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The Solar Oscillation Equation and Some of its Particular Solutions

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Abstract

Background: The phenomenon that sun oscillates in characteristic eigen frequencies witnessed on the solar surface has fascinated many solar physicists to go for exact measurements of the internal properties of our nearest star that we are most familiar with. The present work bears the objective to understand the dynamic and nonlinear profile of solar oscillation on the basis of some particular solutions of the fundamental equation of solar oscillation. After the detection of global solar oscillation during 1970's different physical aspects of solar oscillation started to draw the attention of several astrophysicists both from the perspectives of theoretical investigation as well as observational evidences. **Methods:** The functional forms of the wave representation term appeared in the wave-equation of our present system has been qualitatively studied by performing linear stability analysis, where the 'Jacobian' provides the basis of understanding of the nature of equilibrium points. Analytical means or approaches have been employed to determine the solutions of the said wave equation without using any kind of computational method or simulation at different regions of the number line. **Findings:** The corresponding equilibrium points of the wave-equations followed from the solar oscillation equation at various chosen functional forms of the complicated wave representation term have been nicely calculated. Sincere analytical attempts have also been taken for the first time to derive the solutions of the solar oscillation equation segmented into several mathematically chosen scenarios by us. **Novelty:** The analytic solutions of the solar oscillation equation obtained for different particular functional structures explain the existence of nonlinear acoustic modes of frequency in the observed solar oscillations of long period. This result will surely motivate any further quest of chaotic oscillations in both radial and non-radial modes.

Keywords: Solar oscillation; Equilibrium point; Jacobian; Bessel function; Eigenvalues

1 Introduction

The objective or purpose of the study of oscillations observed on the solar surface [1–4] is to get information about the solar interior with a detailed map of the solar internal structure and internal rotation. This requires an understanding of the relations between the internal properties of the Sun and those of the observed oscillations, particularly their frequencies. The solar oscillations are the resultant of resonances taking place at some particular frequencies and so they are also regarded as resonant modes [5]. These small amplitude solar oscillations can be studied in the light of an eigenvalue problem which on solving fetches a discrete set of eigen frequencies, with every eigen frequency bearing an eigen mode. These modes are the tools to capture concrete information about the solar internal structure and its dynamics. As the sun is spherically symmetric so equilibrium measurement of the solar quantities will exhibit solar radial dependence. Thus, a single mode can be fully illustrated by three integers namely n, l and m, where n implies the radial order or the number of nodes present along the sun's radius and assumes both positive and negative values; l represents the angular degree which estimates the numerical amount of the nodal lines present at the spherical surface and always bears non-negative integral values; m denotes the azimuthal order or the number of lines passing through the poles and the numerical value ranges from $-l$ to $+l$ incorporating zero. Modes with l=0 are known as radial modes and those with $l \geq 1$ are called as non-radial modes. Non-radial modes have been studied earlier to investigate the nonlinear effects in solar oscillations by using nonlinear mode coupling scheme [6]. Any single mode of oscillation usually has a feeble speed of oscillation of about few centimeters per second which makes it very hard to detect. But as they overlap they get a significant enhancement in the speed of oscillation. The speed of oscillation in this process of overlapping can enhance up to few hundred meters per second and this makes them easily detectable. Global helioseismology deals with the analysis of these modes which form clear crests in two-dimensional power spectra [7–12]. Christensen-Dalsgaard and Gough [13] modelled the potential of employing individual mode frequencies to comprehend the internal structure of the Sun. The solar oscillations as observed by Evans and Michard [14] and Leighton et al. [15] in the form of quasi-periodic intensity with a periodicity of about 5 minutes [12,16] are basically like sound waves in the solar gases. But these oscillations are very complex in nature and are assumed to behave adiabatically [17]. Inside the Sun, the sound waves originate in the convective zone [16], where heat is transferred by the movement of the gas itself rather than by any radiation. Largely; the two important sequences of modes of the solar oscillations according to the individual dominant restoring force are acoustic waves and gravity waves. Pressure generally dominates the acoustic or p-modes and buoyancy rules the gravity, which consists of two parts; one is the internal gravity modes or g-modes and the other is surface gravity modes or f-modes. In radial oscillations, near-surface convection is the principal driving process which generates p-modes as discussed in 3D solar atmosphere model [18]. It has been observed recently that nonlinear unsteadiness of g-modes inside the sun has led to the rapid reduction of luminosity L through a dynamical process [19]. The governing ordinary differential equation of solar oscillation [20] can be typically categorized as a half-linear differential equation. There have been uses of semi-linear differential equation to understand oscillation in a physical system [21–23].

2 Theory

The solar oscillation equation can be approximated into the form as shown below [2]:

$$\left.\begin{aligned} \frac{d^2\psi}{dr^2} &= -K(r)\psi(r) \\ K(r) &= \frac{\omega^2}{c^2}\left[1-\frac{\omega_0^2}{\omega^2}-\frac{s_l^2}{\omega^2}\left(1-\frac{N^2}{\omega^2}\right)\right] \end{aligned}\right\} \tag{1}$$

Here $\psi(r)$ is given by [24,25]:

$$\psi(r) = c^2\rho^{\frac{1}{2}}\,\nabla.\delta r \tag{2}$$

where δr is the small displacement of fluid elements due to the perturbation around stationary equilibrium model, ρ is the density at a radial distance r and c is the adiabatic sound speed whose squared magnitude is given by:

$$c^2 = \frac{\Gamma_1 p}{\rho} \tag{3}$$

where p is the pressure at a radial distance r and the isentropic quantity Γ_1 is introduced as below:

$$\Gamma_1 = \left(\frac{\partial lnp}{\partial ln\rho}\right)_S \tag{4}$$

where S is the entropy.

Also in (1), ω is the frequency of oscillation, ω_0 is the adiabatic cut-off frequency given by:

$$\omega_0^2 = \frac{c^2}{4H^2}\left(1 - 2\frac{dH}{dr}\right) \tag{5}$$

where H is the density scale height defined by:

$$H = -\left(\frac{d\ln\rho}{dr}\right)^{-1} \tag{6}$$

Again the remaining terms, S_l and N as used in (1) are known as Lamb frequency and buoyant frequency respectively, whose squared magnitudes are mathematically expressed as:

$$\begin{aligned} S_l^2 &= \frac{l(l+1)c^2}{r^2} \cong k_h^2 c^2 \\ N^2 &= g\left(\frac{1}{\Gamma_1 p}\frac{dp}{dr} - \frac{1}{\rho}\frac{d\rho}{dr}\right) \end{aligned} \tag{7}$$

In (7) l stands for the number of wavelengths around the solar circumference, k_h represents the wave number along horizontal direction and g denotes the gravitational acceleration in the Sun.

Some particular functional forms of $K(r)$ are considered here and the corresponding solutions of (1) are proposed below:

Case-I: $K(r)$ is a constant:

The solution in this case will be given by:

$$\left.\begin{aligned} \psi(r) &= A\cos(\sqrt{K}\times r) + B\sin(\sqrt{K}\times r),\ \text{for } K>0 \\ &= A'e^{(\sqrt{-K}\times r)} + B'e^{(-\sqrt{-K}\times r)},\ \text{for } K<0 \\ &= A''r + B'',\ \text{for } K=0 \end{aligned}\right\} \tag{8}$$

Here all A , B , A' , B', A'', B'' are suitable constants of integration.

In (1) the substitution is introduced as $\frac{d\psi}{dr} = \phi$. This fetches a pair of equations as below:

$$\left.\begin{aligned} \frac{d\psi}{dr} &= \phi = f_1(\psi,\phi) \\ \frac{d\phi}{dr} &= -K\psi = f_2(\psi,\phi) \end{aligned}\right\} \tag{9}$$

Now, for equilibrium point of the above system we must have $\frac{d\psi}{dr} = 0$ and $\frac{d\phi}{dr} = 0$ and this gives $\phi = 0$ and $-K\psi = 0$ respectively, giving $\phi = 0$ and $\psi = 0$. Again, $\phi = 0$ implies $\frac{d\psi}{dr} = 0$ which further leads to the fact that $\psi = constant$. This is analogous to $\psi = 0$ which brings $\phi = constant$. So, the equilibrium point can be obtained here as $(\psi^*, \phi^*) = (0,\ 0)$. The Jacobian matrix (J) at the said equilibrium point takes the form as:

$$J_{(0,0)} = \begin{pmatrix} \frac{\partial f_1(\psi,\phi)}{\partial\psi} & \frac{\partial f_1(\psi,\phi)}{\partial\phi} \\ \frac{\partial f_2(\psi,\phi)}{\partial\psi} & \frac{\partial f_2(\psi,\phi)}{\partial\phi} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix}$$

Here, the corresponding characteristic equation takes the form as shown below:

$\det\left(J_{(0,0)} - \lambda I\right) = 0$

or, $\begin{vmatrix} -\lambda & 1 \\ -K & -\lambda \end{vmatrix} = 0$; where I $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the Identity Matrix

which gives

$$\lambda^2 + K = 0 \tag{10}$$

Sub Case-I(a): K >0:

From (10) we have $\lambda_1 = +i\sqrt{K}$ and $\lambda_2 = -i\sqrt{K}$

Thus the two eigenvalues obtained here are $+i\sqrt{K}$ and $-i\sqrt{K}$.

Since the roots of the above equation expressed by the eigenvalues λ_1 and λ_2 are purely imaginary and as well as complex conjugates, then it can be easily said that the equilibrium point or the fixed point $(\psi^*, \phi^*) = (0,\ 0)$ denotes a 'centre-type' or an 'elliptic equilibrium point' which is neutrally stable.

Sub Case-I(b): K <0:

From (10) we have $\lambda_1 = -\sqrt{-K}$ and $\lambda_2 = +\sqrt{-K}$. Here both the roots λ_1 and λ_2 are non-zero, real valued with opposite signs. Since here the eigenvalues satisfy this condition $\lambda_1 < 0 < \lambda_2$, it can be reckoned that the equilibrium point $(\psi^*, \phi^*) = (0,\ 0)$ here represents a 'hyperbolic equilibrium point' or 'saddle' of the system. This type of fixed point is generally unstable in nature.

Sub Case-I(c): K $=0$:

Here the Jacobian matrix (J) at the equilibrium point $(\psi^*, \phi^*) = (0,\ 0)$ becomes $J_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ leading to $\lambda_1 = 0 = \lambda_2$. Thus both the roots of the characteristic equation described by the corresponding eigenvalues λ_1 and λ_2 exhibit null values. In this case the equilibrium point is not an isolated point rather there are infinite numbers of such equilibrium points or fixed points in the neighbourhood.

Case-II: K$(r) = \alpha r + \beta$ where α and β are real constants:

Sub Case $-$ *II* (a) : $\alpha > 0$ *and* β *is unrestricted in sign such that* $K > 0$**:**

From (1) one can have:

$$\frac{d^2\psi}{dr^2} = -(\alpha r + \beta)\psi(r) \tag{11}$$

The substitution $\xi = \alpha^{-\frac{2}{3}}(\alpha r + \beta)$ (12)

fetches:

$$\frac{d\xi}{dr} = \alpha^{\frac{1}{3}} \tag{13}$$

Again, $\frac{d\psi}{dr} = \frac{d\psi}{d\xi} \times \frac{d\xi}{dr}$

or, $\frac{d\psi}{dr} = \alpha^{\frac{1}{3}} \times \frac{d\psi}{d\xi}$ [from (13)] (14)

And lastly,

$\frac{d^2\psi}{dr^2} = \alpha^{\frac{1}{3}} \frac{d}{dr}\left(\frac{d\psi}{d\xi}\right) = \alpha^{\frac{2}{3}} \times \frac{d^2\psi}{d\xi^2}$ [from (13)] (15)

Using (12) and (15) in (11) it gives:

$$\frac{d^2\psi}{d\xi^2} = -\xi\psi \tag{16}$$

The solution of (16) takes the following form:[26,27]:

$\psi = C_1\sqrt{\xi} \times J_{1/3}\left(\frac{2}{3}\xi^{\frac{3}{2}}\right) + C_2\sqrt{\xi} \times Y_{1/3}\left(\frac{2}{3}\xi^{\frac{3}{2}}\right)$

or, $\psi = \frac{1}{\alpha^{\frac{1}{3}}} \times \sqrt{(\alpha r + \beta)}\Big(C_1 J_{1/3}\left(\frac{2}{3\alpha}(\alpha r + \beta)^{\frac{3}{2}}\right\} + C_2 Y_{1/3}\left(\frac{2}{3\alpha}(\alpha r + \beta)^{\frac{3}{2}}\right\}\Big]$

[Using (12)] (17)

where J stands for the Bessel function of first kind and Y for the Bessel function of second kind and C_1 and C_2 are suitable constants of integration.

Sub Case $-$ *II* (b) : $\alpha < 0$ *and* β *is unrestricted in sign such that* $K < 0$**:**

On substituting $-\alpha = \alpha^{'}(> 0)$ and $-\beta = \beta^{'}$ in (11), one gets:

$$\frac{d^2\psi}{dr^2} = (\alpha' r + \beta')\psi(r) \tag{18}$$

where $\alpha^{'} > 0$ and $(\alpha^{'} r + \beta^{'}) > 0$.

Next the substitution $\xi' = \alpha'^{-\frac{2}{3}}\left(\alpha' r + \beta'\right)$ (19)
leads to the following expressions:

$$\frac{d\xi'}{dr} = \alpha'^{\frac{1}{3}} \tag{20}$$

Again, $\frac{d\psi}{dr} = \frac{d\psi}{d\xi'} \times \frac{d\xi'}{dr}$
or, $\frac{d\psi}{dr} = \alpha'^{\frac{1}{3}} \times \frac{d\psi}{d\xi'}$ [from (20)] (21)
And lastly,
$\frac{d^2\psi}{dr^2} = \alpha'^{\frac{1}{3}} \frac{d}{dr}\left(\frac{d\psi}{d\xi'}\right) = \alpha'^{\frac{2}{3}} \times \frac{d^2\psi}{d\xi'^2}$ [from (20)] (22)
Using (19) and (22) in (18) it brings:

$$\frac{d^2\psi}{d\xi'^2} = -\xi'\psi \tag{23}$$

The solution of (23) can be proposed as follows:[26,27]
$\psi = C_1'\sqrt{\xi'} \times I_{1/3}\left(\frac{2}{3}\xi'^{\frac{3}{2}}\right) + C_2'\sqrt{\xi'} \times K_{1/3}\left(\frac{2}{3}\xi'^{\frac{3}{2}}\right)$
or, $\psi = \frac{1}{(-\alpha)^{\frac{1}{3}}} \times \sqrt{(-\alpha r - \beta)}\left(C_1' I_{1/3}\left(\frac{2}{3(-\alpha)}(-\alpha r - \beta)^{\frac{3}{2}}\right\} + \right.$
$\left.\left(C_2' K_{1/3}\left(\frac{2}{3(-\alpha)}(-\alpha r - \beta)^{\frac{3}{2}}\right\}\right]$ [Using (19)] (24)
where I stands for the modified Bessel function of first kind and K for the modified Bessel function of second kind and C_1' and C_2' are suitable constants of integration.
Case-III: $K(r) = \frac{\gamma}{r^2}$ where γ is a real constant:
Here from (1) one can have:

$$\frac{d^2\psi}{dr^2} = -\frac{\gamma}{r^2}\psi(r) \tag{25}$$

Sub Case-III (a): $\gamma > 0$:
The corresponding solution is given below:[26,27]
$\psi = \sqrt{r}\left(D_1 r^{\frac{\sqrt{1-4\gamma}}{2}} + D_2 r^{\frac{-\sqrt{1-4\gamma}}{2}}\right)$ when $1 > 4\gamma$ i.e. $0 < \gamma < \frac{1}{4}$
$\psi = \sqrt{r}\left(D_1' + D_2' lnr\right)$ when $1 = 4\gamma$ i.e. $\gamma = \frac{1}{4}$ $\psi = \sqrt{r}\left(D_1''sin\left(\left(\frac{\sqrt{4\gamma-1}}{2}\right) \times lnr\right\} + D_2''cos\left(\left(\frac{\sqrt{4\gamma-1}}{2}\right) \times lnr\right\}\right]$
when $1 < 4\gamma$ i.e. $\gamma > \frac{1}{4}$ (26)
where D_1, D_2, D_1', D_2', D_1'' and D_2'' are suitable constants of integration.
Sub Case-III (b): $\gamma < 0$:
The corresponding solution is given below:[26,27]
$\psi = \sqrt{r}\left(E_1 r^{\frac{\sqrt{1-4\gamma}}{2}} + E_2 r^{\frac{-\sqrt{1-4\gamma}}{2}}\right)$ when $\gamma < 0$ (27)
where E_1 and E_2 are suitable constants of integration.

3 Discussion

Since the identification of global solar oscillation in 1970's the researches (both analytic and observational) on different perspectives of solar oscillation has been very much active. The analysis of the present work starts from the solar oscillation equation as proposed by Christensen-Dalsgaard[20]. Earlier wave solutions and estimation of critical points have been done by solving Helmholtz equation with proper boundary conditions in case of solar Rossby waves[28]. Here in the present work in certain cases, whenever possible, stability analysis of the corresponding equilibrium points of the system is performed to demonstrate the intrinsic nature and underlying dynamics of the physical process lying behind this solar phenomenon in phase space. Such stability analysis through investigation of eigenvalues has been also performed earlier to study the oscillation modes of power system driven by renewable energy sources [29]. The study of equilibrium points with their corresponding eigenvalues are evident too in models illustrating the closed system namely Coimbrator, consisting of two photochemical steps [30] and works explaining the properties of DC traction power network and its subsequent causes leading to low frequency oscillation [31]. Importantly, it is to note that the main objective of this work lies in establishing some analytic solutions for different particular structures of the solar oscillation equation. As here all the solutions as obtained are exact we have not put forward any numerical table for the solutions at different points. Most importantly, question of providing comparison table with different methods to show the effectiveness of the present approach does not arise as any exact solution of the concerned model always bangs on target and it is always the supreme and ultimate one.

4 Conclusion

The functional forms that we have dealt in this work have fetched analytical solutions for the adiabatic solar oscillation equation. The wave representation term K(r) plays the key role in the solar oscillation equation. Here in the present work suitable particular functional forms of this K has been taken into consideration and the corresponding particular solutions for the solar oscillation equation are proposed. In Case-I $K(r)$ is considered to be a constant throughout the interior of the solar configuration. This indicates a possible radial homogeneity in solar oscillation. Again the particular functional structure illustrated by Case-II i.e. $K(r) = \alpha r + \beta$, shows that α is a scalar which is giving a weightage to the radial distance and β is working as a bias i.e. a shift from the core. This model eventually gives a linear change in $K(r)$with the radial distance r. Again, the functional form regulating Case-III i.e. $K(r) = \frac{\gamma}{r^2}$, shows that $K(r)$ is obeying the inverse square law which can be frequently observed in several physical phenomena. Here this indicates that $K(r)$is rapidly changing as it moves away from the solar core towards the solar surface following spherical symmetry. The solutions as proposed in all such cases discussed here indicate a highly nonlinear profile of the phenomenon of solar oscillation which eventually corroborates the recent findings of nonlinearity observed in the three-minute velocity oscillations (or very dispersive acoustic waves of long wavelength) in the upper regions of the solar chromospheres [4,32,33]. This upshot will certainly stimulate the search for chaotic oscillations in both radial and non-radial modes.

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