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RESEARCH ARTICLE

A fourth order orthogonal spline collocation method Interface boundary value problem

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Abstract

Objective: A higher order numerical scheme for two-point boundary interface problem with Dirichlet and Neumann boundary condition on two different sides is propounded. **Methods:** Orthogonal cubic spline collocation techniques have been used (OSC) for the two-point interface boundary value problem. To approximate the solution a piecewise Hermite cubic basis functions have been used. **Findings:** Remarkable features of the OSC are accounted for the numerous applications, theoretical clarity, and convenient execution. The stability and efficiency of orthogonal spline collocation methods over B-splines have made the former more preferable than the latter. As against finite element methods, determining the approximate solution and the coefficients of stiffness matrices and mass is relatively fast as the evaluation of integrals is not a requirement. The systematic incorporation of boundary and interface conditions in OSC adds to the list of advantages of preferring this method. **Novelty:** As against the existing methodologies it becomes clear from our findings that OSC is dominantly computationally superior. A computational treatment has been implemented on the two-point interface boundary value problem with super-convergent results of derivative at the nodal points, being the noteworthy finding of the study.

Keywords: Helmholtz problem; Orthogonal spline collocation techniques (OSC); Discontinuous data; Super Convergence; Piecewise cubic Hermite basis functions; Almost block diagonal (ABD) structure

1 Introduction

The1D- Helmholtz equation under consideration is:

$$y'' + k^2 q(x)\, y = f(x), \quad x \in [a,b] \tag{1.1}$$

with boundary conditions

$$\alpha_a y(a) + \beta_a y'(a) = g_0, \quad \alpha_b y(b) + \beta_b y'(b) = g_1 \tag{1.2}$$

where α_a, β_a ,α_b , β_b, g_0, g_1 are known constants and k^2 is the wave number. We assume the coefficient $q(x)$ to be piece-wise constant or piece-wise continuous with finite jump across the interface $= x_i$, $x_i \in (a,b)$.

For convenience, we assume that $\omega^2 = k^2 q(x)$ which is a piece-wise constant or piece-wise continuous across the interface $x = x_i$, and assume that solution $y(x)$ satisfies the natural jump conditions across the interface $(y] = 0$, $\left(y'\right] = 0$.

The jump conditions across the interface are:

$[y]_{x=x_i} = \lim_{x\to x_i^+} y(x) - \lim_{x\to x_i^-} y(x) = y^+(x_i) - y^-(x_i)$.

Let

$$\pi\ : a = x_0 < x_1 < x_2 < \cdots < x_N < x_{N+1} = b$$

denote a partition of I, and set $I_j = \left(x_{j-1},\, x_j\right],\quad j = 1,\, \ldots,\, N+1, h_j = x_j - x_{j-1}\ and\ h = max_j h_j$.

The Helmholtz equation is used in many physical applications such as acoustics, elastic waves, and electromagnetic waves. The present study intends to provide an efficient numerical skill for Helmholtz problem.

Existing literature of theoretical and numerical treatment to Helmholtz equation using finite difference methods[1–3], finite element methods[4] and for existence uniqueness results can be found at[5–7]. 1D and 2D Helmholtz equation have been treated by Xiufang Feng[8] and Xiufang Feng et al.[9] respectively by using high order compact finite difference methods.

This paper treats 1D-Helmholtz equation with piece-wise constant or piecewise continuous functions by employing OSC to it. The stability and efficiency of orthogonal spline collocation methods over B-splines have made the former more preferable than the latter. As against finite element methods, determining the approximate solution and the coefficients of stiffness matrices and mass is relatively fast as the evaluation of integrals is not a requirement. The systematic incorporation of boundary and interface conditions in OSC adds to the list of advantages of preferring this method.

We show that the OSC handle the interface conditions effectively with less discretization. To accomplish the fourth-order accuracy, we utilize piece-wise Hermite cubic basis functions for approximating the solution. This article can be outlined as: Section 2 uses OSC to approximate the solution. Section 3 deals with numerical experiments. Discontinuous data has been used and the solution has been approximated using piece-wise Hermite cubic basis functions. Grid refinement analysis is performed and the order of convergence for —-norm and —-norm is found. Section 4 hosts the conclusion.

2 Orthogonal spline collocation methods

Here, we employ OSC to approximate the solutions of interface boundary value problem (1.1).

Let

$H^m(I) = \{v : v \in C^{m-1}(I)\ \ and\ v^m\ is\ a\ piecewise\ continuous\ function\ on\ I\}$,

with norm

$$\|v\|_{H^m(I)} = \left(\sum_{i=1}^{m} \|v_i\|^2_{L^2(I)}\right)^{\frac{1}{2}}$$

,

where,

$$\|v\|_{L^2(I)} = \left(\int |v(x)|^2\right)^{\frac{1}{2}} = \|v\|_{H^0(I)}.$$

Also set

$H_0^1(I) = H^1 \cap ((v|\, v(a) = v(b) = 0\}$.

Let

$\pi\ : a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b,$

represents a partition of I, and set

$I_j = \left(x_{j-1},\, x_j\right],\quad j = 1,2,\, \ldots,\, N,\qquad h_j = x_j - x_{j-1}\ \ and\ \ h = max_j h_j.$

we assume that$x_i \in \pi$. In the OSC, the approximate solution, y_h, lies in a space C^1 piece-wise polynomials of degree ≥ 3. Here we choose the space of piece-wise Hermite cubics, $M_1^3(\pi)$ as:

$M_1^3(\pi) = \left((v|\, v \in C^1(I),\, (v|_{I_j} \in P_3,\ \ j = 1,2,\ldots,\, N\right\},$

where $C^1(I)$ denotes the space of functions which are one times continuously differentiable on I,P_3 represents the set of polynomials of degree ≤ 3 and $(v|_{I_j}$ denotes the restriction of the function v to the interval I_j.

We denote by $M_1^{3,0}(\pi)$ the space

$M_1^3(\pi) \cap ((v|\, v(a) = v(b) = 0\}$.

It is to see that $M_1^3(\pi)$ and $M_1^{3,0}(\pi)$ are linear spaces of dimensions $2N+2$ and $2N$, respectively.

We consider the collocation points$(\xi_j\}_{j=1}^{2N}$, where

$$\xi_{2i-1} = x_{i-1} + \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right)h_i, \quad and \quad \xi_{2i} = x_{i-1} + \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right)h_i, \quad i = 1, 2, \ldots, N.$$

These are the composite two-point Gauss quadrature points.

We now introduce the standard basis for the space $M_1^3(\pi)$. We define the functions $v_j(x), s_j(x) \in M_1^3(\pi)$, $i, j = 0, 1, \ldots, N$, by

$v_j(x_i) = \delta_{ij},\ v_j'(x_i) = 0$ and $s_j(x_i) = 0,\ s_j'(x_i) = \delta_{ij},\ i, j = 0, \ldots, N,$

where $\delta_{ij} = 1,\ if\ i = j$ and $\delta_{ij} = 0,\ if\ i \neq j$. Then the set $(v_i\}_{i=0}^{N} \cup (s_i\}_{i=0}^{N}$ forms a basis for $M_1^3(\pi)$, which we order in the form $(v_0, s_0, v_1, s_1, \ldots, v_N, s_N\}$.

The function v_i and s_i are called value function and slope function, respectively, associated with the node $x_i \in \pi$. With the basis $(v_0, s_0, v_1, s_1, \ldots, v_N, s_N\}$ for $M_1^3(\pi)$, we set

$$y_h(x) = \sum_{j=0}^{N} \{\alpha_j v_j(x) + \beta_j s_j(x)\} \tag{2.1}$$

where,

$\alpha_j = y_h(x_j),\ \beta_j = y_h'(x_j),\quad j = 0, 1 \ldots\ldots., N.$

The coefficients are then evaluated with the restriction that y_h satisfies (1.1) at the collocation points$(\xi_j\}_{j=1}^{2N}$, and the boundary conditions (1.2) so that

The orthogonal spline collocation approximation for problem (1.1) - (1.2) is stated as:

Approximate $y_h \in M_1^3(I)$ so that

$$\alpha_a y(a) + \beta_a y'(a) = g_a,$$

$$L y_h(\xi_i) = f(\xi_i), \quad i = 1, 2, \ldots, 2N \tag{2.2}$$

$$\alpha_b y(b) + \beta_b y'(b) = g_b.$$

As only four basis functions $v_{i-1}, s_{i-1}, v_i, s_i$ are non-zero on $[x_{i-1}, x_i]$, the coefficient matrix of collocation equations structures out as

$$\begin{bmatrix} D_0 & & & & & \\ S_1 & T_1 & & & & \\ & S_2 & T_2 & & & \\ & & \ddots & \ddots & & \\ & & & S_N & T_N \\ & & & & D_1 \end{bmatrix} \tag{2.3}$$

where $D_0 = [\alpha_a, \beta_a]$, $D_1 = [\alpha_b, \beta_b]$, and, for $j = 1, 2, \ldots\ldots.., N$,

$$S_i = \begin{pmatrix} Lv_{i-1}(\xi_{2i-1}) & Ls_{i-1}(\xi_{2i-1}) \\ Lv_{i-1}(\xi_{2i}) & Ls_{i-1}(\xi_{2i}) \end{pmatrix}, \quad T_i = \begin{pmatrix} Lv_i(\xi_{2i-1}) & Ls_i(\xi_{2i-1}) \\ Lv_i(\xi_{2i}) & Ls_i(\xi_{2i}) \end{pmatrix}.$$

3 Numerical experiments

In order to arrive at an approximate solution, piece-wise Hermite cubic basis functions will be considered for the experimentation and as for the determination of the order of convergence of the numerical method we will emphasize on grid refinement analysis.

The approximate solution $y_h(x) \in M_1^3$ on each subinterval $(x_{i-1},\, x_i]\,,\quad i = 1,2,\ldots,N$ is:

$$y_h(x) = \sum_{j=0}^{N} \left\{ \alpha_j v_j(x) + \beta_j s_j(x) \right\}, \tag{3.1}$$

where,

$\alpha_j = y_h(x_j)\,,\;\; \beta_j = y_h'(x_j),\quad j = 0,\ldots\ldots,\, N.$

Since

$v_j(x_i) = \delta_{ij},\;\; v_j'(x_i) = 0$ and $s_j(x_i) = 0,\;\; s_j'(x_i) = \delta_{ij},\;\; i,j = 0,\ldots,N,$

where $\delta_{ij}$is the kronecker delta function with $\delta_{ij} = 1,\; if\ i = j$ and $\delta_{ij} = 0,\; if\ i \neq j$. The expressions for value functions and slope functions, we refer to[3].

Taking derivatives of (3.1) wrt x, we have,

$$y_h'(x) = \sum_{j=0}^{N} \left\{ \alpha_j v_j'(x) + \beta_j s_j'(x) \right\}$$

and

$$y_h''(x) = \sum_{j=0}^{N} \left\{ \alpha_j v_j''(x) + \beta_j s''_j(x) \right\} \tag{3.2}$$

$\{\xi_i\}_1^{2N}$ are the collocation points on $(x_{i-1},x_i]\,,\quad i = 1,2,\ldots,N$ are two-point Gauss-Legendre quadrature points defined by

$$\xi_{2i-1} = x_{i-1} + \frac{1}{2}\left(1 - \frac{1}{\sqrt{3}}\right) h_i, \quad and \quad \xi_{2i} = x_{i-1} + \frac{1}{2}\left(1 + \frac{1}{\sqrt{3}}\right) h_i, \quad i = 1,2,\ldots,N.$$

Now substituting equations (3.1) - (3.2) in (1.1) and the resulting equation, we calculate on $(x_{i-1},x_i]$ at $x = \xi_1$, we get

$$\alpha_0 v''_0(\xi_1) + \beta_0 s''_0(\xi_1) + \alpha_1 v''_1(\xi_1) + \beta_1 s''_1(\xi_1) + \omega^2 \alpha_0 v_0(\xi_1) + \omega^2 \beta_0 s_0(\xi_1) + \omega^2 \alpha_1 v_1(\xi_1) + \omega^2 \beta_1 s_1(\xi_1) = f(\xi_1) \tag{3.3}$$

Similarly, at $x = \xi_2$ we have the following expression

$$\alpha_0 v_0''(\xi_2) + \beta_0 s''_0(\xi_2) + \alpha_1 v_1''(\xi_2) + \beta_1 s''_1(\xi_2) + \omega^2 \alpha_0 v_0(\xi_2) + \omega^2 \beta_0 s_0(\xi_2) + \omega^2 \alpha_1 v_1(\xi_2) + \omega^2 \beta_1 s_1(\xi_2) = f(\xi_2). \tag{3.4}$$

Let $x = x_i$be the interface point on I = (a, b). By observing equations (3.3) - (3.4) the collocation equations on the sub-intervals $[x_0, x_1]$, $[x_1, x_2]$, $\cdots$, $[x_{i-1}, x_i]$, can be expressed in the matrix form as

$$A_i y_i + B_i y_{i+1} = f_i \tag{3.5}$$

where A_i and B_i structure out as:

$$A_i = \left(\begin{array}{cc} v''_{i-1}(\xi_{2i-1}) + \omega_-^2 v_{i-1}(\xi_{2i-1}) & s''_{i-1}(\xi_{2i-1}) + \omega_-^2 s_{i-1}(\xi_{2i-1}) \\ v''_{i-1}(\xi_{2i}) + \omega_-^2 v_{i-1}(\xi_{2i}) & s''_{i-1}(\xi_{2i-1}) + \omega_-^2 s_{i-1}(\xi_{2i-1}) \end{array} \right],$$

$$B_i = \begin{bmatrix} v''_i(\xi_{2i-1}) + \omega_-^2 v_i(\xi_{2i-1}) & s''_i(\xi_{2i-1}) + \omega_-^2 s_i(\xi_{2i-1}) \\ v''_i(\xi_{2i}) + \omega_-^2 v_i(\xi_{2i}) & s''_i(\xi_{2i-1}) + \omega_-^2 s_i(\xi_{2i-1}) \end{bmatrix}$$

f_i structures as $\begin{bmatrix} f_i^-(\xi_{2i-1}) \\ f_i^-(\xi_{2i}) \end{bmatrix}$ and $y_i = [\alpha_{i-1},\, \beta_{i-1}]^T,\; y_{i+1} = [\alpha_i,\, \beta_i]^T,\; i = 1,2,\ldots,N.$

In the similar manner, the collocation equations on the sub-intervals $[x_i, x_{i+1}]$, $[x_{i+1}, x_{i+2}]$, $\cdots$ $[x_{N-1}, x_N]$, can be expressed in the form of a matrix as

$$A_i y_i + B_i y_{i+1} = f_i. \tag{3.6}$$

where A_i and B_i structure out as:

$$A_i = \begin{bmatrix} v''_{i-1}(\xi_{2i-1}) + \omega_+^2 v_{i-1}(\xi_{2i-1}) & s''_{i-1}(\xi_{2i-1}) + \omega_+^2 s_{i-1}(\xi_{2i-1}) \\ v''_{i-1}(\xi_{2i}) + \omega_+^2 v_{i-1}(\xi_{2i}) & s''_{i-1}(\xi_{2i-1}) + \omega_+^2 s_{i-1}(\xi_{2i-1}) \end{bmatrix},$$

$$B_i = \begin{bmatrix} v''_{i}(\xi_{2i-1}) + \omega_+^2 v_{i}(\xi_{2i-1}) & s''_{i}(\xi_{2i-1}) + \omega_+^2 s_{i}(\xi_{2i-1}) \\ v''_{i}(\xi_{2i}) + \omega_+^2 v_{i}(\xi_{2i}) & s''_{i}(\xi_{2i-1}) + \omega_+^2 s_{i}(\xi_{2i-1}) \end{bmatrix}$$

f_i structures as $\begin{bmatrix} f_i^+(\xi_{2i-1}) \\ f_i^+(\xi_{2i}) \end{bmatrix}$.

Combining (3.5) - (3.6), we obtain an ABD linear system of order 2 N + 2 for

$$\begin{bmatrix} L_b & & & & & & & \\ A_1 & B_1 & & & & & & \\ & A_2 & B_2 & & & & & \\ & & \ddots & & & & & \\ & & A_i & B_i & & & & \\ & & & A_{i+1} & B_{i+1} & & & \\ & & & & \ddots & & & \\ & & & & & A_N & B_N \\ & & & & & & R_b \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{i-1} \\ \vdots \\ y_N \\ y_{N+1} \end{bmatrix} = \begin{bmatrix} g_0 \\ f_1 \\ \vdots \\ f_i \\ \vdots \\ f_N \\ g_1 \end{bmatrix}$$

where L_b and R_b are contributed by lateral boundaries, left and right. The matrix system has been solved using the almost block diagonal (ABD) solver of MATLAB. While existing methods require more number operations ($\approx n^3$) to achieve fourth-order accuracy, the OSC requires only 'n' operations.

4 Numerical Example

Example 1: The problem under consideration is as follows:

$$y''(x) + \omega^2 y(x) = f(x), \qquad x \in (-\pi, \pi)$$

with boundary conditions i.e., Dirichlet in one side and Neumann on the other side $y(-\pi) = 0, \;\; y'(\pi) = 0$

where

$$f(x) = \begin{cases} 4(x-\pi)\cos x + \left(\omega_-^2 - 1\right)(x-\pi)^2 \sin x + 2\sin x, & x \in [-\pi, 0], \\ 4(x-\pi)\cos x + \left(\omega_+^2 - 1\right)(x-\pi)^2 \sin x + 2\sin x, & x \in [0, \pi]. \end{cases}$$

The exact solution is given by $y(x) = (x-\pi)^2 sinx$.

The order of convergence computes out to be:

$$\text{Order} \approx \frac{\log\left(\frac{\left\|y - y_{h_i}\right\|_{L^\infty}}{\left\|y - y_{h_{i+1}}\right\|_{L^\infty}}\right)}{\log\left(\frac{h_i}{h_{i+1}}\right)}, \quad i = 1, 2, \ldots, 5$$

where y: exact solution, y_{h_i}: numerical solution with step size h_i.

The following table describes the errors in max-norm and order of convergence at nodal points.

Table 1. L^{∞}**error for Example 1**

	$\omega^2=1,\ \omega^2=25$		$\omega^2=5,\ \omega^2=100$	
N	–	order	–	order
8	2.2685e-01		5.5156e-02	
16	1.3846e-03	7.3561e+00	3.3897e-03	4.0243e+00
32	1.0897e-04	3.6674e+00	1.6551e-04	4.3562e+00
64	6.8095e-06	4.0003e+00	1.1538e-05	3.8424e+00
128	4.2561e-07	3.9999e+00	7.2997e-07	3.9824e+00
256	2.6628e-08	3.9985e+00	4.5619e-08	4.0001e+00

Table 2. L^{∞}**error of derivative for Example 1**

	$\omega^2=1,\ \omega^2=25$		$\omega^2=5,\ \omega^2=100$	
N	–	order	–	order
8	3.3212e-01		8.6859e-02	
16	6.7995e-03	5.6102e+00	7.2906e-03	3.5746e+00
32	4.2674e-04	3.9940e+00	1.1388e-03	2.6786e+00
64	2.7118e-05	3.9761e+00	8.8430e-05	3.6868e+00
128	1.7192e-06	3.9794e+00	5.6226e-06	3.9752e+00
256	1.0749e-07	3.9994e+00	3.5136e-07	4.0002e+00

N.B: Since it is a numerical scheme, so the convergence depends upon the large value N. Initially it may get some deviation but at the higher value of N, it will converge to 4^{th} order, which has been inferred from the above mentioned table.

Example 2: We consider the following problem

$$y''(x)\ +\ (p'(x)/p(x))y'(x)\ +\ (1/p(x))y(x)\ =\ f\,(x)/p(x),\qquad x\in(-1,1)$$

With boundary conditions i.e., Dirichlet on one side and Neumann on the other side

$y(-1)=-sin(1),\ \ y^{'}(1)=cos(1),$

where,

$$p(x)=\begin{cases} x, & x\in[-1,0] \\ (1+x), & x\in[0,1]\end{cases}$$

The exact solution is given by $y(x)=xsinx$.

The expression for $f(x)$ can be computed using $y(x)=xsinx$.

Table 3. L^{∞} **error of value and derivative for Example 2**

	$p_1=x$	$p_2=(1+x)$	$p_1=x$	$p_2=(1+x)$
N	–	order	–	order
8	1.4615e-05		1.1332e-05	
16	8.7474e-07	4.0624e+00	6.7401e-07	4.0714e+00
32	5.3506e-08	4.0311e+00	4.1092e-08	4.0358e+00
64	3.3080e-09	4.0157e+00	2.5361e-09	4.0182e+00
128	2.0561e-10	4.0079e+00	1.5749e-10	4.0092e+00
256	1.2815e-11	4.0040e+00	9.8099e-12	4.0049e+00

5 Conclusion

An OSC to 1D- Helmholtz equation with discontinuous coefficients has been established in the study. Discontinuous data has been experimented on, using numerical methodologies. Fourth-order convergence at the grid points for $\|y-y_h\|_{L^{\infty}}$-norm and $\|y' - y'_h\|_{L^\infty}$-norm has been found. As against the methods that exist, OSC handles the discontinuous coefficients potently and gives optimal order of convergence for $\|y - y_h\|_{L^\infty}$-norm and super-convergent result for $\|y - y_h\|_{L^\infty}$-norm. Despite having theorized and having computed the OSC for a single point in the interface we can extend our theory to a finite set of points.

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References

1) Bao G, Sun W. A Fast Algorithm for the Electromagnetic Scattering from a Large Cavity. *SIAM Journal on Scientific Computing*. 2005;27(2):553–574. Available from: https://dx.doi.org/10.1137/s1064827503428539.
2) Hoang-Long N, Taguchi D. On the Euler-Maruyama Approximation for One Dimensional Stochastic Differential Equations with Irregular Coefficients. *IMA Journal of Numerical Analysis*. 2017;37(4):1864–1883. doi:10.1093/imanum/drw058.
3) Wang Y, Du K, Sun W. A second order method for the electromagnetic scattering from a large cavity. *Numerical Mathematics Theory, Methods and Applications*. 2008;1(4):357–382. Available from: https://www.global-sci.org/v1/nmtma/volumes/v1n4/pdf/14-357.pdf.
4) Ito K, Qiao Z, Toivanen J. A domain decomposition solver for acoustic scattering by elastic objects in layered media. *Journal of Computational Physics*. 2008;227(19):8685–8698. Available from: https://dx.doi.org/10.1016/j.jcp.2008.06.015.
5) Aitbayev R. Existence and uniqueness for a two-point interface boundary value problem. *Electronic Journal of Differential Equations*. 2013;2013(242):1–15. Available from: http://ejde.math.txstate.edu.
6) Bhal SK, Danumjaya P. A fourth-order orthogonal spline collocation solution to 1D-Helmholtz equation with discontinuity. *The Journal of Analysis*. 2019;27(2):377–390. Available from: https://dx.doi.org/10.1007/s41478-018-0082-9.
7) Kumar BS, Danumjaya P, Kumar A. A fourth-order orthogonal spline collocation method to fourth-order boundary value problems. *International Journal for Computational Methods in Engineering Science and Mechanics*. 2019;20(5):460–470. Available from: https://dx.doi.org/10.1080/15502287.2019.1600070.
8) Feng X. A high-order compact scheme for the one-dimensional Helmholtz equation with a discontinuous coefficient. *International Journal of Computer Mathematics*. 2012;89(5):618–624. Available from: https://dx.doi.org/10.1080/00207160.2011.648184.
9) Feng X. High Order Compact Finite Difference Schemes for the Helmholtz Equation with Discontinuous Coefficients. *Journal of Computational Mathematics*. 2011;29(3):324–340. Available from: https://dx.doi.org/10.4208/jcm.1010-m3204.