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Some New Classes of (*k, d*) Graceful 3 Distance Trees and 3 Distance Unicyclic Graphs

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Abstract

Objectives: To identify a new family of $(k,\ d)$ graceful graphs. **Methods :** The methodology involves mathematical formulation for labeling of the vertices of a given graph and subsequently establishing that these formulations give rise to (k,d) graceful labeling. **Findings:** Here we define a three-distance tree as the tree possessing a path such that each vertex of the tree is at most at a distance three from that path. In this paper we identify two families of three distance trees that possess $(k,\ d)$ graceful labeling. Furthermore, we show that the three distance unicyclic graphs obtained from these three distance trees by joining two end vertices of their central paths are also $(k,\ d)$ graceful. **Novelty:** Here, we give $(k,\ d)$ graceful labeling to two new families of graphs, namely some classes of three distance trees and three distance unicyclic graphs. This effort is the first of its kind which involves exploration of 3-distance $(k,\ d)$ graceful graphs.

Keywords: (k; D); graceful labelling; Hairy cycle; Firecracker; Three distance tree; Three distance unicyclic graphs

1 Introduction

Acharya and Hegde[1] defined (k, d) − graceful labeling of a graph G with q edges as a surjective mapping of the vertex set of G into the set $\{0,\ 1,\ 2,\ \ldots,\ k+(q-1)d\}$ for some positive integers k and d. A $(1,\ 1)-$graceful labeling is called a graceful labeling and a $(k,\ 1)-$graceful labeling is called a k-graceful labeling. Bu and Zhang[2] established that $K_{m,n}$ is $(k,\ d)-$ graceful for all k and d and K_n is $(k,\ d)-$graceful if and only if $k = d$. Hegde and Shetty[3] showed that a tree T which can be transformed into a path by carrying out successive elementary transformations and the tree formed from T by subdividing each edge of T is $(k,\ d)-$graceful for all k and d. Some more results on graph labeling problems are found in some recent papers[4–10]. For details of the literature involving (k, d) graceful graphs one may refer to the latest dynamic survey on graph labeling problems by Gallian[2].

From the literature survey it is found that there exist only some specific classes of graphs, namely $K_{m,n}$, K_n, and transformed trees which admit $(k,\ d)$ graceful labeling.

So, there is huge scope to explore in this area. In this paper we give (k, d)- graceful labeling to some new classes of three distance trees and three distance unicyclic graphs. Before deriving our results, we would like to have a recap of some of the existing graph theoretic terminologies and some new terminologies required for proving our results.

Definition 1.1[2] By a firecracker we mean a tree possessing a path known as the central path such that each vertex of the path is attached to the center of some star. Here we denote a firecracker by $P_n \odot K_{m_i,1,1},.\ m_i \geq 0,\ i = 1, 2, \ldots, n$, where the i vertex of the path P_n is attached to the center of the star $K_{m_i,1}$.

Definition 1.2 A three distance tree T is a tree which contains a path H such that each vertex of $T \setminus H$ is at a distance at most three from H. We call the path H as the central path of T. Figure 1 represents a three-distance tree. A three distance unicyclic graph is a graph consisting of one cycle C_n such that each vertex of the graph is at distance at most three from C_n. Figure 2 represents a three distant unicyclic graph.

The three distance $(k,\ d)-$graceful trees in this paper are obtained by attaching leaves to the leaves and central path of firecrackers. The three distance unicyclic $(k,\ d)-$ graphs in this paper are obtained by joining the end vertices of the central paths of three distance trees.

Here we use the method involving mathematical formulation for obtaining labeling of the vertices of a graph and then show that such a labeling is a *(k ,d)* - graceful labeling of that graph.

2 Results and Discussions

Constustion [i] 3.1 consider the fire cracker $T = P_n \odot K_{m_i,1,1},.,\ i = 1, 2, \ldots, n$, whose vertices on the central P_n are $c_1,\ c_2,\ c_3,\ \cdots,\ c_n$. The vertices $T \setminus P_n$ adjacent to c_i are $c_{i,i},\ i = 1, 2, \ldots, n$ T $c_{i,i}$, are $c_{i,i,\ j_i},\ j_i = 1, 2, \ldots, m_i,\ i = 1, 2, \ldots, n$. Constructa 3-distance tree T by attaching leaves to the vertices $c_{i,i,\ j_i}$ and denote them by $c_{i,i,\ j_i,\ t_{i,\ j_i}},\ t_{i,\ j_i} = 1, 2, \ldots, s_{i,\ j_i}$, where $s_{i,\ j_i}$ is the number of leaves adjacent to $c_{i,i,\ j_i}$. All the vertices $c_{i,i,\ j_i}$ need not be attached to leaves. Say, out of m_i vertices attached to $c_{i,i,}$, r_i of the m attached to leaves. Let $r = max_{i=1}^{n} (r_i)$. Assume that $m_i \geq r$ for each i. Let $(E(T)| = q$. Obviously, $q = 2n - 1 + \sum_{i=1}^{n} \left(m_i + \left(\sum_{j_i}^{r_i} s_{i,\ j_i}\right)\right]$.

Theorem 3.1 The three distant trees in Construction 3.1 admit $(k,\ d)$ graceful labeling with $d \nmid k$.

Proof: Consider the three distant trees T in Construction 3.1. Define the mapping $f : V(T) \longrightarrow \{0, 1, 2, 3, 4, \ldots, k + (q-1)d\}$ as follows.

For $i = 1,2, \ldots, n$, $f(c_i) = \left(\begin{array}{ll} \frac{(i-1)(r+2)}{2} d & if\ i\ is\ odd \\ k + \left(q - \frac{i}{2}(r+2)\right) d & if\ i\ is\ even \end{array} \right.$

and $f(c_{i,i}) = \left(\begin{array}{ll} k + \left(q - \frac{(i-1)(r+2)}{2} - 1\right) d & if\ i\ is\ odd \\ \left(\frac{i}{2}(r+2) - 1\right] d & if\ i\ is\ even \end{array} \right.$

For $i = 1,2, \ldots, n,\ j_i = 1,2, \ldots, m_i$,

$$f(c_{i,\ i,\ j_i}) = \left(\begin{array}{ll} \left(\begin{array}{ll} j_i d & if \quad j_i \leq r \\ (q + r - j_i] d & if \quad j_i > r \end{array} \right\} & if \quad i = 1 \\ \left(\begin{array}{ll} \left(\frac{(i-1)(r+2)}{2} + j_i\right] d & if \quad j_i \leq r \\ \left(q - \frac{(i-1)(r+2)}{2} - \sum_{p=1}^{i-1} \left(m_p - r + \sum_{j_p=1}^{r} s_{p,\ j_p}\right) + r - j_i\right] d & if \quad j_i > r \end{array} \right\} & if \quad i\ is\ odd\ and\ i > 1 \\ \left(\begin{array}{ll} k + \left(q - \frac{(i-2)(r+2)}{2} - 1 - j_i\right] d & if \quad j_i \leq r \\ k + \left(\frac{i(r+2)}{2} + \sum_{p=1}^{i-1} \left(m_p - r + \sum_{j_p=1}^{r} s_{p,\ j_p}\right) + j_i - 2\right] d & if \quad j_i > r \end{array} \right\} & if \quad i\ is\ even \end{array} \right.$$

For $i = 1,2, \ldots, n,\ j_i = 1,2, \ldots, m_i, t_{i,\ j_i} = 1, 2, \ldots, s_{i,\ j_i},\ f\left(c_{i,i,\ j_i,\ t_{i,\ j_i}}\right) =$

$$\left(\begin{array}{ll} \left(\begin{array}{ll} k + (m_1 - r + t_{1,1}) d & if \quad j_i = 1 \\ k + \left(m_1 - r + \sum_{z=1}^{j_1-1} s_{1,z} + j_1 - 1 + t_{1,j_1}\right) d & if \quad j_i > 1 \end{array} \right\} & if \quad i = 1 \\ \left(\begin{array}{ll} k + \left(\frac{(i-1)(r+2)}{2} + \sum_{p=1}^{i-1} \left(m_p - r + \sum_{z=1}^{j_p} s_{p,z}\right) + m_i - r + t_{i,1}\right) d & if \quad j_i = 1 \\ k + \left(\frac{(i-1)(r+2)}{2} + j_i + \sum_{p=1}^{i-1} \left(m_p - r + \sum_{z=1}^{j_p} s_{p,z}\right) + m_i - r + \sum_{z=1}^{j_i-1} s_{i,z} + t_{i,j_i} - 1\right) d & if \quad j_i > 1 \end{array} \right\} & if \quad i > 1\ is\ odd \\ \left(\begin{array}{ll} \left(q - \frac{(i-2)(r+2)}{2} - \sum_{p=1}^{i-1} \left(m_p - r + \sum_{z=1}^{j_p} s_{p,z}\right) - m_i + r - t_{i,1} - 1\right) d & if \quad j_i = 1 \\ \left(q - \frac{(i-2)(r+2)}{2} - j_i - \sum_{p=1}^{i-1} \left(m_p - r + \sum_{z=1}^{j_p} s_{p,z}\right) - m_i + r - \sum_{z=1}^{j_i-1} s_{i,z} - t_{i,j_i}\right) d & if \quad j_i > 1 \end{array} \right\} & if \quad i\ is\ even \end{array} \right.$$

Defining the labeling g on $E(T)$ by $g(u,\ v) = |f(u) - f(v)|$, we have

For $i = 1, 2, \ldots, n-1,\ g(c_i,\ c_{i+1}) = k + (q - i(r+2)]d$;

for $i = 1, 2, \ldots, n,\ j_i = 1, 2, \ldots, m_i,\ t_{i,\,i,\,j_i} = 1, 2, \ldots, s_{i,\,i,\,j_i}$,

$$g(c_i, c_{i,i}) = \begin{cases} k+(q-(i-1)(r+2)-1)d & if\ i\ is\ odd \\ k+(q-i(r+2)-1]d & if\ i\ is\ even \end{cases};$$

$$g\left(c_{i,i,j_i},\ c_{i,i,j_i,t_{i,\,i,\,j_i}}\right)$$

$$= \begin{cases} \left.\begin{cases} k+(m_1-r+t_{1,1}-1)d & if \quad j_i=1 \\ k+\left(m_1-r+\sum_{z=1}^{j_1-1} s_{1,z}+t_{1,j_1}-1\right)d & if \quad j_i>1 \end{cases}\right\} & if \qquad i=1 \\ \left.\begin{cases} k+\left(\sum_{p=1}^{i-1}\left(m_p-r+\sum_{z=1}^{j_p} s_{p,\,z}\right)+m_i-r+t_{i,1}-1\right)d & if \quad j_i=1 \\ k+\left(\sum_{p=1}^{i-1}\left(m_p-r+\sum_{z=1}^{j_p} s_{p,\,z}\right)+m_i-r+\sum_{z=1}^{j_i-1} s_{1,\,z}+t_{i,\,j_i}-1\right)d & if \quad j_i>1 \end{cases}\right\} & if \quad i>1\ \ is\ odd \end{cases}$$

We find that $g(u,\ v)$ assumes values $\{k,\ k+d,\ k+2d,\ \ldots,\ k+(q-1)d\}$. Therefore, the labels of the edges of T constitute the set $\{k,\ k+d,\ k+2d,\ \ldots,\ k+(q-1)d\}$ and hence the mapping f is a $(k,\ d)$ graceful labeling of T, i.e. T is a $(k,\ d)$ graceful tree with $d \nmid k$.

Theorem 3.2 The three distant unicyclic graphs obtained from the three distant trees in Construction 3.1 by joining the vertices c_1 and c_n admit $(k,\ d)$ graceful labeling with $d \nmid k\ if\ n \equiv (0\ mod\ 4)$.

Proof: Consider the three distant unicyclic graph G in Theorem 3.2. Define the mapping

$f\ :\ V(G) \longrightarrow \{0,\ 1,\ 2,\ 3,\ 4,\ ,\ k+(q-1)d\}$ as follows.

For $i = 1,2,\ \ldots,\ n,\ p_i = 1,\ 2,\ \cdots,\ l_i,\ \ j_i = 1,\ 2,\ \cdots,\ m_i,\ t_{i,j_i} = 1,\ 2,\ \cdots,\ s_{i,j_i}$,

$$f(c_i) = \begin{cases} \frac{(i-1)(r+2)}{2}d & if \quad i\ is\ odd\ and\ i<\frac{n}{2} \\ \left(\frac{(i-1)(r+2)}{2}+1\right]d & if \quad i\ is\ odd\ and\ i>\frac{n}{2} \\ k+\left(q-\frac{i}{2}(r+2)\right]d & if \qquad i\ is\ even \end{cases};$$

$$f(c_{i,i}) = \begin{cases} k+\left(q-\frac{1}{2}(i-1)(r+2)-1\right] & if \qquad i\ is\ odd \\ \left(\frac{i(r+2)}{2}-1\right]d & if \quad i\ is\ even\ and\ i\leq\frac{n}{2} \\ \frac{i}{2}(r+2)d & if \quad i\ is\ even\ and\ i>\frac{n}{2} \end{cases};$$

$$f(c_{i,i,\,j_i}) =$$

$$\begin{cases} \left.\begin{cases} j_1 d & if \quad j_i\leq r \\ (q+r-j_1)d & if \quad j_i>r \end{cases}\right\} & if \quad i=1 \\ \left.\begin{cases} \left(\frac{(i-1)(r+2)}{2}+j_i\right]d & if \quad j_i\leq r \\ \left(q-\frac{(i-1)(r+2)}{2}-\sum_{p=1}^{i-1}\left(m_p-r+\sum_{j_p=1}^{r} s_{p,\,j_p}\right)+r-j_i\right] & if \quad j_i>r \end{cases}\right\} & if \quad i\ is\ odd\ and\ 1<i<\frac{n}{2} \\ \left.\begin{cases} \left(\frac{(i-1)(r+2)}{2}+j_i+1\right]d & if \quad j_i\leq r \\ \left(q-\frac{(i-1)(r+2)}{2}-\sum_{p=1}^{i-1}\left(m_p-r+\sum_{j_p=1}^{r} s_{p,\,j_p}\right)+r-j_i\right] & if \quad j_i>r \end{cases}\right\} & if \quad i\ is\ odd\ and\ i>\frac{n}{2} \\ \left.\begin{cases} k+\left(q-\frac{(i-2)(r+2)}{2}-j_i-1\right] & if \quad j_i\leq r \\ k+\left(\frac{i(r+2)}{2}-\sum_{p=1}^{i-1}\left(m_p-r+\sum_{j_p=1}^{r} s_{p,\,j_p}\right)+j_i-2\right] & if \quad j_i>r \end{cases}\right\} & if \quad i\ is\ even\ and\ 1<i\leq\frac{n}{2} \\ \left.\begin{cases} k+\left(q-\frac{(i-2)(r+2)}{2}-j_i-1\right] & if \quad j_i\leq r \\ k+\left(\frac{i(r+2)}{2}-\sum_{p=1}^{i-1}\left(m_p-r+\sum_{j_p=1}^{r} s_{p,\,j_p}\right)+j_i-1\right] & if \quad j_i>r \end{cases}\right\} & if \quad i\ is\ even\ and\ i>\frac{n}{2} \end{cases};$$

$$f\left(c_{i,i,\,j_i,t_{i,j_i}}\right) =$$

$$\begin{cases} \left.\begin{cases} k+(m_1-r+t_{1,1})d & if \quad j_i=1 \\ k+\left(m_1-r+\sum_{z=1}^{j_1-1}s_{1,z}+j_1-1+t_{1,j_1}\right)d & if \quad j_i>1 \end{cases}\right\} & if \quad i=1 \\ \left.\begin{cases} k+\left(\frac{(i-1)(r+2)}{2}+\sum_{p=1}^{i-1}\left(m_p-r+\sum_{z=1}^{j_p}s_{p,z}\right)+m_i-r+t_{i,1}\right)d & if \quad j_i=1 \\ k+\left(\frac{(i-1)(r+2)}{2}+j_i+\sum_{p=1}^{i-1}\left(m_p-r+\sum_{z=1}^{j_p}s_{p,z}\right)+m_i-r+\sum_{z=1}^{j_i-1}s_{i,z}+t_{i,j_i}-1\right)d & if \quad j_i>1 \end{cases}\right\} & if \quad 1<i<\frac{n}{2} i \text{ is odd} \\ \left.\begin{cases} k+\left(\frac{(i-1)(r+2)}{2}+\sum_{p=1}^{i-1}\left(m_p-r+\sum_{z=1}^{j_p}s_{p,z}\right)+m_i-r+t_{i,1}+1\right)d & if \quad j_i=1 \\ k+\left(\frac{(i-1)(r+2)}{2}+j_i+\sum_{p=1}^{i-1}\left(m_p-r+\sum_{z=1}^{j_p}s_{p,z}\right)+m_i-r+\sum_{z=1}^{j_i-1}s_{i,z}+t_{i,j_i}\right)d & if \quad j_i>1 \end{cases}\right\} & if \quad i \text{ is odd and } i>\frac{n}{2} \\ \left.\begin{cases} \left(q-\frac{(i-2)(r+2)}{2}-\sum_{p=1}^{i-1}\left(m_p-r+\sum_{z=1}^{j_p}s_{p,z}\right)-m_i+r-t_{i,1}\right)d & if \quad j_i=1 \\ \left(q-\frac{(i-2)(r+2)}{2}-j_i-\sum_{p=1}^{i-1}\left(m_p-r+\sum_{z=1}^{j_p}s_{p,z}\right)-m_i+r-\sum_{z=1}^{j_i-1}s_{i,z}-t_{i,j_i}\right)d & if \quad j_i>1 \end{cases}\right\} & if \quad i \text{ is even} \end{cases}$$

Defining the labeling g on E(G) by g(u, v) = |f (u) − f (v)|, we have

$$for\, i=1,2,\cdots,n,\ g(c_i,\ c_{i+1})=\begin{cases} k+(q-i(r+2)]d & if \quad i<\frac{n}{2} \\ k+(q-i(r+2)-1]d & if \quad i\geq\frac{n}{2} \end{cases};$$

for $i=1,2,\cdots,n, j_i=1,2,\cdots,m_i,t_{i,i,\ j_i}=1,2,\cdots,\ s_{i,i,\ j_i}$

$$g(c_i,c_{i,i})=\begin{cases} k+(q-(i-1)(r+2)-1]d & if \quad i \text{ is odd and } i<\frac{n}{2} \\ k+(q-(i-1)(r+2)-2]d & if \quad i \text{ is odd and } i>\frac{n}{2} \\ k+(q-i(r+2)-1]d & if \quad i \text{ is even and } i\leq\frac{n}{2} \\ k+(q-i(r+2)]d & if \quad i \text{ is even and } i>\frac{n}{2} \end{cases};$$

$$\begin{cases} \left.\begin{cases} k+(q-j_1-1)d & if \quad j_i\leq r \\ k+(j_1-r-1)d & if \quad j_i>r \end{cases}\right\} & if \quad i=1 \\ \left.\begin{cases} k+(q-(i-1)(r+2)-j_1-1)d & if \quad j_i\leq r \\ k+\left(\sum_{p=1}^{i-1}\left(m_p-r+\sum_{j_p=1}^{r}s_{p,j_p}\right)-r+j_i-1\right)d & if \quad j_i>r \end{cases}\right\} & if \quad i \text{ is odd and } i<\frac{n}{2} \\ \left.\begin{cases} k+(q-(i-1)(r+2)-j_1-2)d & if \quad j_i\leq r \\ k+\left(\sum_{p=1}^{i-1}\left(m_p-r+\sum_{j_p=1}^{r}s_{p,j_p}\right)-r+j_i-1\right)d & if \quad j_i>r \end{cases}\right\} & if \quad i \text{ is odd and } i>\frac{n}{2} \\ \left.\begin{cases} k+(q-(i-1)(r+2)-j_1)d & if \quad j_i\leq r \\ k+\left(\sum_{p=1}^{i-1}\left(m_p-r+\sum_{j_p=1}^{r}s_{p,j_p}\right)-r+j_i-1\right)d & if \quad j_i>r \end{cases}\right\} & if \quad i \text{ is even and } i\leq\frac{n}{2} \\ \left.\begin{cases} k+(q-(i-1)(r+2)-j_1-1)d & if \quad j_i\leq r \\ k+\left(\sum_{p=1}^{i-1}\left(m_p-r+\sum_{j_p=1}^{r}s_{p,j_p}\right)-r+j_i-1\right)d & if \quad j_i>r \end{cases}\right\} & if \quad i \text{ is even and } i>\frac{n}{2} \end{cases}$$

$$g\left(c_{i,i,j_i},c_{i,i,j_i,t_{i,j_i}}\right)=$$

$$\begin{cases} \left.\begin{cases} k+(m_1-r+t_{1,1}-1)d & if \quad j_i\leq r \\ k+\left(m_1-r+\sum_{z=1}^{j_1-1}s_{1,z}+t_{1,j_1}-1\right)d & if \quad j_i>r \end{cases}\right\} & if \quad i=1 \\ \left.\begin{cases} k+\left(\sum_{p=1}^{i-1}\left(m_p-r+\sum_{z=1}^{j_p}s_{p,z}\right)+m_i-r+t_{i,1}-1\right)d & if \quad j_i\leq r \\ k+\left(\sum_{p=1}^{i-1}\left(m_p-r+\sum_{z=1}^{j_p}s_{p,z}\right)+m_i-r+\sum_{z=1}^{j_i-1}s_{1,z}+t_{i,j_i}-1\right)d & if \quad j_i>r \end{cases}\right\} & if \quad i>1 \end{cases}$$

Find that $g(u,\ v)$ assumes values $\{k,\ k+d,\ k+2d,\ \ldots,\ k+(q-1)d\}$. Therefore, the labels of the edges of T constitute the set $\{k,\ k+d,\ k+2d,\ \ldots,\ k+(q-1)d\}$ and hence the mapping f is a $(k,\ d)$ graceful labeling of G, i.e. G is a $(k,\ d)$ graceful graph when $d \nmid k$.

Theorem 3.3: The 3-distance tree derived from the 3-distance tree in Theorem 3.1 with some or all vertices of $P_n$attached to some leaves is $(k,\ d)$ graceful with $d \nmid k$.

Proof: Let T be a tree derived from the 3-distance tree in Theorem 3.1 with some or all the vertices of P_n may be adjacent to some leaves. Let the vertices on the central path P_n be $c_1,c_n,\ \cdots,c_n$. Let c_i be attached to l_i leaves apart from the vertex $c_{i,i}$.

Let the l_i leaves adjacent to c_i be $x_{i,p_i}, p_i = 1, 2, \ldots, l_i$. The remaining descriptions and notations involving the tree T are the same as those in Theorem 3.2. Define the mapping $f : V(T) \dashrightarrow \{0, 1, 2, 3, 4, , k + (q-1)d\}$ as follows.

For $i = 1, 2, \ldots, n,\ p_i = 1, 2, \cdots, l_i,\ \ j_i = 1, 2, \cdots, m_i,\ t_{i,j_i} = 1, 2, \cdots, s_{i,j_i}$,

$$f(c_i) = \left(\begin{array}{ll} \frac{(i-1)(r+2)}{2} d & if\ i\ is\ odd \\ k + \left(q - \frac{i}{2}(r+2)\right) d & if\ i\ is\ even \end{array} \right. ;\ f(c_{i,i}) = \left(\begin{array}{ll} k + \left(q - \frac{(i-1)(r+2)}{2} - 1\right) d & if\ i\ is\ odd \\ \left(\frac{i}{2}(r+2) - 1\right] d & if\ i\ is\ even \end{array} \right. ;$$

$$f(x_{i,p_i}) = \left(\begin{array}{ll} k + (p_1 - 1) d & if\quad i = 1 \\ k + \left(\frac{(i-1)(r+2)}{2} + \sum_{p=1}^{i-1} \left(l_p + m_p - r + \sum_{j_p=1}^{r} s_{p,z} \right) - l_i + r - j_i + 1 \right] d & if\quad i > 1\ is\ odd \\ \left(q - \frac{i(r+2)}{2} - \sum_{p=1}^{i-1} \left(l_p + m_p - r + \sum_{j_p=1}^{r} s_{p,z} \right) - m_i + r - \sum_{z=1}^{j_i} s_{i,z} - l_i - p_i + 1 \right] d & if\quad i\ is\ even \end{array} \right.$$

$$f(c_{i,i,j_i}) = \left(\begin{array}{ll} \left(\begin{array}{lll} j_i d & if & j_i \le r \\ (q + r - l_i - j_i] d & if & j_i > r \end{array} \right\} & if\quad i = 1 \\ \left(\begin{array}{lll} \left(\frac{(i-1)(r+2)}{2} + j_i \right] d & if & j_i \le r \\ \left(q - \frac{(i-1)(r+2)}{2} - \sum_{p=1}^{i-1} \left(l_p + m_p - r + \sum_{j_p=1}^{r} s_{p,j_p} \right) - l_i + r - j_i + 1 \right] d & if & j_i > r \end{array} \right\} & if\quad i > 1\ is\ odd \\ \left(\begin{array}{lll} k + \left(q - \frac{(i-2)(r+2)}{2} - 1 - j_i \right] d & if & j_i \le r \\ k + \left(\frac{i(r+2)}{2} + \sum_{p=1}^{i-1} \left(l_p + m_p - r + \sum_{j_p=1}^{r} s_{p,j_p} \right) + j_i - 2 \right] d & if & j_i > r \end{array} \right\} & if\quad i\ is\ even \end{array} \right.$$

$$f\left(c_{i,i,j_i,t_{i,j_i}}\right) =$$

$$\left(\begin{array}{ll} \left(\begin{array}{lll} k + (l_1 + m_1 - r + t_{1,1}) d & if & j_i = 1 \\ k + \left(l_1 + m_1 - r + \sum_{z=1}^{j_1-1} s_{1,z} + j_1 - 1 + t_{1,j_1} \right) d & if & j_i > 1 \end{array} \right\} & if\quad i = 1 \\ \left(\begin{array}{lll} k + \left(\frac{(i-1)(r+2)}{2} + \sum_{p=1}^{i-1} \left(l_p + m_p - r + \sum_{z=1}^{j_p} s_{p,z} \right) + l_i + m_i - r + t_{i,1} \right) d & if & j_i = 1 \\ k + \left(\frac{(i-1)(r+2)}{2} + j_i + \sum_{p=1}^{i-1} \left(l_p + m_p - r + \sum_{z=1}^{j_p} s_{p,z} \right) + l_i + m_i - r + \sum_{z=1}^{j_i-1} s_{i,z} + t_{i,j_i} - 1 \right) d & if & j_i > 1 \end{array} \right\} & if\quad i > 1\ is\ odd \\ \left(\begin{array}{lll} \left(q - \frac{(i-2)(r+2)}{2} - \sum_{p=1}^{i-1} \left(l_p + m_p - r + \sum_{z=1}^{j_p} s_{p,z} \right) - l_i - m_i + r - t_{i,1} - 1 \right) d & if & j_i = 1 \\ \left(q - \frac{(i-2)(r+2)}{2} - j_i - \sum_{p=1}^{i-1} \left(l_p + m_p - r + \sum_{z=1}^{j_p} s_{p,z} \right) - l_i - m_i + r - \sum_{z=1}^{j_i-1} s_{i,z} - t_{i,j_i} \right) d & if & j_i > 1 \end{array} \right\} & if\quad i\ is\ even \end{array} \right.$$

Defining the labeling g on $E(T)$ by $g(u, v) = |f(u) - f(v)|$, we have

for $i = 1, 2, \ldots, n-1,\ g(c_i, c_{i+1}) = k + (q - i(r+2)] d$;

for $i = 1, 2, \ldots, n,\ j_i = 1, 2, \ldots, m_i,\ t_{i,i,j_i} = 1, 2, \ldots, s_{i,i,j_i}$,

$$g(c_i, c_{i,i}) = \left(\begin{array}{ll} k + (q - (i-1)(r+2) - 1) d & if\ i\ is\ odd \\ k + (q - i(r+2) - 1] d & if\ i\ is\ even \end{array} \right. ;$$

$$g(c_i, x_{i,p_i}) = \left(\begin{array}{ll} k + (p_1 - 1) d & if\quad i = 1 \\ k + \left(\sum_{p=1}^{i-1} \left(l_p + m_p - r + \sum_{j_p=1}^{r} s_{p,z} \right) - l_i + r - j_i + 1 \right] d & if\quad i > 1\ is\ odd \\ k + \left(\sum_{p=1}^{i-1} \left(l_p + m_p - r + \sum_{j_p=1}^{r} s_{p,z} \right) + m_i - r + \sum_{z=1}^{j_i} s_{i,z} + l_i + p_i - 1 \right] & if\quad i\ is\ even \end{array} \right. ;$$

$$g(c_{i,i}, c_{i,i,j_i}) = \left(\begin{array}{ll} \left(\begin{array}{lll} k + (q - j_1 - 1) d & if & j_i \le r \\ k + (j_1 - r - 1) d & if & j_i > r \end{array} \right\} & if\quad i = 1 \\ \left(\begin{array}{lll} k + (q - (i-1)(r+2) - j_i - 1) d & if & j_i \le r \\ k + \left(\sum_{p=1}^{i-1} \left(m_p - r + \sum_{j_p=1}^{r} s_{p,j_p} \right) - r + j_i - 1 \right) d & if & j_i > r \end{array} \right\} & if\quad i\ is\ odd\ and\ i > 1 \\ \left(\begin{array}{lll} k + (q - (i-1)(r+2) - j_i) d & if & j_i \le r \\ k + \left(\sum_{p=1}^{i-1} \left(m_p - r + \sum_{j_p=1}^{r} s_{p,j_p} \right) - r + j_i - 1 \right) d & if & j_i > r \end{array} \right\} & if\quad i\ is\ even \end{array} \right. ;$$

$$g\left(c_{i,i,j_i},\, c_{i,i,j_i,t_{i,\,i,\,j_i}}\right)$$

$$= \begin{cases} \left.\begin{cases} k+(m_1-r+t_{1,1}-1)\,d & if \quad j_i=1 \\ k+\left(m_1-r+\sum_{z=1}^{j_1-1} s_{1,z}+t_{1,j_1}-1\right)d & if \quad j_i>1 \end{cases}\right\} & if \qquad i=1 \\ \left.\begin{cases} k+\left(\sum_{p=1}^{i-1}\left(m_p-r+\sum_{z=1}^{j_p} s_{p,\,z}\right)+m_i-r+t_{i,1}-1\right)d & if \quad j_i=1 \\ k+\left(\sum_{p=1}^{i-1}\left(m_p-r+\sum_{z=1}^{j_p} s_{p,\,z}\right)+m_i-r+\sum_{z=1}^{j_i-1} s_{1,\,z}+t_{i,\,j_i}-1\right)d & if \quad j_i>1 \end{cases}\right\} & if \quad i>1 \;\; is\ odd \end{cases}.$$

Find that g(u, v) assumes values $\{k,\ k\ +\ d,\ k\ +\ 2d,\ \ldots,\ k\ +\ (q\ -\ 1)d\}$. Therefore, the labels of the edges of T constitute the set $\{k,\ k\ +\ d,\ k\ +\ 2\,d,\ \ldots,\ k\ +\ (q\ -\ 1)d\}$ and hence the mapping f is a $(k,\ d)$ graceful labeling of T , i.e. T is a $(k,\ d)$ graceful tree with $d \nmid k$.

Theorem 3.4 The three distant unicyclic graphs obtained from the three distant trees in Theorem 3.4 by joining the vertices c_1 and c_n admit $(k,\ d)$ graceful labeling with $d \nmid k$ if $n\ \equiv\ (0\ mod\ 4)$.

Proof: Consider the three distant unicyclic graph G in Theorem 3.4. Define the mapping

$f\ :\ V\ (G) - \rightarrow\ \{0,\ 1,\ 2,\ 3,\ 4,\ ,\ k\ +\ (q\ -\ 1)d\}$ as follows.

For $i\ =\ 1,2,\ \ldots,\ n,\ p_i = 1,\ 2,\ \cdots,\ l_i,\ \ j_i = 1,\ 2,\ \cdots,\ m_i,\ t_{i,j_i} = 1,\ 2,\ \cdots,\ s_{i,j_i},$

$$f\left(c_i\right) = \begin{cases} \frac{(i-1)(r+2)}{2}d & if \quad i\ is\ odd\ and\ i<\frac{n}{2} \\ \left(\frac{(i-1)(r+2)}{2}+1\right]d & if \quad i\ is\ odd\ and\ i>\frac{n}{2} \\ k+\left(q-\frac{i}{2}\,(r+2)\right]d & if \qquad i\ is\ even \end{cases}$$

$$f\left(c_{i,i}\right) = \begin{cases} k+\left(q-\frac{1}{2}\,(i-1)\,(r+2)-1\right] & if \qquad i\ is\ odd \\ \left(\frac{i(r+2)}{2}-1\right]d & if \quad i\ is\ even\ and\ i\le\frac{n}{2} \\ \frac{i}{2}\,(r+2)\,d & if \quad i\ is\ even\ and\ i>\frac{n}{2} \end{cases};$$

$$f\left(x_{i,p_i}\right) = \begin{cases} k+(p_1-1)\,d \quad if \quad i=1 \\ k+\left(\frac{(i-1)(r+2)}{2}+\sum_{p=1}^{i-1}\left(l_p+m_p-r+\sum_{z=1}^{j_p} s_{p,z}\right)+p_i-1\right]d \quad if \quad 1<i<\frac{n}{2}\ ,\ i\ odd \\ k+\left(\frac{(i-1)(r+2)}{2}+\sum_{p=1}^{i-1}\left(l_p+m_p-r+\sum_{z=1}^{j_p} s_{p,z}\right)+p_i\right]d \quad if \quad i\ is\ odd,\ i>\frac{n}{2} \\ \left(q-\frac{i(r+2)}{2}-\sum_{p=1}^{i-1}\left(l_p+m_p-r+\sum_{z=1}^{j_p} s_{p,z}\right)-m_i+r+\sum_{z=1}^{j_i} s_{i,z}-p_i+1\right]d \quad if \quad i\ is\ even \end{cases};$$

$$f\left(c_{i,i,\,j_i}\right) =$$

$$\begin{cases} \left.\begin{cases} j_1 d & if \quad j_i\le r \\ (q+r-j_1)\,d & if \quad j_i>r \end{cases}\right\} \quad if \quad i=1 \\ \left.\begin{cases} \left(\frac{(i-1)(r+2)}{2}+j_i\right]d & if \quad j_i\le r \\ \left(q-\frac{(i-1)(r+2)}{2}-\sum_{p=1}^{i-1}\left(l_p+m_p-r+\sum_{j_p=1}^{r} s_{p,\,j_p}\right)-l_i+r-j_i\right] & if \quad j_i>r \end{cases}\right\} \quad if \quad i\ is\ odd,\ 1\ <i\ <\frac{n}{2} \\ \left.\begin{cases} \left(\frac{(i-1)(r+2)}{2}+j_i+1\right]d & if \quad j_i\le r \\ \left(q-\frac{(i-1)(r+2)}{2}-\sum_{p=1}^{i-1}\left(l_p+m_p-r+\sum_{j_p=1}^{r} s_{p,\,j_p}\right)-l_i+r-j_i\right] & if \quad j_i>r \end{cases}\right\} \quad if \quad i\ is\ odd,\ i>\frac{n}{2} \\ \left.\begin{cases} k+\left(q-\frac{(i-2)(r+2)}{2}-j_i-1\right] & if \quad j_i\le r \\ k+\left(\frac{i(r+2)}{2}-\sum_{p=1}^{i-1}\left(l_p+m_p-r+\sum_{j_p=1}^{r} s_{p,\,j_p}\right)+j_i-2\right] & if \quad j_i>r \end{cases}\right\} \quad if \quad i\ is\ even,\ 1\ <i\ \le\frac{n}{2} \\ \left.\begin{cases} k+\left(q-\frac{(i-2)(r+2)}{2}-j_i-1\right] & if \quad j_i\le r \\ k+\left(\frac{i(r+2)}{2}-\sum_{p=1}^{i-1}\left(l_p+m_p-r+\sum_{j_p=1}^{r} s_{p,\,j_p}\right)+j_i-1\right] & if \quad j_i>r \end{cases}\right\} \quad if \quad i\ is\ even,\ i>\frac{n}{2} \end{cases}$$

$$f\left(c_{i,i,\,j_i,t_{i,j_i}}\right) =$$

$$\begin{cases} \left.\begin{cases} k+(l_1+m_1-r+t_{1,1})\,d & if \quad j_i=1 \\ k+\left(l_1+m_1-r+\sum_{z=1}^{j_1-1}s_{1,z}+j_1-1+t_{1,j_1}\right)d & if \quad j_i>1 \end{cases}\right\} & if \quad i=1 \\ \left.\begin{cases} k+\left(\frac{(i-1)(r+2)}{2}+\sum_{p=1}^{i-1}\left(l_p+m_p-r+\sum_{z=1}^{j_p}s_{p,z}\right)+l_i+\ m_i-r+t_{i,1}\right)d & if \quad j_i=1 \\ k+\left(\frac{(i-1)(r+2)}{2}+j_i+\sum_{p=1}^{i-1}\left(l_p+m_p-r+\sum_{z=1}^{j_p}s_{p,z}\right)+l_i+m_i-r+\sum_{z=1}^{j_i-1}s_{i,z}+t_{i,j_i}-1\right)d & if \quad j_i>1 \end{cases}\right\} & if \quad i \text{ is odd},\ 1\ <i< \\ \left.\begin{cases} k+\left(\frac{(i-1)(r+2)}{2}+\sum_{p=1}^{i-1}\left(l_p+m_p-r+\sum_{z=1}^{j_p}s_{p,z}\right)+l_i+m_i-r+t_{i,1}+1\right)d & if \quad j_i=1 \\ k+\left(\frac{(i-1)(r+2)}{2}+j_i+\sum_{p=1}^{i-1}\left(l_p+m_p-r+\sum_{z=1}^{j_p}s_{p,z}\right)+l_i+m_i-r+\sum_{z=1}^{j_i-1}s_{i,z}+t_{i,j_i}\right)d & if \quad j_i>1 \end{cases}\right\} & if \quad i \text{ is odd and } i>\frac{n}{2} \\ \left.\begin{cases} \left(q-\frac{(i-2)(r+2)}{2}-j_i-\sum_{p=1}^{i-1}\left(l_p+m_p-r+\sum_{z=1}^{j_p}s_{p,z}\right)-l_i-m_i+r-t_{i,1}-1\right)d & if \quad j_i=1 \\ \left(q-\frac{(i-2)(r+2)}{2}-j_i-\sum_{p=1}^{i-1}\left(l_p+m_p-r+\sum_{z=1}^{j_p}s_{p,z}\right)-l_i-m_i+r-\sum_{z=1}^{j_i-1}s_{i,z}-t_{i,j_i}\right)d & if \quad j_i>1 \end{cases}\right\} & if \quad i \text{ is even} \end{cases}$$

Defining the labeling g on $E(T)$ by $g(u,\ v)\ =\ |f(u)\ -\ f(v)|$, we have

$for\, i=1,2,\cdots,n-1,\ g(c_i,\ c_{i+1})=\begin{cases} k+(q-i(r+2)]\,d & if \quad i<\frac{n}{2} \\ k+(q-i(r+2)-1]\,d & if \quad i\geq\frac{n}{2} \end{cases};$

for $i=1,2,\cdots,n,\ j_i=1,2,\cdots,m_i, t_{i,i,\ j_i}=1,2,\cdots,\ s_{i,i,\ j_i}$

$$g(c_i,c_{i,i})=\begin{cases} k+(q-(i-1)(r+2)-1]\,d & if \quad i \text{ is odd and } i<\frac{n}{2} \\ k+(q-(i-1)(r+2)-2]\,d & if \quad i \text{ is odd and } i>\frac{n}{2} \\ k+(q-i(r+2)-1]\,d & if \quad i \text{ is even and } i\leq\frac{n}{2} \\ k+(q-i(r+2)]\,d & if \quad i \text{ is even and } i>\frac{n}{2} \end{cases};$$

$$g(c_i,x_{i,p_i})=\begin{cases} k+(p_1-1)\,d & if \quad i=1 \\ k+\left(\sum_{p=1}^{i-1}\left(l_p+m_p-r+\sum_{z=1}^{j_p}s_{p,z}\right)+p_i-1\right]d & if \quad i \text{ is odd and } i>1 \\ k+\left(\sum_{p=1}^{i-1}\left(l_p+m_p-r+\sum_{z=1}^{j_p}s_{p,z}\right)+m_i-r-\sum_{z=1}^{j_i}s_{i,z}+p_i-1\right] & if \quad i \text{ is even} \end{cases}$$

$$\begin{cases} \left.\begin{cases} k+(q-j_1-1)\,d & if \quad j_i\leq r \\ k+(j_1-r-1)\,d & if \quad j_i>r \end{cases}\right\} & if \quad i=1 \\ \left.\begin{cases} k+(q-(i-1)(r+2)-j_1-1)\,d & if \quad j_i\leq r \\ k+\left(\sum_{p=1}^{i-1}\left(m_p-r+\sum_{j_p=1}^{r}s_{p,j_p}\right)-r+j_i-1\right)d & if \quad j_i>r \end{cases}\right\} & if \quad i \text{ is odd},\ i<\frac{n}{2} \\ \left.\begin{cases} k+(q-(i-1)(r+2)-j_1-2)\,d & if \quad j_i\leq r \\ k+\left(\sum_{p=1}^{i-1}\left(m_p-r+\sum_{j_p=1}^{r}s_{p,j_p}\right)-r+j_i-1\right)d & if \quad j_i>r \end{cases}\right\} & if \quad i \text{ is odd},\ i>\frac{n}{2} \\ \left.\begin{cases} k+(q-(i-1)(r+2)-j_1)\,d & if \quad j_i\leq r \\ k+\left(\sum_{p=1}^{i-1}\left(m_p-r+\sum_{j_p=1}^{r}s_{p,j_p}\right)-r+j_i-1\right)d & if \quad j_i>r \end{cases}\right\} & if \quad i \text{ is even},\ i\leq\frac{n}{2} \\ \left.\begin{cases} k+(q-(i-1)(r+2)-j_1-1)\,d & if \quad j_i\leq r \\ k+\left(\sum_{p=1}^{i-1}\left(m_p-r+\sum_{j_p=1}^{r}s_{p,j_p}\right)-r+j_i-1\right)d & if \quad j_i>r \end{cases}\right\} & if \quad i \text{ is even},\ i>\frac{n}{2} \end{cases}$$

$$g\left(c_{i,i,j_i},c_{i,i,j_i,t_{i,j_i}}\right)=$$

$$\begin{cases} \left.\begin{cases} k+(m_1-r+t_{1,1}\ -1)\,d & if \quad j_i\leq r \\ k+\left(m_1-r+\sum_{z=1}^{j_1-1}s_{1,z}+t_{1,j_1}-1\right)d & if \quad j_i>r \end{cases}\right\} & if \quad i=1 \\ \left.\begin{cases} k+\left(\sum_{p=1}^{i-1}\left(m_p-r+\sum_{z=1}^{j_p}s_{p,z}\right)+m_i-r+t_{i,1}-1\right)d & if \quad j_i\leq r \\ k+\left(\sum_{p=1}^{i-1}\left(m_p-r+\sum_{z=1}^{j_p}s_{p,z}\right)+m_i-r+\sum_{z=1}^{j_i-1}s_{1,z}+t_{i,j_i}-1\right)d & if \quad j_i>r \end{cases}\right\} & if \quad i>1 \end{cases}$$

Find that $g(u,\ v)$ assumes values $\{k,\ k\ +\ d,\ k\ +\ 2d,\ \ldots,\ k\ +\ (q\ -\ 1)d\}$. Therefore, the labels of the edges of T constitute the set $\{k,\ k\ +\ d,\ k\ +\ 2d,\ \ldots,\ k\ +\ (q\ -\ 1)d\}$ and hence the mapping f is a $(k,\ d)$ graceful labeling of G, i.e. G is a $(k,\ d)$ graceful graph when $d\nmid k$.

Discussion: Figures 2, 3 and 4 illustrate our results, i.e., Theorems 3.1 – 3.4

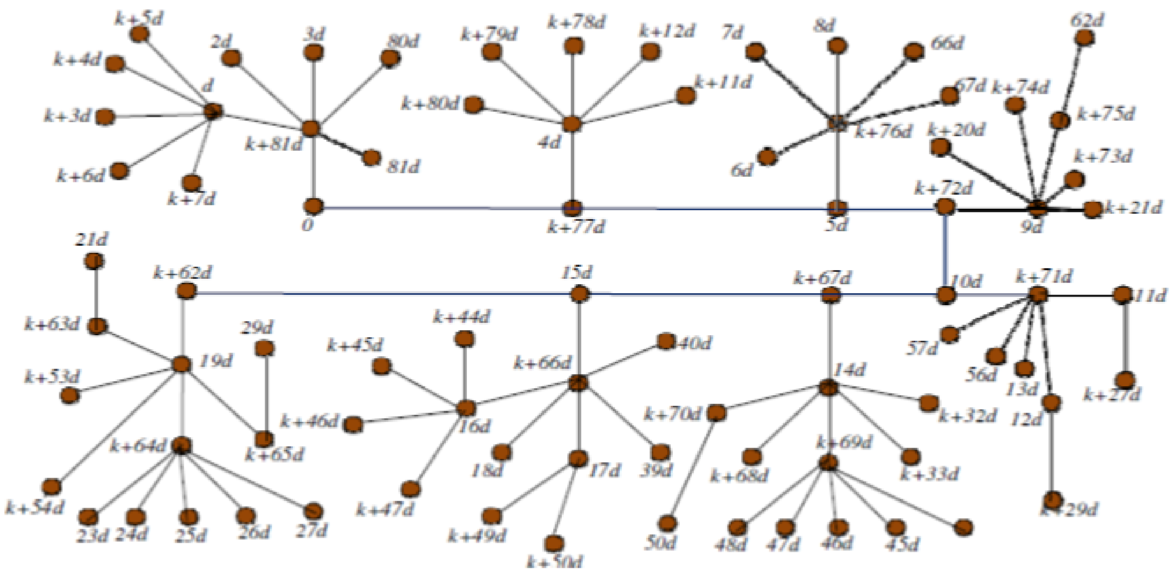

Fig 1. A three distance tree of the type in Theorem 3.1 with a graceful labeling. Here $n = 8, m_1 = 5, m_2 = 5, m_3 = 5, m_4 = 5, m_5 = 5, m_6 = 5, m_7 = 5, m_8 = 5, r =$
$3, s_{1,1} = 5, s_{4,1} = 1, s_{5,1} = 1, s_{5,2} = 1, s_{6,1} = 1, s_{6,2} = 5, s_{7,1} = 4, s_{7,2} = 2, s_{8,1} =$
$1, s_{8,2} = 5, s_{8,3} = 1$

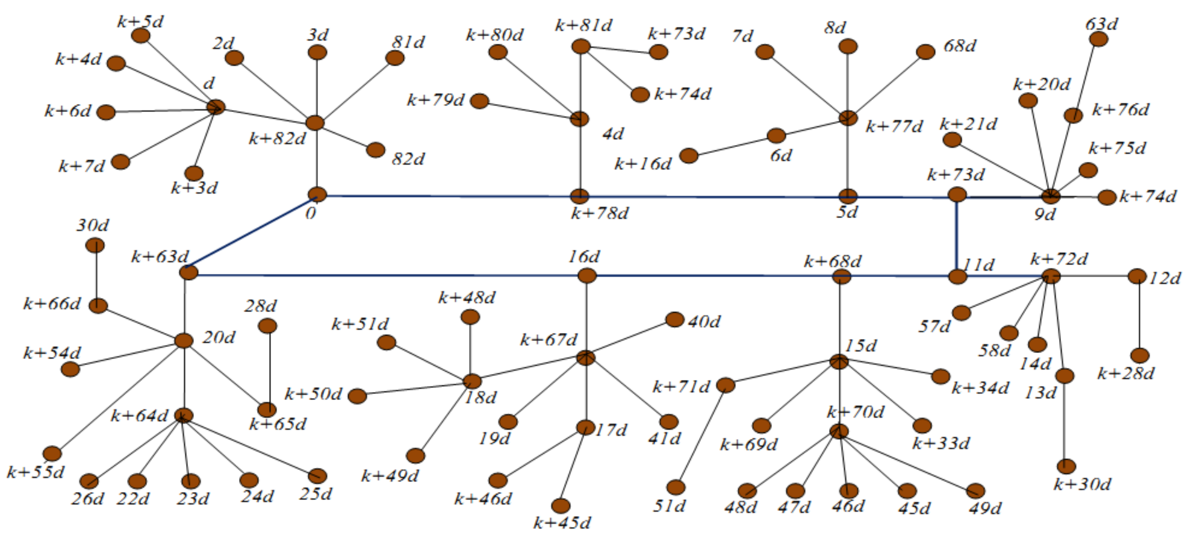

Fig 2. A three distance unicyclic graph of the type in Theorem 3.2 with a $(k,\ d)$ graceful labeling. Here $n = 8, m_1 = 5, m_2 = 3, m_3 = 4, m_4 =$
$5, m_5 = 5, m_6 = 5, m_7 =$
$5, m_8 = 5, r = 3, s_{1,1} = 5, s_{2,1} = 2, s_{3,1} = 1 s_{4,1} = 1, s_{5,1} = 1, s_{5,2} = 1, s_{6,1} =$
$1, s_{6,2} = 5, s_{7,1} = 4, s_{7,2} = 2, s_{8,1} = 1, s_{8,2} = 5, s_{8,3} = 1$

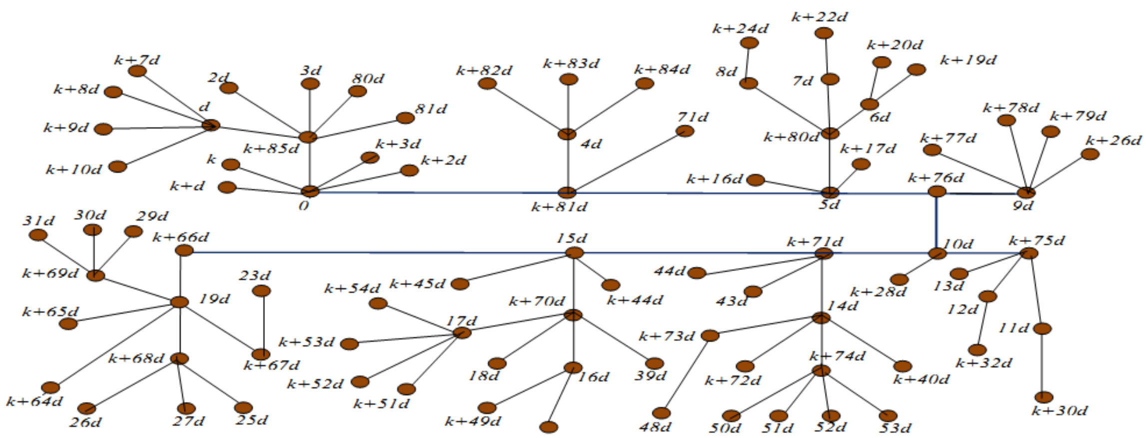

Fig 3. A three distance tree of the type in Theorem 3.3 with a $(k,\ d)$ graceful labeling. Here $n\ =\ 8,\ m_1 = 5,\ m_2\ =\ 3,\ m_3 = 3,\ m_4 = 4,\ m_5 = 3,\ m_6\ =\ 4,\ m_7 = 4,\ m_8 = 5,\ r\ =\ 3,\ s_{1,1} = 5,\ s_{3,1} = 2,\ s_{3,2} = 1,\ s_{3,3} = 1,\ s_{5,1} = 1,\ s_{5,2} = 1,\ s_{6,1} =\ 4,\ s_{6,2} = 1,\ s_{7,1} = 2,\ s_{7,2} = 4,\ s_{8,1} = 3,\ s_{8,2} = 3,\ s_{8,3} = 1,\ l_1 = 1,\ l_2 = 1,\ l_3 = 2,\ l_4 = 0,\ l_5 = 1,\ l_6 = 2,\ l_7 = 2,\ l_8 = 0.$

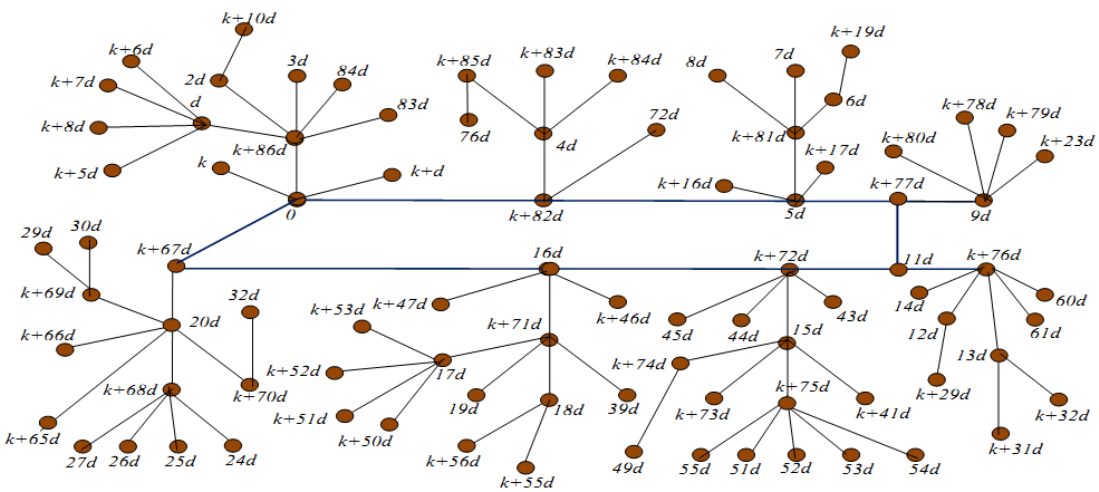

Fig 4. A three distance unicyclic graph of the type in Theorem 3.4 with a $(k,\ d)$ graceful labeling. Here $n\ =\ 8,\ m_1\ =\ 5,\ m_2\ =\ 3,\ m_3 = 3,\ m_4 =\ 4,\ m_5 =\ 5,\ m_6\ =\ 4,\ m_7 =\ 4,\ m_8 =\ 4,\ r\ =\ 3,\ s_{1,1} =\ 4,\ s_{2,1} =\ 1, s_{3_{,1}} =\ 1,\ s_{5,1} =\ 1,\ s_{5,2} =\ 1,\ s_{6,1} =\ 5,\ s_{6,2} =\ 1,\ s_{7,1} = 4,\ s_{7,2} =\ 2,\ s_{8,1} =\ 3,\ s_{8,2} =\ 2,\ s_{8,3} =\ 4,\ l_1 =\ 2,\ l_2 =\ 1,\ l_3 =\ 2,\ l_4 =\ 0,\ l_5 =\ 0,\ l_6 =\ 3,\ l_7 =\ 2,\ l_8 =\ 0.$

3 Conclusion

In this article we give $(k \quad d)$ graceful labeling to a class three distance trees which are obained from a firecracker by attaching leaves either to the leaves of the firecracker or the vertices on the central path of the firecracker or both. Find that in Construction 3.1 we assume the condition that $m_i \geq r$ for each i Moreover, here we give $(k \quad d)$ graceful labeling to a class of three distance unicyclic graphs obtained by joining end vertices of the central path of a $(k \quad d)$ graceful three distance tree mentioned above. Our effort is the first of its kind where we give $(k \quad d)$graceful labeling to a family of three distance three and three distance unicyclic graph. However, the future advancement of this result requires to cover all three distance trees and three distance unicyclic graphs, i.e., by generalizing our results dropping the assumption $m_{i \geq r}$ for each i

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