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Secure Domination Cover Pebbling Number of Join of graphs

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Abstract

Objectives: To find the secure domination cover pebbling number for the join of two graphs $G(p, q)$ and $G'(p', q')$. **Methods:** We define Secure domination cover pebbling number, $f_{sdp}(G)$, of a graph G as the minimum number of pebbles that must be placed on $V(G)$ such that, after a sequence of pebbling moves, the set of vertices with pebbles forms a secure dominating set for G. **Findings:** We found the secure domination cover pebbling number for the join of two graphs $G(p, q)$ and K_n. Also, the secure domination cover pebbling number for the join of two graphs $G(p, q)$ and $G'(p', q')$ is determined when the cardinality of the secure dominating set is 2, 3 and 4. A generalization for the secure domination cover pebbling number of path P_n is also found.
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1 Introduction

Let $G(V,E)$ be a connected, simple graph. Let the vertices of $G(V,E)$ be distributed by p pebbles. Then, the removal of two pebbles from a vertex and placing one pebble on an adjacent vertex is said to be a pebbling move. The pebbling number of a graph G, $f(G)$, is the least n such that however n pebbles are placed on the vertices of G, we can move a pebble to any vertex by a sequence of pebbling moves[(1)]. For a survey of additional results refer[(2)]. The objective of this network optimization model is for the transportation of resources that are consumed in transit. Secure domination cover pebbling number is combination of cover pebbling and secure domination number. Secure domination cover pebbling number, $f_{sdp}(G)$, of a graph G is the minimum number of pebbles that must be placed on $V(G)$, such that after a sequence of pebbling moves the set of vertices with pebbles forms a secure dominating set regardless of the initial configuration[(3)]. It has wide applications; for instance, by using the concept of secure domination cover pebbling number we can find the minimum number of guards that are required to protect a country. The detailed application of secure domination cover pebbling number is mentioned in[(3)]. In this paper, the secure domination cover pebbling number for the join of two graphs is determined and also generalization for the secure domination cover pebbling number of path P_n is found.

2 Preliminaries

Definition 1 [(4)] A set of vertices S in G is a dominating set in G, if every vertex in G is either in S or adjacent to some element in S. The minimum number of vertices in the set S is called domination number and is denoted by $\gamma(G)$.

Definition 2 [(5)] A dominating set S in G is called a secure dominating **set** in G denoted by $\gamma_s(G)$, if for every $v \in V(G)\backslash S$, there exists $u \in S \cap N(v)$ such that $(S\backslash\{u\}) \cup \{v\}$ is a dominating set where $N(v) = \{t \in V(G) : vt \in E(G)\}$. The minimum cardinality of a secure dominating set is called the secure domination number of G.

Definition 3 [(5)] The cover pebbling number of a graph G is the minimum number of pebbles required to place a pebble on every vertex simultaneously under any initial configuration.

Definition 4 [(4)] The domination cover pebbling number, $\psi(G)$, of a graph G is the minimum number of pebbles required such that after a sequence of pebbling moves, the set of vertices with pebbles forms a domination set of G, regardless of the initial configuration of pebbles.

Definition 5 [(6)] Let $G_1(V_1,E_1)$ and $G_2(V_2,E_2)$ be a connected simple graph. Then $G_1 \cup G_2$ is the graph $G(V,E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ and $G_1 + G_2$ is $G_1 \cup G_2$ together with the edges joining elements of V_1 to elements of V_2.

Result 1. [(1)] f(P_{t+1})=2^t

Result 2. [(3)] f_{sdp}(G)=1 iff G is complete graph

Notation: $p(v)$ denotes the number of pebbles placed at the vertex v.

3 Main Results

Theorem 1 $f_{sdp}(G+G') = 1$ iff G and G' are complete graphs.

Proof. Let $f_{sdp}(G+G') = 1$. Then, by Result 2, it is obvious that $G+G'$ is complete which implies that G and G' are complete. Conversely, if G and G' are complete graphs, then $G+G'$ is also complete. Thus, by Result 2, we get $f_{sdp}(G+G') = 1$.

Theorem 2 Let $G(p,q)$ be a connected graph which is not complete. The secure domination cover pebbling number $f_{sdp}(G+K_n) = p+n+3, n \geq 2$.

Proof. Let $x \in V(G)$ and $y \in V(K_n)$. Then, $\{x,y\}$ forms the secure dominating set [(7)]. If we have at least one pebble each on x and y, then there is nothing to prove. Now, assume that $p(x) = 0$ and $p(y) = 0$. If we are able to place at least 2 pebbles on any adjacent vertices of x and y then the result holds. If not, we can place a pebble on x by a pebbling move in any one of the following cases:

(i) any non-adjacent vertex of x in G has at least 4 pebbles on it

(ii) there exists at least 2 vertices with 2 pebbles each in G

(iii) if there exists at least one vertex in K_n with at least 2 pebbles on it.

Similarly, if there exists a vertex in $G+K_n$ with at least 2 pebbles then we can place a pebble on y by a pebbling move. Finally, consider the case where we have a single pebble on each vertex of $G+K_n$ except one say, $'u'$ in G and if we place 6 pebbles on the exceptional vertex $'u'$, then the non-pebbled vertices in the secure dominating set will be forced to have a pebble by a pebbling move. The result is obvious if the exceptional vertex $'u'$ is in K_n. Therefore, $f_{sdp}(G+K_n) = p+n+3$.

Theorem 3 Let $G(p,q)$ and $G'(p',q')$ be connected graphs which are not complete. Then $f_{sdp}(G+G') = p+p'+5$ if either $\gamma_s(G) = 2$ or $\gamma_s(G') = 2$.

Proof. Let $x,y \in G$. Then, $\{x,y\}$ forms the secure dominating set [(7)]. If there exists at least one pebble each on x and y then there is nothing to prove. Without loss of generality, assume that $p(x) = 0$ and $p(y) = 0$. We can place a pebble on x and y by a pebbling move in any one of the following cases:

(i) if the adjacent vertices of x and y have at least 2 pebbles

(ii) if there exists a minimum of two vertices in G' each having at least 2 pebbles on it

(iii) if there exists a minimum of two vertices in G each having at least 4 pebbles on it

(iv) if there exists at least one vertex in G with at least 8 pebbles

(v) if there exists at least one vertex in G' with at least 4 pebbles

So, consider the case where all the vertices of $G+G'$ with at least one pebble except one say u in G. By placing 8 pebbles on the exceptional vertex u, the non-pebbled vertices in the secure dominating set are forced to have a pebble on it by a pebbling move. The result is obvious if the exceptional vertex is in G'.

Similarly, we can prove $f_{sdp}(G+G') = p+p'+5$ if $\gamma_s(G') = 2$ as in the previous case.

Corollary 1 Let $G(p,q)$ be a connected graph which is not complete. Then

(a) $f_{sdp}(G+K_1) = p+4$ if it satisfies any one of the following conditions:

(i) $\gamma(G) = 1$

(ii) if there exists $u \in V(G)$ such that $< V(G) \backslash N_G(u) >$ is complete.

(b) $f_{sdp}(G+K_1) = p+5$ if $\gamma_s(G) = 2$.

Proof. (a) Let $V(K_1) = \{x\}$ and let $y \in V(G)$. Then, $\{x,y\}$ forms the secure dominating set[7] for (i) and (ii) and the result follows from Theorem 2.

(b) Let $x,y \in G(p,q)$ and $\{x,y\}$ forms the secure dominating set[7] and the result follows from Theorem 3.

Theorem 4 Let $G(p,q)$ and $G^{'}(p^{'},q^{'})$ be connected graphs which are not complete and suppose $\gamma_s(G+G^{'}) = 3$. Then, $f_{sdp}(G+G^{'}) = p+p^{'}+8$ if either $\gamma_s(G) = 3$ or $\gamma_s(G^{'}) = 3$.

Proof. Consider the case where $\gamma_s(G) = 3$. Let $u,v,w \in G(p,q)$, then $\{u,v,w\}$ forms the secure dominating set[7]. If there exists at least one pebble on u,v and w then the result is obvious by a pebbling move. Now, assume that $p(u) = 0, p(v) = 0$, and $p(w) = 0$. It is possible to place a pebble for all vertices in the secure dominating set in any one of the following cases:

(i)If the adjacent vertices of each u,v and w in G have at least 2 pebbles

(ii)If there exists a vertex in G with 12 pebbles

(iii)If there exists a minimum of 3 vertices in $G^{'}$ with at least 2 pebbles each (iv)If there exists a vertex in $G^{'}$ with 6 pebbles

(v)If there exists a minimum of 3 vertices in G with at least 4 pebbles each.

Now, let us consider the case where all the vertices except one say x in G with a single pebble on it. By placing 12 pebbles on the exceptional vertex x, the non-pebbled vertices in the secure dominating set are forced to have a pebble by a pebbling move. The result is obvious if the exceptional vertex is in $G^{'}$. The case where any of the vertices in $G+G^{'}$ with zero pebble in the above discussed case is an obvious result.

Similarly, we can prove $f_{sdp}(G+G^{'}) = p+p^{'}+8$ if $\gamma_s(G^{'}) = 3$.

Theorem 5 Let $G(p,q)$ and $G^{'}(p^{'},q^{'})$ be connected graphs which are not complete and suppose $\gamma_s(G+G^{'}) = 3$. Then, $f_{sdp}(G+G^{'}) = p+p^{'}+6$ if either $\psi(G) = 2$ or $\psi(G^{'}) = 2$.

Proof. (i) Let $u,v \in G(p,q)$ and $w \in G^{'}(p^{'},q^{'})$, then $\{u,v,w\}$ forms the secure dominating set[7]. If there exists at least one pebble on u,v and w then there is nothing to prove. Now, consider the case where $p(u) = 0, p(v) = 0$ and $p(w) = 0$. Then, we can place a pebble on u and v by a pebbling move if any one of the following conditions holds:

(i) if the adjacent vertices of u and v have at least 2 pebbles

(ii) if there exists a minimum of 2 vertices in $G^{'}$ with at least 2 pebbles each

If there exists a vertex in G with at least 8 pebbles

(iv) if there exists a minimum of 2 vertices in G with at least 4 pebbles each

Also, we can place a pebble on w by a pebbling move if any one of the following conditions holds:

(i) if the adjacent vertex of w has at least 2 pebbles on it

(ii) if there exists a vertex in $G^{'}$ with at least 4 pebbles.

Now, let us consider the case where all vertices in $G+G^{'}$ except one say, x in G has at least one pebble. Then, by placing 10 pebbles on the exceptional vertex x, the non-pebbled vertices in the secure dominating set are forced to have a pebble by a pebbling move and the result follows.

Theorem 6 Let $G(p,q)$ and $G^{'}(p^{'},q^{'})$ be connected graphs which are not complete and suppose $\gamma_s(G+G^{'}) = 4$. Then $f_{sdp}(G+G^{'}) = p+p^{'}+7$.

Proof. Let $u,v \in G(p,q)$ and $x,y \in G^{'}(p^{'},q^{'})$, then obviously $\{u,v,x,y\}$ forms the secure dominating set[7]. If there exists at least one pebble on u,v,x and y then there is nothing to prove. Without loss of generality, assume that $p(u) = 0, p(v) = 0, p(x) = 0$ and $p(y) = 0$. We can place a pebble on u and v by a pebbling move if it satisfies any one of the following conditions:

(i) if there exists at least two pebbles on adjacent vertices of u and v

(ii) if there exists a minimum of two vertices in G with at least 4 pebbles each

If there exists a vertex in G with 8 pebbles

(iv) if there exists a minimum of two vertices in $G^{'}$ with at least 2 pebbles each

(v) if there exists a vertex in $G^{'}$ with at least 4 pebbles

We can place a pebble on x and y by a pebbling move if it satisfies any one of the following conditions:

(i) if there exists at least two pebbles on adjacent vertices of x and y

(ii) if there exists a minimum of two vertices in G with at least 2 pebbles each

If there exists a vertex in G with 4 pebbles

(iv) if there exists a minimum of two vertices in $G^{'}$ with at least 4 pebbles each

(v) if there exists a vertex in $G^{'}$ with at least 8 pebbles

Consider the case where all vertices in $G+G^{'}$ except one say p, has a single pebble on it. Thus, by placing 12 pebbles on the exceptional vertex p, the non-pebbled vertices in the secure dominating set are forced to have a pebble by a pebbling move and the result follows.

Remark 1 Let $G(p,q)$ and $G^{'}(p^{'},q^{'})$ be connected graphs which are not complete. Then $p+p^{'}+5 \leq f_{sdp}(G+G^{'}) \leq p+p^{'}+7$.

Proof. Since $2 \leq \gamma_s(G+G^{'}) \leq 4$[8], the result follows from Theorems 3, 4 and 5.

Theorem 7 For $k \geq 1$,

$$f_{sdp}(P_{7k+r}) = \begin{cases} \frac{42[2^{7k}-1]}{2^7-1}, r=0 \\ \frac{42[2^{7k}-1]}{2^7-1} + \sum_{r=1}^{5} 2^{7k+r-1}, r = odd \\ \frac{42[2^{7k}-1]}{2^7-1} + \sum_{r=2}^{6} 2^{7k+r-2}, r = even \end{cases}$$

Proof. Let $V(P_n) = \{v_1, v_2, ..., v_n\}$ and $E(P_n) = \{v_1v_2, v_2v_3, ..., v_{n-1}v_n\}$. For $k = 0$, we have $f_{sdp}(P_r) = 1$ for $r = 1,2$, $f_{sdp}(P_r) = 15$ for $r = 3,4$, $f_{sdp}(P_r) = 21$ for $r = 5,6$ and the result is obvious.

Case 1: When n =7k, $k \geq 1$.

Then $\cup_{j=0}^{k-1}\{v_{7j+2}, v_{7j+4}, v_{7j+6}\}$ forms the secure dominating set[5] . Consider the configuration of placing all the pebbles on v_1. We need at least $2^{n-2}+2^{n-4}+2^{n-6}+2^{n-9}+2^{n-11}+...+2$ pebbles to securely dominate $\cup_{j=0}^{k-1}\{v_{7j+2}, v_{7j+4}, v_{7j+6}\}$. Thus, under this configuration, $f_{sdp}(P_{7k}) \geq \frac{42[2^{7k}-1]}{2^7-1}$, $k \geq 1$.

Now we use induction to show that $f_{sdp}(P_{7k}) \leq \frac{42[2^{7k}-1]}{2^7-1}$, $k \geq 1$.

The result is obvious for $k = 1$. Let us assume that the assertion is true for all P_{7t}, where $1 \leq t \leq k-1$. Consider an arbitrary configuration of P_{7k}, $k \geq 1$ having $\frac{42[2^{7k}-1]}{2^7-1}$ pebbles. Clearly, we can place a pebble on $\{v_{n-1}, v_{n-3}, v_{n-5}\}$ in a finite number of moves with a maximum of $21(2^{7k-6})$ pebbles under any configuration by Result 2. We need to securely dominate $P_{7(k-1)}$ with the remaining $\frac{42[2^{7k}-1]}{2^7-1} - 21(2^{7k-6}) = \frac{42[2^{7(k-1)}-1]}{2^7-1}$ pebbles. By the hypothesis, the remaining number of pebbles is sufficient enough to dominate $P_{7(k-1)}$. Thus, $f_{sdp}(P_{7k}) \leq \frac{42[2^{7k}-1]}{2^7-1}$, $k \geq 1$.

Therefore, we have $f_{sdp}(P_{7k}) = \frac{42[2^{7k}-1]}{2^7-1}$, $k \geq 1$.

Case 2: When $n = 7k+r$, $k \geq 1$, $r = 1,2$.

Then $\cup_{j=0}^{k-1}\{v_{7j+2}, v_{7j+4}, v_{7j+6}\} \cup \{v_{7k+1}\}$ forms the secure dominating set[5]. Here, we have only one extra vertex v_{7k+1} in the secure dominating set when we compare Case 1. By using Result 2, we need a maximum of 2^{7k} pebbles to place a pebble on v_{7k+1} under any configuration of pebbles on the vertices. Thus, by adding 2^{7k} pebbles to $f_{sdp}(P_{7k}), k \geq 1$ we get a secure domination cover solution for $P_{7k+r}, k \geq 1, r = 1,2$.

Case 3: When $n = 7k+r$, $k \geq 1$, $r = 3,4$.

Then $\cup_{j=0}^{k-1}\{v_{7j+2}, v_{7j+4}, v_{7j+6}\} \cup \{v_{7k+1}, v_{7k+3}\}$ forms the secure dominating set[5]. Here when we compare to Case 2, we have one extra vertex u_{7k+3} in the secure dominating set. In order to place a pebble on v_{7k+3}, by Result 2, we need a maximum of 2^{7k+2} pebbles under any configuration of pebbles on the vertices. So on adding 2^{7k+2} pebbles to $f_{sdp}(P_{7k+t})$, $t = 1,2$ we get a secure domination cover solution for P_{7k+r}, $k \geq 1$, $r = 3,4$.

Case 4: When $n = 7k+r$,$k \geq 1$, $r = 5,6$.

Then $\cup_{j=0}^{k-1}\{v_{7j+2}, v_{7j+4}, v_{7j+6}\} \cup \{v_{7k+1}, v_{7k+3}, v_{7k+5}\}$ forms the secure dominating set[5]. Here we have v_{7k+5} as the extra vertices in the secure dominating set when compared to Case 3. But by Result 2, we know that 2^{7k+4} pebbles is sufficient enough to place a pebble on v_{7k+5} under any configuration. Thus, by adding 2^{7k+4} pebbles to P_{7k+r}, $k \geq 1$, $r = 3,4$, we get a secure domination cover solution.

4 Conclusion

Secure domination cover pebbling number is one of the areas of research in graph theory. In this paper, we have found the secure domination cover pebbling number for the join of two graphs $G(p,q)$ and K_n. Also, the secure domination cover pebbling number for the join of two graphs $G(p,q)$ and $G^{'}(p^{'},q^{'})$ is determined when the cardinality of the secure dominating set is 2, 3 and 4. A generalization for the secure domination cover pebbling number of path P_n is also found.

Given below are some interesting open problems for secure domination cover pebbling number:

Problem 1: Find the secure domination cover pebbling number for other graph operations such as sum, product, lexico product, etc.,

Problem 2: Find secure domination cover pebbling number for other families of graphs and networks.

Problem 3: Finding the complexity of secure domination cover problem.

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