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RESEARCH ARTICLE

Common Fixed-point Theorem for Four Weakly Compatible Self-maps Satisfying (E.A) — Property on a Complete S-metric Space

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Abstract

Objectives: To establish a fixed-point theorem on a complete S -metric space. **Methods:** By using (E.A)-property of self-maps and applying the concept of strong comparison function. **Findings:** Obtained a unique common fixed-point theorem for four self-maps of a complete S-metric space and validated it with a suitable example. **Novelty:** By utilizing weak compatibility together with (E.A)-property, a unique common fixed point is obtained for four self-maps which is more robust generalization of the existing theorems in the literature which are proved by using common limit range property.

Mathematics Subject Classification: 47H10, 54H25

Keywords: (EA)-property; Weak compatibility; Comparison function; Fixed point; S - metric space

1 Introduction

Fixed point theory is an important branch of non-linear analysis due to its potential application. In proving fixed point theorems, we use completeness, continuity, convergence and various other topological aspects. Banach's Contraction Principle or Banach's fixed point theorem is one of the most important results in non-linear analysis. This theorem has been generalized in many directions by generalizing the underlying space or by viewing it as a common fixed-point theorem along with other self-maps.

Commutativity plays an important role in proving common fixed-point theorems. As it is a stronger requirement, in[(1)], the idea of compatibility was proposed as a generalization of commutativity. Later in[(2)], the notion of weak compatibility was introduced as a further generalization of compatibility. Clearly, compatible mappings are weakly compatible but not conversely. Several researchers employed common limit range property to obtain common fixed points which require the containment of ranges. In[(3)], the idea of (E.A)-property is initiated to establish common fixed point results.

In the past few years, several generalizations of metric spaces have appeared in several papers, like G -metric spaces, partial metric spaces, cone metric spaces and S- metric spaces. These generalizations were used to extend the scope of the study of fixed-point theory. We can refer[(4)] to a new approach of such generalizations.

Among all generalizations, S-metric spaces introduced in [5] evinced a lot of interest in many researchers as they unified, extended, generalized and refined several existing results onto these S -metric spaces [6].

In the present paper, we establish a common fixed-point theorem for four weakly compatible self-maps of a S - metric space which satisfy the (E.A)-property. Clearly, (E.A)-property is weaker than the common limit range property in the sense that the common limit range property implies (E.A)-property, while (E.A)-property need not imply a common limit range property in general. As a result, the finding is more refined than that demonstrated in [7]. We support the conclusions with a relevant example.

2 Methodology

In this section, we shall give few definitions and lemmas required to prove our main result.

2.1 Definition [5] :

Let M be a non-empty set. A function $S : M^3 \to [0,\infty)$ is said to be a S - metric on M, if for each $v, \omega, \vartheta, \lambda \in M$

(i) $S(v,\omega,\vartheta) \geq 0$

(ii) $S(v,\omega,\vartheta) = 0$ if and only if $v = \omega = \vartheta$

(iii) $S(v,\omega,\vartheta) \leq S(v,v,\lambda) + S(\omega,\omega,\lambda) + S(\vartheta,\vartheta,\lambda)$

Then *(M, S)* is know as a S-metric space.

2.2 Lemma [5] :

In a S-metric space, we have $S(v,v,\omega) = S(\omega,\omega,v)$, for all $v,\omega \in M$

2.3 Definition [5] :

Let *(M, S)* be a S-metric space. A sequece (v_n) in M is said to be convergent, if there is $v \in M$ such that $S(v_n, v_n, v) \to 0$ as $n \to \infty$. That is, for each $U' \rangle 0$ there exists a $n_0 \in \bullet$ such that for all $n \geq \mathrm{n}_0$, we have $S(v_n, v_n, v) < U$. We denote it by $\lim_{n\to\infty} v_n = v$.

2.4 Definition [5] :

Let *(M, S)* be a S-metric space. A sequece (v_n) in M is said to be Cauchy sequence, if for each $\dot{U} > 0$ there exists a $n_0 \in \bullet$ such that $S(v_n, v_n, v_m) < U$ for any n, $\mathrm{m} \geq \mathrm{n}_0$

2.5 Definition [5] :

A S-metric space (M, S) is said to be complete, if every Cauchy sequence in M converges to some point in it.

2.6 Lemma [5]:

In a S-metric space (M, S), if there exist two sequences (v_n) and (ω_n) such that $\lim_{n\to\infty} v_n = v$ and $\lim_{n\to\infty} \omega_n = \omega$ then $\lim_{n\to\infty} S(v_n, v_n, \omega_n) = S(v,v,\omega)$

2.7 Definition [8]:

A mapping $f : [0,\infty) \to [0,\infty)$ is called as a comparison function, if $f(\tau_1) \leq f(\tau_2)$ whenever $\tau_1 \leq \tau_2$ and $(f^n(\tau))$ converges to 0, for every $\tau > 0$.

2.8 Remark [8]:

If $f : [0,\infty) \to [0,\infty)$ is a comparison function, then $f(\tau) \leq \tau$, for all $\tau > 0$, $f(0) = 0$, and f is right continuous at 0.

2.9 Definition [2] :

A comparison function f is called as a strong comparison function, if the series $\sum_{n=1}^{\infty} f^n(\tau)$ converges, for every $\tau > 0$.

2.10 Definition[2] :

In a S-metric space (M, S), two self-maps L, K are called weakly compatible, if $LKv = KLv$ whenever $Lv = Kv$, for any $v \in M$.

2.11 Definition[3] :

In a S-metric space (M, S), two self-maps L, K are said to satisfy (E.A)-property, if there exists a sequence (v_n) in M such that $\lim_{n\to\infty} Lv_n = \lim_{n\to\infty} Kv_n = \tau$, for some $\tau \in M$.

2.12 Definition[9] :

In a S -metric space *(M, S)*, two pairs of self-maps *(L, H)* and *(K, J)* of M are said to satisfy common (E.A)-property, if there exist two sequences (v_n) and (ω_n) in M such that $\lim_{n\to\infty} Lv_n = \lim_{n\to\infty} Hv_n = \lim_{n\to\infty} K\omega_n = \lim_{n\to\infty} J\omega_n = \tau$, for some $\tau \in M$.

2.13 Remark[7] :

Let A_f be the set of all functions $g : [0,\infty)^4 \to [0,\infty)$ which satisfy the following properties.

1. g is continuos
2. for any $\rho, \dot{O} \in [0,\infty)$, if

(i) $\rho \leq g(\rho, \dot{O}, \dot{O}, \dot{O})$ or
(ii) $\rho \leq g(\dot{O}, \rho, \dot{O}, \dot{O})$ or
(iii) $\rho \leq g(\dot{O}, \dot{O}, \rho, \dot{O})$
Then $\rho \leq f(\dot{O})$, where f is a strong comparison function.

3 Results and Discussion

In this section, we prove a common fixed-point theorem for four self-maps of a complete S-metric space which satisfy (E.A)-property. With the aid of strong comparison function, a unique common fixed point is obtained.

3.1 Theorem:

Let *(M,S)* be a complete S-metric space and F be a non-empty subset of M. Let $L, K : F \to F$ be self-maps. If there exist some $g \in A_f$ such that for all $v, \omega \in M$

$$S(Lv,Lv,K\omega) \leq g\left(S(Hv,Hv,J\omega), S(Hv,Hv,Lv), S(K\omega,K\omega,J\omega), \frac{S(K\omega,K\omega,J\omega)S(J\omega,J\omega,Lv)}{1+S(Hv,Hv,K\omega)}\right) \tag{1}$$

where $H, J : F \to M$ satisfies the following conditions

(i) $L(F) \subseteq J(F)$ and $K(F) \subseteq H(F)$
(ii) the pairs(L,H) and (K,J) satisfying common (E.A)-property,
(iii) the pairs (L,H) and (K,J) are weakly compatible,
(iv) $H(F)$ or $J(F)$ is closed in *M*,
then *H, J, K* and *L* have a unique common fixed point in *M*.

Proof. From the (E.A)-property of the pairs (L,H) and (K,J), there exist two sequences (v_n) and (ω_n) in F such that $\lim_{n\to\infty} Lv_n = \lim_{n\to\infty} Hv_n = \lim_{n\to\infty} K\omega_n = \lim_{n\to\infty} J\omega_n = \eta$, for some $\eta \in F$. We have $J\xi = \eta$, for some $\xi \in F$
We now claim that $K\xi = \eta$, for if $K\xi \neq \eta$
on substituting $v = v_n$ and $\omega = \xi$ in (1), we otain

$$S(Lv_n,Lv_n,K\xi) \leq g\left(\begin{array}{c} S(Hv_n,Hv_n,J\xi), S(Hv_n,Hv_n,Lv_n), S(K\xi,K\xi,J\xi), \\ \dfrac{S(K\xi,K\xi,J\xi)S(J\xi,J\xi,Lv_n)}{1+S(Hv_n,Hv_n,K\xi)} \end{array}\right) \tag{2}$$

On taking the limit as $n \to \infty$

$$\begin{aligned} S(\eta,\eta,K\xi) &\leq g\left(S(\eta,\eta,\eta), S(\eta,\eta,\eta), S(K\xi,K\xi,\eta), \frac{S(K\xi,K\xi,\eta)S(\eta,\eta,\eta)}{1+S(\eta,\eta,K\xi)}\right) \\ &= g(0,0,S(\eta,\eta,K\xi),0) \end{aligned}$$

from condition (iii) of Remark 2.13., $S(\eta,\eta,K\xi) \leq f(0) = 0$
$S(\eta,\eta,K\xi) = 0$ which gives $K\xi = \eta$.
Hence

$$K\xi = J\xi = \eta \tag{3}$$

We have $H\varsigma = \eta$, for some $\varsigma \in F$ as $K(F) \subseteq H(F)$.
It follows that $L\varsigma = \eta$. For if $L\varsigma \neq \eta$,
on taking $v = \varsigma, \omega = \omega_n$ in (1), we have

$$S(L\zeta,L\zeta,K\omega_n) \leq g\left(\begin{array}{c} S(H\zeta,H\zeta,J\omega_n), S(H\zeta,H\zeta,L\zeta), S(K\omega_n,K\omega_n,J\omega_n), \\ \dfrac{S(K\omega_n,K\omega_n,J\omega_n)\,S(J\omega_n,J\omega_n,L\zeta)}{1+S(H\zeta,H\zeta,K\omega_n)} \end{array}\right) \tag{4}$$

On taking the limit as $n \to \infty$
$S(L\varsigma,L\varsigma,\eta) \leq g\left(S(\eta,\eta,\eta), S(\eta,\eta,L\varsigma), S(\eta,\eta,\eta), \dfrac{S(\eta,\eta,\eta)S(\eta,\eta,L\varsigma)}{1+S(\eta,\eta,\eta)}\right)$
$= g(0, S(L\varsigma,L\varsigma,\eta), 0, 0)$
Again from condition (ii) of Remark 2.13., $S(L\varsigma,L\varsigma,\eta) \leq f(0) = 0$
from which we get $S(L\zeta,L\zeta,\eta) = 0$, giving
$L\varsigma = \eta$
Hence

$$L\varsigma = H\varsigma = \eta \tag{5}$$

Therefore from (3) and (5), we get
$K\xi = J\xi = L\varsigma = H\varsigma = \eta$
And from the weakly compatibility of the pairs *(K, J)* and *(L, H)*, we obtain
$K\eta = J\eta$ and $L\eta = H\eta$
We now prove that $\eta$is a fixed point of L.
On taking $v = \eta, \omega = \xi$ in (1), we have
$S(L\eta,L\eta,K\xi) \leq g\left(S(H\eta,H\eta,J\xi), S(H\eta,H\eta,L\eta), S(K\xi,K\xi,J\xi), \dfrac{S(K\xi,K\xi,J\xi)S(J\xi,J\xi,L\eta)}{1+S(H\eta,H\eta,K\xi)}\right)$
$S(L\eta,L\eta,\eta) \leq g\left(S(L\eta,L\eta,\eta), S(L\eta,L\eta,L\eta), S(\eta,\eta,\eta), \dfrac{S(\eta,\eta,\eta)S(\eta,\eta,L\eta)}{1+S(L\eta,L\eta,\eta)}\right)$
$= g(S(L\eta,L\eta,\eta), 0, 0, 0)$.
From condition (i) of Remark 2.13.,
We get $S(L\eta,L\eta,\eta) \leq f(0) = 0$
which gives $L\eta = \eta$
Hence

$$H\eta = L\eta = \eta \tag{6}$$

We now prove $K\eta = \eta$
On taking $v = \varsigma, \omega = \eta$ in (1), we have
$S(L\varsigma,L\varsigma,K\eta) \leq g\left((H\varsigma,H\varsigma,J\eta), S(H\varsigma,H\varsigma,L\varsigma), S(K\eta,K\eta,J\eta), \dfrac{S(K\eta,K\eta,J\eta)S(J\eta,J\eta,L\varsigma)}{1+S(H\varsigma,H\varsigma,K\eta)}\right)$
$S(\eta,\eta,K\eta) \leq g\left(S(\eta,\eta,K\eta), S(\eta,\eta,\eta), S(K\eta,K\eta,K\eta), \dfrac{S(K\eta,K\eta,K\eta)S(K\eta,K\eta,\eta)}{1+S(\eta,\eta,K\eta)}\right)$
$= g(S(\eta,\eta,K\eta), 0, 0, 0)$.
From condition (i) of Remark 2.13., $S(\eta,\eta,K\eta) \leq f(0) = 0$
giving $S(\eta,\eta,K\eta) = 0$
from which we get $K\eta = \eta$

Hence

$$J\eta = K\eta = \eta \tag{7}$$

From (6) and (7), it follows that $H\eta = J\eta = K\eta = L\eta = \eta$

proving η is a common fixed point of *H, J, K* and *L*.

For if, $\zeta(\neq \eta) \in F$ is such that

$H\zeta = J\zeta = K\zeta = L\zeta = \zeta$

On substituting $v = \eta, \omega = \zeta$ in (1), we get

$S(L\eta, L\eta, K\zeta) \leq g\left(S(H\eta, H\eta, J\zeta), S(H\eta, H\eta, L\eta), S(K\zeta, K\zeta, J\zeta), \frac{S(K\zeta,K\zeta,J\zeta)S(J\zeta,J\zeta,L\eta)}{1+S(H\eta,H\eta,K\zeta)}\right)$

which gives

$$S(\eta,\eta,\zeta) \leq g\left(S(\eta,\eta,\zeta), S(\eta,\eta,\eta), S(\zeta,\zeta,\zeta), \frac{S(\zeta,\zeta,\zeta)S(\zeta,\zeta,\eta)}{1+S(\eta,\eta,\zeta)}\right)$$

$$= g(S(\eta,\eta,\zeta),0,0,0)$$

Again from condition (i) of Remark 2.13., $S(\eta,\eta,\zeta) \leq f(0) = 0$

which imply $S(\eta,\eta,\zeta) = 0$

proving $\eta = \zeta$

As an illustration, we have the following example.

3.2 Example:

Let M = [2, 11]. If $S(v,\omega,\vartheta) = d(v,\vartheta) + d(\omega,\vartheta)$ where $d(v,\omega) = \max\{v,\omega\}$, then *S* is a *S*-metric on *M*. Now define mappings *H, J, K* and *L* on *M* such that

$L(v) = \begin{cases} 2 & \text{if } v \in \{2\} \cup (3,11] \\ 3 & \text{if } v \in (2,3] \end{cases}$, $K(v) = \begin{cases} 2 & \text{if } v \in \{2\} \cup (3,11] \\ 4 & \text{if } v \in (2,3] \end{cases}$

$H(v) = \begin{cases} 2 & \text{if } v = 2 \\ 4+v & \text{if } v \in (2,3] \\ \frac{v+1}{2} & \text{if } v \in (3,11] \end{cases}$, $J(v) = \begin{cases} 2 & \text{if } v = 2 \\ 7 & \text{if } v \in (2,3] \\ \frac{v+1}{2} & \text{if } v \in (3,11] \end{cases}$

Let $g(v,\omega,\vartheta,\gamma) = \max\{v,\omega,\vartheta,\gamma\}$ and $f(\tau) = \frac{3\tau}{4}$.

On taking $v_n = 3 + \frac{1}{n}$ and $\omega_n = 2$, we have

$\lim_{n\to\infty} Lv_n = \lim_{n\to\infty} Hv_n = \lim_{n\to\infty} K\omega_n = \lim_{n\to\infty} J\omega_n = 2$,

proving that the pairs *(L.H) (K,J)* and satisfy common (E.A)-property. (L.H) and (K,J) commute at , which is the coincidence point.

$L(M) = \{2,3\}, J(M) = [2,6] \cup \{7\}, K(M) = \{2,4\}$ and $H(M) = [2,7]$.

Observe that $L(M) \subseteq J(M)$ and $K(M) \subseteq H(M)$

We now verify the contractive condition (1) stated in Theorem 3.1 in various cases as shown below

Case (i) Let $v,\omega \in (2,3]$

$S(Lv,Lv,K\omega) = 8, S(Hv,Hv,J\omega) = 14, S(Hv,Hv,Lv) = 14, S(K\omega,K\omega,J\omega) = 14$

$$\frac{S(K\omega,K\omega,J\omega)S(J\omega,J\omega,Lv)}{1+S(Hv,Hv,K\omega)} = \frac{196}{15}$$

which yields $8 \leq g\left(14,14,14,\frac{196}{15}\right)$

Case (ii) Let $v \in (2,3], \omega \in (3,11]$

$S(Lv,Lv,K\omega) = 6, S(Hv,Hv,J\omega) = 14, S(Hv,Hv,Lv) = 14, S(K\omega,K\omega,J\omega) = 12$

$$\frac{S(K\omega,K\omega,J\omega)S(J\omega,J\omega,Lv)}{1+S(Hv,Hv,K\omega)} = \frac{144}{15}$$

which yields $8 \leq g\left(14,14,12,\frac{144}{15}\right)$

In other cases, that is either if $v,\omega \in (3,11]$, or if $v \in (3,11], \omega \in (2,3]$ can be verified in a similar manner.

Clearly 2 is the unique common fixed point of *H, J, K* and *M*.

4 Conclusion

This article establishes a common fixed point theorem for four self-maps utilizing the (E.A) property and the weak compatibility feature of self-maps. By utilizing the common limit range (CLR) property, various fixed point theorems were established. However, utilizing the (E.A)-property, we are able to prove the existence and uniqueness of the common fixed point for four self-maps with a relatively weaker hypothesis in our current Theorem 3.1. Our theorem strengthens and broadens the one proven in[7], since it is based on the generalized metric space known as the S-metric space. Additionally, example 3.2 is given to illustrate the findings.

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