_n j_n^T$

This gives the Cholesky factorization of the matrix $\widehat{j_n}$.

For example, the Cholesky factorization of the inverses symmetric Jacobsthal matrix is given by $\widehat{j_n}^{-1} = \left(j_n^T\right)^{-1} j_n^{-1} = \left(j_n^{-1}\right)^T j_n^{-1}$.

$$\widehat{j_6}^{-1} = \begin{bmatrix} 1415582 & -353903 & 88506 & -22248 & 6048 & -1152 \\ -353903 & 88478 & -22127 & 5562 & -1512 & 288 \\ 88506 & -22127 & 5534 & -1391 & 378 & -72 \\ -22248 & 5562 & -1391 & 350 & -95 & 18 \\ 6048 & -1512 & 378 & -95 & 26 & -5 \\ -1152 & 288 & -72 & 18 & -5 & 1 \end{bmatrix}$$

3.3.1 Eigenvalues of symmetric Jacobsthal-Lucas matrix

In this section we study about the eigenvalues of the symmetric Jacobsthal-Lucas matrix $\widehat{j_n}$

Definition:3.3.2

Let $F = \{p = (p_1, p_2, \dots, p_n) \in R_n; p_1 \geq p_2 \geq \cdots \geq p_n\}$

Let $p, q \in F$, $p \prec q$ if $\begin{cases} \sum_{i=1}^{k} p_{[i]} \leq \sum_{i=1}^{k} q_{[i]} \\ \sum_{i=1}^{k} p_{[i]} = \sum_{i=1}^{k} q_{[i]} \\ k = 1, 2, \dots, n-1 \end{cases}$

When $p \prec q$, p is said to be majorized by q.

Hardy, J E Littlewood, and G Polya introduced the concept of majorization techniques.

Let Ψ be an $n \times n$ Hermitian matrix then it is positive definite if and only if $det\Psi$ >0.

Clearly $det\,\widehat{j_n} = det\left(j_n j_n^T\right) = 1$ and $\widehat{j_n}$is a positive definite matrix and its eigenvalues are all positive. Note that $det\, j_n = 1$ and $det\,\widehat{j_n} = 1$.

Proposition 3.3.3

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of $\widehat{j_n}$. Then $n\lambda_n \leq j_{2n} \leq n\lambda_1$.

Proof. Let $c_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ (20)

Since $\left(\frac{c_n}{n}, \frac{c_n}{n}, \dots, \frac{c_n}{n}\right) \prec (\lambda_1, \lambda_2, \dots, \lambda_n)$, we get $\lambda_n \leq \frac{c_n}{n} \leq \lambda_1$. (21)

Hence the result is completed.

Let $\rho = \frac{1}{n}\left(1 + 26 + 350 + \cdots + 26 + 18^2\left(\frac{4^{2n-6}-1}{15}\right)\right) = \frac{15^2 + 990(n-1) + 18^2(4^{2n-2}-1)}{225n} = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\lambda_i}$

Then we have $(\rho, \rho, \dots, \rho) \prec \left(\frac{1}{\lambda_n}, \frac{1}{\lambda_{n-1}}, \dots, \frac{1}{\lambda_1}\right)$ (22)

Theorem 3.3.4

For $(\lambda_1,\lambda_2,\ldots,\lambda_n)\in F$, we have $\left(\frac{1}{n-1}\left(c_n-\frac{1}{\rho}\right),\ldots,\frac{1}{n-1}\left(c_n-\frac{1}{\rho}\right),\frac{1}{\rho}\right)\prec(\lambda_1,\lambda_2,\ldots,\lambda_n)$.

Proof. Let E_n be a square matrix of order n as follows:

$$E_n=\begin{bmatrix}\frac{1-\phi_{1n}}{n-1} & \frac{1-\phi_{1n}}{n-1} & \ldots & \frac{1-\phi_{1n}}{n-1} & \phi_{1n}\\ \frac{1-\phi_{2n}}{n-1} & \frac{1-\phi_{2n}}{n-1} & \ldots & \frac{1-\phi_{2n}}{n-1} & \phi_{2n}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \frac{1-\phi_{nn}}{n-1} & \frac{1-\phi_{nn}}{n-1} & \ldots & \frac{1-\phi_{nn}}{n-1} & \phi_{nn}\end{bmatrix} \tag{23}$$

where $\phi_{ij}=\frac{1-\phi_{in}}{n-1}$, $i=1,2,\ldots,n,\quad j=1,2,\ldots,\ n-1$

and $\phi_{in}=\frac{1}{n\rho\lambda_i}$, $i=1,2,\ldots,\ n$.

Then for $i=1,2,\ldots,n$, we have $\sum_{i=1}^{n}\phi_{in}=\sum_{i=1}^{n}\frac{1}{n\rho\lambda_i}$ (24) $(n-1)\frac{1-\phi_{in}}{n-1}+\phi_{in}=1$ (25)

and $\sum_{i=1}^{n}\frac{1-\phi_{in}}{n-1}=\frac{1}{n-1}\left(n-\left(\sum_{i=1}^{n}\phi_{in}\right)\right)=1$ (26)

Hence E_n is a doubly stochastic matrix.

And also in addition, we have $\sum_{i=1}^{n}\lambda_i\phi_{in}=\frac{1}{\rho}$ (27)

and $\sum_{i=1}^{n}\lambda_i\left(\frac{1-\phi_{in}}{n-1}\right)=\frac{1}{n-1}\left(c_n-\left(\sum_{i=1}^{n}\lambda_i\phi_{in}\right)\right)=\frac{1}{n-1}\left(c_n-\frac{1}{\rho}\right)$ (28) Therefore, we get

$$\left(\frac{1}{n-1}\left(c_n-\frac{1}{\rho}\right),\ldots,\frac{1}{n-1}\left(c_n-\frac{1}{\rho}\right),\frac{1}{\rho}\right)=(\lambda_1,\lambda_2,\ldots,\lambda_n)E_n \tag{29}$$

$\left(\frac{1}{n-1}\left(c_n-\frac{1}{\rho}\right),\ldots,\frac{1}{n-1}\left(c_n-\frac{1}{\rho}\right),\frac{1}{\rho}\right)=(\lambda_1,\lambda_2,\ldots,\lambda_n)$

The following lemma gives the lower bounds of the eigenvalues of Jacobsthal-Lucas symmetric matrix $\widehat{j}_n$

Lemma 3.3.5

For $k=1,2,\ldots,n$, we have $\frac{1}{\mu_k}\leq\lambda_k$, where $\mu_k=\frac{15^2+990(k-1)+18^2(4^{2k-2}-1)}{225}$ is the sum of the diagonal elements of $\widehat{j}_n^{-1}$.

Proof. From $\widehat{j}_6^{-1}$ we have,

$\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\cdots+\frac{1}{\lambda_k}\leq 1+26+350+\cdots+26+18^2\left(\frac{16^{n-3}-1}{15}\right)=\mu_k$ (30)

Clearly, the sum of the eigenvalues of the inverse symmetric Jacobsthal-Lucas matrix is positive and we get $\frac{1}{\mu_k}\leq\lambda_k$.

Theorem 3.3.6

For $k=1,\ldots,n-1,n-2$, we have $\frac{1}{\mu_{n-k}}\leq\lambda_{n-k}\leq\frac{1}{n-1}\left(\frac{n-k-1}{\rho}+kc_n\right)-\sum_{i=1}^{n}\frac{1}{\mu_{n-i}}$.

In particular, $\frac{1}{\mu_n}\leq\lambda_n\leq\frac{1}{\rho}$.

Proof. From Theorem 3.3.4 we have $\frac{1}{n-1}\left(s_n-\frac{1}{\rho}\right)\leq\lambda_1$ and $\lambda_n\leq\frac{1}{\rho}$. Also from lemma 3.3.5,

we have $\frac{1}{\mu_n}\leq\lambda_n$ and $det\,\widehat{j}_n=det\left(j_n j_n{}^T\right)=1\ =\lambda_1\lambda_2\ldots\lambda_n$.

Then we get $\lambda_1\prod_{i=2}^{n}\frac{1}{\mu_i}\leq\lambda_1\lambda_2\ldots\lambda_n=1$ and $\lambda_1\leq\prod_{i=2}^{n}\mu_i$. From Theorem 3.6 we have

$$\lambda_n+\lambda_{n-1}+\cdots+\lambda_{n-k}\leq\frac{1}{\rho}+\frac{k}{n-1}\left(c_n-\frac{1}{\rho}\right)$$

$=\frac{1}{n-1}\left(\frac{n-k-1}{\rho}+kc_n\right)$.

By lemma 3.3.5, we have

$$\lambda_{n-k}\leq\frac{1}{n-1}\left(\frac{n-k-1}{\rho}+kc_n\right)-(\lambda_n+\lambda_{n-1}+\cdots+\lambda_{n-k+1})$$

$\leq\frac{1}{n-1}\left(\frac{n-k-1}{\rho}+kc_n\right)-\sum_{i=0}^{k-1}\frac{1}{\mu_{n-i}}$.

Then we have $\frac{1}{\mu_{n-k}}\leq\lambda_{n-k}\leq\frac{1}{n-1}\left(\frac{n-k-1}{\rho}+kc_n\right)-\sum_{i=0}^{k-1}\frac{1}{\mu_{n-i}}$.

4 Conclusion

We present the decomposition of Jacobsthal matrix and Jacobsthal-Lucas symmetric matrix and their inverses using factor matrix. These matrices support to study about the medical image processing and cryptography for coding and decoding messages. Also we studied about the upper and lower of the eigen values of Jacobsthal-Lucas symmetric matrix.

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