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An Analysis of a Two-State Markovian Retrial Queueing Model with Priority Customers

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Abstract

Objective: This study considered a system of retrial queues with two types of customers: high-priority and low-priority. This study deals to find the time dependent probabilities of exact number of arrivals and departures from the system when server is free or busy. Numerical solution and graphical representation will also be presented. **Method:** For this model, we solved difference differential equations recursively and used Laplace transformation to obtain the transient state probabilities of exact number of arrivals and departures from the system when server is free or busy. **Findings:** Time-dependent probabilities of exact number of arrivals (primary arrivals, arrivals in high priority queue, arrivals in low priority queue) in the system and exact number of departures (primary departures, departures from high priority queue, departures from low priority queue) from the system by a given time for when the server is idle and when the server is busy are obtained. Various interesting performance measures along with some special cases are also obtained. Conversion of two state model into single state model was discussed. Numerical illustrations are also presented using MATLAB programming along with the busy period probabilities of the system and server. **Novelty:** In past research, models considered arrivals and departures from the orbit whereas in present model arrivals and departures from the system are studied along with the concept of retrial and priority customers. **Applications**: Priority retrial queues are used in many applications like real time systems, operating systems, manufacturing system, simulation and medical service systems.

Keywords: Arrivals; Departures; Probability; Priority; Retrial

1 Introduction

Retrial queues have been extensively studied, since they arise in various systems such as telephone switching systems, telecommunication, call centers and computer networks. The characteristic feature of retrial queue is that when arriving customers find the server is unavailable, then the customer makes a new attempt to get service after a random amount of time. For instance-in call centers, there are certain number of servers that answer the customer calls. When a customer call arrives, it will be served immediately if server is available. If all servers are busy with other calls then the customer will be put on hold, and will be asked to wait until a server become available. Some customers are patient enough to wait for a server to become available, while others will hang up or abandon after waiting for some time or immediately. Whenever the server is available then the customer gets the service from the server and leaves the system permanently. The customers who redial and try to access the call center represent the concept of retrial.

For a complete survey on retrial queues, refer the work of Artalejo[(1)] and the monographs by Falin and Templeton[(2)], Artalejo and Gomez-Corral[(3)]. Melikov et al.[(4)] discussed retrial queues with unreliable servers and delayed feedback. Goel and Kulshrestha[(5)] described queueing based spectrum management in cognitive radio networks with retrial and heterogeneous service classes.

There are two common situations in priority discipline, one is preemptive and other is non-preemptive. In the case of preemptive, the customer with higher priority is allowed to receive service immediately even if a customer with lower priority is already in service; that is, the service of lower priority customer should be preempted or stopped and to be restart again after the service of high priority customer. In the second case of priority discipline which is called non-preemptive, the higher priority customer goes to the front of the queue but cannot get service until the customer presently in service has completed his service, even though the customer in service has a lower priority.

An obvious example of such a situation deals with the classification of patients arriving at an emergency room of a hospital. Another example appears in a health care system in which the patients who require an immediate surgery or transplant have provided a priority over the patients, who has been otherwise waiting in a line. In many communication systems, a priority is given to certain classes or calls to improve the grade of service over other classes. In a wireless communication system, which delivers a wide variety of services, priority rule is often followed. For example: voice calls being delay sensitive are required to have a higher priority than data calls. While it is true that priority systems reduce the waiting times of the higher priority customers, they also necessarily increase the waiting times of the lower priority ones.

Nair et al.[(6)] considered MMAP/(PH,PH)/1 queue with priority loss through feedback in which two single server queueing systems to which customers of two distinct priorities (P_1 *and* P_2) arrive according to a Marked Markovian arrival process. Shajin et al.[(7)] discussed a priority queueing-inventory problem having two types of customers with additional items. Lan and Tang[(8)] suggested unreliable discrete-time retrial queue with probabilistic preemptive priority, balking customers and replacements of repair times. Boualem and Touche[(9)] has developed a non markovian priority retrial queueing model with two types of customers. In this study, a particular interest is devoted to the stochastic monotonicity approach. Raj and Jain[(10)] optimized a model of traffic control in MMAP [2] /PH [2] /S priority queueing model with PH retrial times and preemptive repeat policy.

Pegden & Rosenshine[(11)] gave an abstract idea about two-state for a classical queueing model M/M/1/∞. The measures provided by Pegden & Rosenshine supplies better insights into the behavior of queueing system than the probability of exact number of units in the system at a given time.

Many other researchers already worked on the concept of retrial queues with non preemptive priority system but the major difference is that they deal with total number of customers in the system instead of considering the two-dimensional state retrial queueing system in the form of (i,j). Here ‘i’ represent total number of arrivals in the system and ‘j’ represent total number of departures from the system by time t. One similar study “two-state retrial queueing system with priority customers” has also been presented. In this study, authors found the time dependent probabilities of exact number of arrivals and departures from the orbit instead of finding the probabilities from the system. In my present study, the time-dependent probabilities of exact number of arrivals (primary arrivals, arrivals in high priority queue, arrivals in low priority queue) in the system and exact number of departures (primary departures, departures from high priority queue, departures from low priority queue) from the system by a given time have to be found. The present work is better than the previous research because here we considered the units from the system (including orbit) instead of considering the units only from the orbit and also considered the whole system including orbit gives finer results comparatively to the results consider from the orbit.

This investigation considers a general comparison of other systems in the sense that we have combined different characteristics together covering (i) non-preemptive priority discipline (ii) retrial queueing system (iii) two-dimensional state. The model proposed by us by examining the three above factors simultaneously provided a superior model contrasting the other queueing models.

Singla and Kaur[(12)] obtained transient state probabilities for exact number of arrivals and exact number of departures for retrial queueing model having two identical parallel servers with feedback, however in contrast the present study carried out the probabilities without the concept of feedback. Singla and Kalra[(13)] obtained time dependent probabilities of a two-dimensional multiserver queueing system with repeated attempts and impatience.

The summary of this paper is as follows: Section 2, deals with model description, notations used, mathematical formulation and difference differential equations of the model. Section 3, elucidates the transient state solution of our model. Section 4, demonstrates and verify some important results along with some special cases. Numerical solution and graphical representation are given in section 5 with some busy period probabilities of system and server. In section 6, concluding remarks are given.

2 Methodology

2.1 Model Description

A two-state M/M/1 retrial queueing model with priority subscribers is considered. A primary customer, who enters into the queueing system arrives according to a Poisson process with rate λ_1 If the server is free at the arrival time of primary customer then his service begins immediately and he leaves the system after service completion. However, if customer finds the server occupied at his time of arrival then he leaves the service area and goes to the secondary queue (orbit) and joins high priority queue in orbit with rate λ_2 or low priority queue in orbit with rate λ_3. After some random delay, the higher priority customer repeats his attempts to get service from orbit with rate θ_2 and a lower priority customer with rate θ_3. A low priority unit is retrying for service only when there are no units in the high priority queue. If a low priority unit is being served, then a high priority unit awaits for his service until the completion of the service of low priority unit.

Laplace transformation $\bar{f}(s)$ of $f(t)$ is given by

$\bar{f}(s) = \int_0^\infty e^{-st} f(t)dt$, Re (s) > 0

The Laplace inverse of

$\frac{Q(p)}{P(p)}$ *is* $\sum_{k=1}^{n} \sum_{l=1}^{m_k} \frac{t^{m_k-l}\, e^{a_k t}}{(m_k-l)!(l-1)!} \times \frac{d^{l-1}}{dp^{l-1}} \frac{Q(p)}{P(p)} (p-a_k)^{m_k} \forall$ p=a_k, $a_i \neq a_k$ for i $\neq$ k.

where,

P(p)= $(p-a_1)^{m_1}(p-a_2)^{m_2} \ldots\ldots\ldots\ldots (p-a_n)^{m_n}$

Q(p) is a polynomial of degree <m_1+m_2+m_3+............m_n-1.

If L^{-1}\{f(s)\} = F(t) and L^{-1}\{g(s)\} = G(t), then

L^{-1}\{f(s) g(s)\} = $\int_0^t F(u)\, G(t-u)$du = F * G, F * G is called the convolution of F and G.

The Laplace inverse of $\bar{N}^{a,b,c}_{n_{1,n_{2,}n_3}}(s) = \frac{1}{(s+a)^{n_1}(s+b)^{n_2}(s+c)^{n_3}}$ is

$$N^{a,b,c}_{n_{1,n_{2,}n_3}}(t) = \sum_{l=1}^{n_3} \sum_{m=1}^{l} \frac{e^{-at}\ t^{n_3-l}(-1)^{m+1}\left(\frac{l-1}{m-1}\right)\left(\prod_{g_1=0}^{l-m-1}(n_1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(n_2+g_2)\right)}{(n_3-l)!(m-1)!\,(b-a)^{n_2+m-1}(c-a)^{n_1+l-m}}$$

$$+\sum_{l=1}^{n_2} \sum_{m=1}^{l} \frac{e^{-bt}\ t^{n_2-l}(-1)^{m+1}\left(\frac{l-1}{m-1}\right)\left(\prod_{g_1=0}^{l-m-1}(n_1+g_1)\right)\left(\prod_{g_2=0}^{m-2}(n_3+g_2)\right)}{(n_2-l)!\,(m-1)!\,(a-b)^{n_3+m-1}(c-b)^{n_1+l-m}}$$

$$+\sum_{l=1}^{n_1} \sum_{m=1}^{l} \frac{e^{-ct}\ t^{n_1-l}(-1)^{m+1}\left(\frac{l-1}{m-1}\right)\left(\prod_{g_1=0}^{l-m-1}(n_2+g_1)\right)\left(\prod_{g_2=0}^{m-2}(n_3+g_2)\right)}{(n_1-l)!\,(m-1)!\,(a-c)^{n_3+m-1}(b-c)^{n_2+l-m}}$$

If $L^{-1}(f(s)\} = F(t)$and $L^{-1}(g(s)\} = G(t)$, then

$L^{-1}(f(s)\, g(s)\} = \int_0^t F(u)\, G(t-u)$du = $F * G, F * G$is called the convolution of F and G.

2.2 The Two-Dimensional State Model

An M/M/1 queue is a stochastic process whose state space is the set $\{0,1,2,3\ldots..\}$ where the values corresponds to the number of arrivals enter in the system being served and depart from the system.This model includes features that are quite general, and as a result, a rather extensive class of well-known and often applied queueing systems can be viewed as simply special cases of the fundamental birth-and-death model.

For distribution of arrivals, service times and retrials we use the following assumptions and notations:

- The primary arrivals follow a Poisson distribution with parameter λ_1
- The high priority arrival calls and low priority arrival calls also follow a Poisson distribution with parameter λ_2 and λ_3 respectively

- The high priority calls and low priority calls repeated their attempt to get service in a Poisson distribution with parameter θ_2 and θ_3 respectively
- Service times are exponentially distributed with parameter μ
- The stochastic process involved viz. arrivals of units, departures of units and retrials are statistically independent.

2.2.1 Definitions

$P^{(0)}_{i,k,j}(t)$ = Probability that there are exactly i arrivals (including primary arrivals, arrivals in high priority queue and arrivals in low priority queue) in the system, k units standing in orbit with higher priority in the system and j departures (including primary departures, departures from high priority queue and departures from low priority queue) from the system by time t when server is idle.

$P^{(1)}_{i,k,j}(t)$ = Probability that there are exactly i arrivals (including primary arrivals, arrivals in high priority queue and arrivals in low priority queue) in the system, k units standing in orbit with higher priority in the system and j departures (including primary departures, departures from high priority queue and departures from low priority queue) from the system by time t when server is busy.

$P_{i,k,j}(t)$= Probability that there are exactly i arrivals (including primary arrivals, arrivals in high priority queue and arrivals in low priority queue) in the system, k units standing in orbit with higher priority in the system and j departures (including primary departures, departures from high priority queue and departures from low priority queue) from the system by time t.

$$P_{i,k,j}(t) = P^{(0)}_{i,k,j}(t) + P^{(1)}_{i,k,j}(t) \forall i\,,\, j \quad i > k \text{ and } i \geq j$$

also

$$P^{(1)}_{i,k,j}(t) = 0, \qquad i \leq j$$

$$P^{(0)}_{i,k,j}(t) = 0, \qquad i < j$$

Initially

$$P^{(0)}_{0,0,0}(0) = 1; P^{(0)}_{i,k,j}(0) = 0 \;\&\; P^{(1)}_{i,k,j}(0) = 0 \quad \forall \quad i,k,j \neq 0$$

2.3 The difference – differential equations governing the system are

$$\frac{d}{dt}P^{(0)}_{0,0,0}(t) = -\lambda_1 P^{(0)}_{0,0,0}(t) \tag{2.1}$$

$$\frac{d}{dt}P^{(0)}_{i,0,j}(t) = -(\lambda_1 + (i-j)\theta_3)P^{(0)}_{i,0,j}(t) + \mu P^{(1)}_{i,0,j-1}(t)$$

$$i, j \geq 0 \;\& i \geq j \tag{2.2}$$

$$\frac{d}{dt}P^{(0)}_{i,k,j}(t) = -(\lambda_1 + (i-k-j)\theta_3 + k\theta_2)P^{(0)}_{i,k,j}(t) + \mu P^{(1)}_{i,k,j-1}(t)$$

$$i > k, \quad j,k > 0 \;\& \; i \geq k+j \tag{2.3}$$

$$\frac{d}{dt}P^{(1)}_{i,0,j}(t) = -(\lambda_2 + \lambda_3 + \mu)P^{(1)}_{i,0,j}(t) + \lambda_3 P^{(1)}_{i-1,0,j}(t) + \lambda_1 P^{(0)}_{i-1,0,j}(t) +$$

$$(i-j)\theta_3 P^{(0)}_{i,0,j} \quad + \theta_2 P^{(0)}_{i,1,j}(t)$$

$$i > 0, \quad j = 0 \; to \; i-1 \;\& \; i \geq j+1 \tag{2.4}$$

$$\frac{d}{dt}P^{(1)}_{i,k,j}(t) = -(\lambda_2 + \lambda_3 + \mu)P^{(1)}_{i,k,j}(t) + \lambda_2 P^{(1)}_{i-1,k-1,j}(t) + \lambda_3 P^{(1)}_{i-1,k,j}(t) + \lambda_1 P^{(0)}_{i-1,k,j}(t) +$$
$$(i-k-j)\theta_3 P^{(0)}_{i,k,j}(t) + (k+1)\theta_2 P^{(0)}_{i,k+1,j}(t)$$

$$i > k,\ \ k > 0,\ \ j = 0\ to\ i-1\ \&\ i\ \geq k+j+1 \tag{2.5}$$

Using the Laplace transformation $\bar{f}$ (s) of $f(t)$ given by

$\bar{f}$ (s) = $\int_0^\infty e^{-st} f(t)$dt, Re (s) > 0

in the equations (2.1) - (2.5) along with the initial conditions, we have

$$(s+\lambda_1)\,\bar{P}^{(0)}_{0,0,0}(s) = P^{(0)}_{0,0,0}(0) \tag{2.6}$$

$$(s+\lambda_1+(i-j)\,\theta_3)\,\bar{P}^{(0)}_{i,0,j}(s) = \bar{P}^{(1)}_{i,0,j-1}(s)\ \ i,j \geq 0\ \&i \geq j \tag{2.7}$$

$$(s+\lambda_1+(i-k-j)\,\theta_3+k\theta_2)\bar{P}^{(0)}_{i,k,j}(s) = \bar{P}^{(1)}_{i,k,j-1}(s)$$

$$i > k,\ \ \ j,k > 0\ \&\ i \geq k+j \tag{2.8}$$

$$(s+\lambda_2+\lambda_3+\mu)\,\bar{P}^{(1)}_{i,0,j}(s) =\ \lambda_3\bar{P}^{(1)}_{i-1,0,j}(s)+\lambda_1\bar{P}^{(0)}_{i-1,0,j}(s)+(i-j)\,\theta_3\bar{P}^{(0)}_{i,0,j}(s)+\theta_2\bar{P}^{(0)}_{i,1,j}(s)$$

$$i > 0,\ \ j = 0\ to\ i-1\ \&\ i\ \geq\ j+1 \tag{2.9}$$

$$(s+\lambda_2+\lambda_3+\mu)\,\bar{P}^{(1)}_{i,k,j}(s) = \lambda_2\bar{P}^{(1)}_{i-1,k-1,j}(s)+\lambda_3\bar{P}^{(1)}_{i-1,k,j}(s)+\lambda_1\bar{P}^{(0)}_{i-1,k,j}(s)$$

$$+(i-k-j)\,\theta_3\bar{P}^{(0)}_{i,k,j}(s)+(k+1)\theta_2\bar{P}^{(0)}_{i,k+1,j}(s)$$

$$i > k,\ \ \ k > 0,\ \ j = 0\ to\ i-1\ \&\ i\ \geq\ k+j+1 \tag{2.10}$$

3 Results of the problem

Solving equations (2.6) to (2.10) recursively, we have

$$\bar{P}^{(0)}_{0,0,0}(s) = \frac{1}{s+\lambda_1} \tag{3.1}$$

$$\bar{P}^{(0)}_{i,0,i}(s) = \frac{\mu}{s+\lambda_1}\bar{P}^{(1)}_{i,0,i-1}(s) \qquad\qquad i \geq 1 \tag{3.2}$$

$$\bar{P}^{(0)}_{i,0,j}(s) = \frac{\mu}{s+\lambda_1+(i-j)\,\theta_3}\bar{P}^{(1)}_{i,0,j-1}(s) \qquad i > 1,\ \ \ j = 1\ to\ i-1\ \&\ i > k+j \tag{3.3}$$

$$\bar{P}^{(0)}_{i,k,j}(s) = \frac{\mu}{s+\lambda_1+k\theta_2}\bar{P}^{(1)}_{i,k,j-1}(s) \qquad\qquad i > 1,\ \ i > j\ \&\ k,\ \ j,k > 0\ \&\ i = k+j \tag{3.4}$$

$$\bar{P}^{(0)}_{i,k,1}(s) = \frac{\mu}{s+\lambda_1+(i-k-j)\theta_3+k\theta_2}\left(\frac{\lambda_2}{s+\lambda_2+\lambda_3+\mu}\bar{P}^{(1)}_{i-1,k-1,0}(s)+\frac{\lambda_3}{s+\lambda_2+\lambda_3+\mu}\bar{P}^{(1)}_{i-1,k,0}(s)\right)$$

$$i>2,\quad k=1\ to\ i-2\ \&\ i>k+j \tag{3.5}$$

$$\bar{P}^{(0)}_{i,k,j}(s) = \frac{\mu}{s+\lambda_1+(i-k-j)\theta_3+k\theta_2}\left(\frac{\lambda_2}{s+\lambda_2+\lambda_3+\mu}\bar{P}^{(1)}_{i-1,k-1,j-1}(s)+\frac{\lambda_3}{s+\lambda_2+\lambda_3+\mu}\bar{P}^{(1)}_{i-1,k,j-1}(s)\right)$$

$$+\frac{\lambda_1}{s+\lambda_2+\lambda_3+\mu}\bar{P}^{(0)}_{i-1,k,j-1}(s)+\frac{(i-k-j+1)\theta_3}{s+\lambda_2+\lambda_3+\mu}\bar{P}^{(0)}_{i-1,k,j-1}(s)+\frac{(k+1)\theta_2}{s+\lambda_2+\lambda_3+\mu}\bar{P}^{(0)}_{i,k+1,j-1}(s) \tag{3.6}$$

$$i>3,\quad j=2\ to\ i-2,\quad k=1\ to\ i-3\ \&\ i>k+j$$

$$\bar{P}^{(1)}_{1,0,0}(s) = \frac{\lambda_1}{s+\lambda_2+\lambda_3+\mu}\bar{P}^{(0)}_{0,0,0}(s) \tag{3.7}$$

$$\bar{P}^{(1)}_{i,0,0}(s) = \frac{\lambda_3}{s+\lambda_2+\lambda_3+\mu}\bar{P}^{(0)}_{i-1,0,0}(s)\quad i>1 \tag{3.8}$$

$$\bar{P}^{(1)}_{i,0,i-1}(s) = \frac{1}{s+\lambda_2+\lambda_3+\mu}\left(\lambda_1\bar{P}^{(0)}_{i-1,0,i-1}(s)+\theta_3\bar{P}^{(0)}_{i,0,i-1}(s)+\theta_2\bar{P}^{(0)}_{i,1,i-1}(s)\right)\quad i>1 \tag{3.9}$$

$$\bar{P}^{(1)}_{i,0,j}(s) = \frac{1}{s+\lambda_2+\lambda_3+\mu}\left(\lambda_3\bar{P}^{(1)}_{i-1,0,j}(s)+\lambda_1\bar{P}^{(0)}_{i-1,0,j}(s)+(i-j)\theta_3\bar{P}^{(0)}_{i,0,j}(s)+\theta_2\bar{P}^{(0)}_{i,1,j}(s)\right)$$

$$i>2,\quad j=1\ to\ i-2\ \&\ i>j+1 \tag{3.10}$$

$$\bar{P}^{(1)}_{i,i-1,0}(s) = \frac{\lambda_2}{s+\lambda_2+\lambda_3+\mu}\bar{P}^{(1)}_{i-1,i-2,0}(s)\quad i>1 \tag{3.11}$$

$$\bar{P}^{(1)}_{i,k,0}(s) = \frac{\lambda_2}{s+\lambda_2+\lambda_3+\mu}\bar{P}^{(1)}_{i-1,k-1,0}(s)+\frac{\lambda_3}{s+\lambda_2+\lambda_3+\mu}\bar{P}^{(1)}_{i-1,k,0}(s)$$

$$i>2,\quad k=1\ to\ i-2\ \&\ i>k+1 \tag{3.12}$$

$$\bar{P}^{(1)}_{i,k,j}(s) = \frac{1}{s+\lambda_2+\lambda_3+\mu}\left(\lambda_2\bar{P}^{(1)}_{i-1,k-1,j}(s)+\lambda_1\bar{P}^{(0)}_{i-1,k,j}(s)+(i-k-j)\theta_3\bar{P}^{(0)}_{i,k,j}(s)+(k+1)\theta_2\bar{P}^{(0)}_{i,k+,1,j}(s)\right)$$

$$i>2,\quad k=1\ to\ i-2\ \&\ i=k+j+1 \tag{3.13}$$

$$\bar{P}^{(1)}_{i,k,j}(s) = \frac{1}{s+\lambda_2+\lambda_3+\mu}\begin{pmatrix} \lambda_2\bar{P}^{(1)}_{i-1,k-1,j}(s)+\lambda_3\bar{P}^{(1)}_{i-1,k,j}(s)+\lambda_1\bar{P}^{(0)}_{i-1,k,j}(s) \\ +(i-k-j)\theta_3\bar{P}^{(0)}_{i,k,j}(s)+(k+1)\theta_2\bar{P}^{(0)}_{i,k+,1,j}(s)\end{pmatrix}$$
$$i>3,\quad k=1\ to\ i-3\ \&\ i>k+j+1 \tag{3.14}$$

Taking the Inverse Laplace transform of equations (3.1) to (3.14), we have

$$P^{(0)}_{0,0,0}(t) = e^{-\lambda_1 t} \tag{3.15}$$

$$P^{(0)}_{i,0,i}(t) = e^{-\lambda_1 t} * P^{(1)}_{i,0,i-1}(t)\, i \geq 1 \tag{3.16}$$

$$P^{(0)}_{i,0,j}(t) = \mu e^{-(\lambda_1+(i-j)\theta_3)t} * P^{(1)}_{i,0,j-1}(t) \qquad i>1,\quad j=1\ to\ i-1\ \&\ i>k+j \tag{3.17}$$

$$P^{(0)}_{i,k,j}(t) = e^{-(\lambda_1+k\theta_2)t} * P^{(1)}_{i,k,j-1}(t) \qquad i>1,\quad i>j\ \&\ k,\quad j,k>0\ \&\ i=k+j \tag{3.18}$$

$$P^{(0)}_{i,k,1}(t) = \mu\lambda_2 e^{-(\lambda_1+(i-k-j)\theta_3+k\theta_2)t}\left(\frac{1}{\mu}-\frac{e^{-\mu t}}{\mu}\right\} * P^{(1)}_{i-1,k-1,0}$$
$$+\mu\lambda_3 e^{-(\lambda_1+(i-k-j)\theta_3+k\theta_2)t}\left(\frac{1}{\mu}-\frac{e^{-\mu t}}{\mu}\right\} * P^{(1)}_{i-1,k,0}(t)$$
$$i>2,\quad k=1\ to\ i-2\ \&\ i>k+j \tag{3.19}$$

$$P^{(0)}_{i,k,j}(t) = \mu\lambda_2 e^{-(\lambda_1+(i-k-j)\theta_3+k\theta_2)t}\left(\frac{1}{\mu}-\frac{e^{-\mu t}}{\mu}\right\} * P^{(1)}_{i-1,k-1,j-1} + \mu\lambda_3 e^{-(\lambda_1+(i-k-j)\theta_3+k\theta_2)t}\left(\frac{1}{\mu}-\frac{e^{-\mu t}}{\mu}\right\} * P^{(1)}_{i-1,k,j-1} +$$
$$\mu\lambda_1 e^{-(\lambda_1+(i-k-j)\theta_3+k\theta_2)t}\left(\frac{1}{\mu}-\frac{e^{-\mu t}}{\mu}\right\} * P^{(0)}_{i-1,k,j-1}(t) +$$
$$(i-k-j+1)\,\theta_3\,\mu\, e^{-(\lambda_1+(i-k-j)\theta_3+k\theta_2)t}\left(\frac{1}{\mu}-\frac{e^{-\mu t}}{\mu}\right\} * P^{(0)}_{i,k,j-1}(t) + k_2\mu\, e^{-(\lambda_1+(i-k-j)\theta_3+k\theta_2)t}\left(\frac{1}{\mu}-\frac{e^{-\mu t}}{\mu}\right\} * P^{(0)}_{i,k+1,j-1}(t)$$
$$i>3,\quad j=2\ to\ i-2,\quad k=1\ to\ i-3\ \&\ i>k+j \tag{3.20}$$

$$P^{(1)}_{1,0,0}(t) = \lambda_1 e^{-(\lambda_2+\lambda_3+)t} * P^{(0)}_{0,0,0}(t) \tag{3.21}$$

$$P^{(1)}_{i,0,0}(t) = \lambda_3 e^{-(\lambda_2+\lambda_3+)t} * P^{(0)}_{i-1,0,0}(t) \qquad i>1 \tag{3.22}$$

$$P^{(1)}_{i,0,i-1}(t) = \lambda_1 e^{-(\lambda_2+\lambda_3+)t} * P^{(0)}_{i-1,0,i-1}(t) + \theta_3 e^{-(\lambda_2+\lambda_3+)t} * P^{(0)}_{i,0,i-1}(t) + \theta_2 e^{-(\lambda_2+\lambda_3+)t} * P^{(0)}_{i,1,i-1}(t) \qquad i>1 \tag{3.23}$$

$$P^{(1)}_{i,0,j}(t) = \lambda_3 e^{-(\lambda_2+\lambda_3+)t} * P^{(1)}_{i-1,0,j}(t) + \lambda_1 e^{-(\lambda_2+\lambda_3+)t} * P^{(0)}_{i-1,0,j}(t)$$
$$+(i-j)\,\theta_3 e^{-(\lambda_2+\lambda_3+)t} * P^{(0)}_{i,0,j}(t) + \theta_2 e^{-(\lambda_2+\lambda_3+)t} * P^{(0)}_{i,1,j}(t)$$

$$i>2,\ \ j=1\ to\ i-2\ \&\ i>j+1 \tag{3.24}$$

$$P^{(1)}_{i,i-1,0}(t)=\lambda_2 e^{-(\lambda_2+\lambda_3+)t} * P^{(0)}_{i-1,i-2,0}(t) \qquad i>1 \tag{3.25}$$

$$P^{(1)}_{i,k,0}(t)=\lambda_2 e^{-(\lambda_2+\lambda_3+)t} * P^{(1)}_{i-1,k-1,0}(t)+\lambda_3 e^{-(\lambda_2+\lambda_3+)t} * P^{(1)}_{i-1,k,0}(t)$$

$$i>2,\ \ k=1\ to\ i-2\ \&\ i>k+1 \tag{3.26}$$

$$P^{(1)}_{i,k,j}(t)=\ \lambda_2 e^{-(\lambda_2+\lambda_3+)t} * P^{(1)}_{i-1,k-1,j}(t)+\ \lambda_1 e^{-(\lambda_2+\lambda_3+)t} * P^{(0)}_{i-1,k,j}$$
$$+(i-k-j)\,\theta_3 e^{-(\lambda_2+\lambda_3+)t} *\ P^{(0)}_{i,k,j}(t)+(k+1)\theta_2 e^{-(\lambda_2+\lambda_3+)t} *\ P^{(0)}_{i,k+1,j}(t)$$

$$i>2,\quad j,k=1\ to\ i-2\ \&\ i=k+j+1 \tag{3.27}$$

$$P^{(1)}_{i,k,j}(t)=\ \lambda_2 e^{-(\lambda_2+\lambda_3+)t} * P^{(1)}_{i-1,k-1,j}(t)+\ \lambda_3 e^{-(\lambda_2+\lambda_3+)t} * P^{(1)}_{i-1,k,j}(t)+$$
$$\lambda_1 e^{-(\lambda_2+\lambda_3+)t} * P^{(0)}_{i-1,k,j}+(i-k-j)\,\theta_3 e^{-(\lambda_2+\lambda_3+)t} *\ P^{(0)}_{i,k,j}(t)+(k+1)\theta_2 e^{-(\lambda_2+\lambda_3+)t} *\ P^{(0)}_{i,k+1,j}(t)$$

$$i>3,\quad j,k=1\ to\ i-3\ \&\ i>k+j+1 \tag{3.28}$$

4 Verification of some important results

4.1

From the abstract solution of two-dimensional state model, it is verified that the sum of all possible probabilities is one. On taking summation over i and j on equations (3.1) - (3.14) and adding, the expression obtained is

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\left(\bar{P}^{(0)}_{i,k,j}(s)+\bar{P}^{(1)}_{i,k,j}(s)\right\}=\frac{1}{s}$$

on taking inverse Laplace transformation it gives

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}\left(P^{(0)}_{i,k,j}(t)+P^{(1)}_{i,k,j}(t)\right\}=1.$$

Which is a verification of obtained results.

4.2 Converting Two-state Model into Single State Model

To convert two-dimensional state model into single state model, probability $Q^{(p)}_{n,k}(t)$ is defined as under:

$Q^{(p)}_{n,k}(t)=$ Probability that there are n customers in orbit, k customers standing in orbit in high priority queue at time t and server is free or busy according as $p=0$ or 1.

The probability of exactly n customers in the system and k customers standing in orbit in high priority queue at time $t$in terms of$P^{(0)}_{i,k,j}(t)$ and $P^{(1)}_{i,k,j}(t)$:

When the server is free, it is defined by probability $Q_{n,k}^{(0)}(t)$:

$$Q_{n,k}^{(0)}(t) = \sum_{j=0}^{\infty} P_{j+n,k,j}^{(0)}(t)$$

Where n = (number of arrivals – number of departures).

When the server is busy, it is defined by probability $Q_{n,k}^{(1)}(t)$:

$$Q_{n,k}^{(1)}(t) = \sum_{j=0}^{\infty} P_{j+n+1,k.j}^{(1)}(t)$$

Where n = (number of arrivals – number of departures – 1).

Using the above definitions, from the equations (2.1) to (2.5), the set of equations in statistical equilibrium are:

$$(\lambda_1 + (n-k)\,\theta_3 + k\theta_2)\,Q_{n,k}^{(0)} = \mu Q_{n,k}^{(1)} \qquad n \geq 0 \tag{4.1}$$

$$(\lambda_2 + \lambda_3 + \mu)\,Q_{n,k}^{(1)} = \lambda_2 Q_{n-1,k-1}^{(1)} + \lambda_3 Q_{n-1,k}^{(1)} + \lambda_1 Q_{n,k}^{(0)} + (n-k+1)\,\theta_3 Q_{n+1,k}^{(0)}$$

$$+ (k+1)\,\theta_2 Q_{n+1,k+1}^{(0)} \qquad n > 1 \tag{4.2}$$

4.3 Particular Cases:

(a) When considering there are no customers standing in high priority queue then in this case $\lambda_2 = 0, \lambda_3 = \lambda_1 = \lambda$ and $\theta_3 = \theta, \theta_2 = 0$. Also letting $P_{i,0,j}^{(0)}(t) = P_{i,j,0}(t)$ and $P_{i,0,j}^{(1)}(t) = P_{i,j,1}(t)$ in equations (3.15) to (3.28), the results match with the Singla and Kalra[14].

$$P_{0,0,0}(t) = e^{-t} \tag{4.3}$$

$$P_{i,1,0}(t) = e^{-(+(i-1))t} * P_{i,0,1}(t) \qquad i \geq 1 \tag{4.4}$$

$$P_{i,i,0}(t) = \left((\lambda\mu)\, e^{-t} \left(\frac{1}{\mu} \left(-\frac{e^{-\mu t}}{\mu} \right\} * P_{i-1,i-1,0}(t) + \left((\mu\theta)\, e^{-t} \left(\frac{1}{\mu} \left(-\frac{e^{-t}}{\mu} \right\} * P_{i,i-1,0}(t) \right] \qquad i \geq 1 \tag{4.5}$$

$$P_{i,0,1}(t) = \lambda^i e^{-\lambda t} \left\{ \frac{1}{(\mu)^i} - e^{-\mu t} \sum_{r=0}^{i-1} \frac{(t)^r}{r!} \frac{1}{(\mu)^{i-r}} \right\} \qquad i \geq 1 \tag{4.6}$$

$$P_{i,i-1,1}(t) = \left(e^{-(+)t} * P_{i-1,i-1,0}(t) + e^{-(+)t} * P_{i,i-1,0}(t) \right) \qquad i > 1 \tag{4.7}$$

$$\begin{aligned}
P_{i,j,0}(t) &= \mu\lambda^{i-j} e^{-(\lambda+(i-j)\theta)t} \left\{ \tfrac{1}{(\mu)^{i-j}} - e^{-\mu t} \textstyle\sum_{r=0}^{l-j-1} \tfrac{(t)^r}{r!} \tfrac{1}{(\mu)^{i-j-r}} \right\} * P_{j,j-1,0}(\mathrm{t}) \\
&+ e^{-(\lambda+(i-j)\theta)t} \textstyle\sum_{k=2}^{i-j} \ldots \mu\lambda^{i-j-k+1} \left\{ \tfrac{1}{(\mu)^{i-j-k+1}} - e^{-\mu t} \textstyle\sum_{r=0}^{i-j-k} \tfrac{(t)^r}{r!} \tfrac{1}{(\mu)^{i-j-k-r+1}} \right\} \\
&* P_{j+k-1,j-1,0}(\mathrm{t}) \quad + e^{-(\lambda+(i-j)\theta)t} \sum_{k=2}^{i-j} (\mu k\theta) \lambda^{i-j-k+1} \left\{ \frac{1}{(\mu)^{i-j-k+2}} \right. \\
&\left. - e^{-\mu t} \sum_{r=0}^{i-j-k+1} \frac{(t)^r}{r!} \frac{1}{(\mu)^{i-j-k-r+2}} \right\} * P_{j+k-1,j-1,0}(\mathrm{t}) \\
&+ e^{-(\lambda+(i-j)\theta)t} \left\{ \frac{1}{\mu} - \frac{e^{-\mu t}}{\mu} \right\} ((i-j+1)\mu\theta) * P_{i,j-1,0}(\mathrm{t}) \\
&+ \mu\lambda^{i-j} e^{-(\lambda+(i-j)\theta)t} \left\{ \frac{1}{(\mu)^{i-j}} - e^{-\mu t} \sum_{r=0}^{i-j-1} \frac{(t)^r}{r!} \frac{1}{(\mu)^{i-j-r}} \right\} * P_{j,j-1,1}(\mathrm{t}) \\
& i > j > 1
\end{aligned} \tag{4.8}$$

$$P_{i,j,1}(t) = \lambda^{i-j-1} e^{-(\lambda+\mu)t} \frac{(t)^{i-j-2}}{(i-j-2)!} * P_{j+1,j,0}(\mathrm{t}) + e^{-(\lambda+\mu)t} \sum_{k=2}^{i-j-1} \lambda^{i-j-k} \frac{(t)^{i-j-k-1}}{(i-j-k-1)!} *$$
$$P_{j+k,j,0}(\mathrm{t}) + e^{-(\lambda+\mu)t} \sum_{k=2}^{i-j-1} k\theta\lambda^{i-j-k} \frac{(t)^{i-j-k}}{(i-j-k)!} * P_{j+k,j,0}(\mathrm{t}) + (i-j)\theta e^{-(\lambda+\mu)t} * P_{i,j,0}(\mathrm{t}) +$$

$$\lambda^{i-j-1} e^{-(\lambda+\mu)t} \frac{(t)^{i-j-2}}{(i-j-2)!} * P_{j+1,j,1}(\mathrm{t}) i \geq j+2, j \geq 1 \tag{4.9}$$

(b) Again assuming k=0, $\lambda_2 = 0, \theta_2 = 0$ then $\lambda_3 = \lambda_1 = \lambda, \theta_3 = \theta$ and take $Q_{n,k}^{(0)} = Q_{n,0}$, $Q_{n,k}^{(1)} = Q_{n,1}$ in equations (4.1) and (4.2), we get

$$(\lambda + n\theta)\, Q_{n,0} = \mu\, Q_{n,1} \qquad n \geq 0 \tag{4.10}$$

$$(\lambda + \mu)\, Q_{n,1} = \lambda\,(Q_{n,0} + Q_{n-1,1}) + (n+1)\,\theta\, Q_{n+1,0} \qquad n > 1 \tag{4.11}$$

these equations coincide with result (1.5) and (1.6) of Falin & Templeton [4].

5 Discussion

5.1 Numerical Solution and Graphical Representation

In this section, some numerical results that describe the model under study are examined. Numerical results are generated using MATLAB programming for the cases $\rho_1 = \left(\frac{\lambda_1}{\mu}\right) = 0.6$, $\rho_2 = \left(\frac{\lambda_2}{\mu}\right) = 0.3, \rho_3 = \left(\frac{\lambda_3}{\mu}\right) = 0.2, \eta_2 = \left(\frac{\theta_2}{\mu}\right)$ =0.5, $\eta_3 = \left(\frac{\theta_3}{\mu}\right) =$ 0.3and $\rho_1 = \left(\frac{\lambda_1}{\mu}\right) = 0.9$, $\rho_2 = \left(\frac{\lambda_2}{\mu}\right) = 0.5, \rho_3 = \left(\frac{\lambda_3}{\mu}\right) = 0.3, \eta_2 \left(\frac{\theta_3}{\mu}\right)$ =0.5, $\eta_3 = \left(\frac{\theta_3}{\mu}\right)$ =0.3. From the numerical results, it is found that the sum of all the probabilities at any instant approaches to 1. The same probabilities are used to verify some important results of the model in the form of graphs.

Various probabilities are plotted against time t through Figures 1, 2, 3, 4 and 5.Figure 1 shows the plots of probabilities $P_{0,0,0}^{(0)}$and $P_{1,0,1}^{(0)}$ against time t for the case $\rho_1 = 0.6, \rho_2 = 0.3, \ \rho_3 = 0.2, \eta_2$ =0.5, $\rho_1 = 0.6, \rho_2 = 0.3, \ \rho_3 = 0.2$, It is clear from the graph that the probability $P_{0,0,0}^{(0)}$ decreases rapidly from the initial value 1 at time $t = 0$. The probability $P_{1,0,1}^{(0)}$ increases rapidly in the starting moments from initial value zero, and then decreases gradually.

Figure 2 shows relative changes in probabilities $P_{3,1,1}^{(0)}$and $P_{3,1,1}^{(1)}$against time t for the case when $\rho_1 = 0.6, \rho_2 = 0.3, \rho_3 = 0.2, \eta_2 = 0.5, \eta_3 = ($. From the figure, it is observed that the probability $P_{3,1,1}^{(1)}$increases rapidly in the starting moments, then decreases with a high rate. Probability $P_{3,1,1}^{(0)}$also increases in the starting moments and then start decreases. It is seen that the probability $P_{3,1,1}^{(1)}$remained higher than probability $P_{3,1,1}^{(0)}$because generally when the server is serving the customers then the probabilities are more comparatively to when the server is not serving the customers.

To study the effect of traffic intensity (customers are arriving per unit service time) on different probabilities of the model, the data of various probabilities are generated for different values of $\rho_1$keeping the other parameter constant. The set of $\rho_1$values for the comparison is {0.3,0.6,0.9}. Figure 3 shows the plot of probability $P_{0,0,0}^{(0)}$ against time t for different value of ρ_1. It is seen that the probability $P_{0,0,0}^{(0)}$ at time $t = 0$ is 1 and the probability $P_{0,0,0}^{(0)}$ start decreasing with increasing time. Behavior of $P_{0,0,0}^{(0)}$ is same for all the values of ρ_1. From the figure, it is also concluded that as ρ_1 increases $P_{0,0,0}^{(0)}$ decreases. As the number of arrivals are increase per unit service time then corresponding the probabilities of server remains free in the system are also less.

In Figure 4, the probability $P_{3,0,2}^{(0)}$ and inFigure 5 , the probability $P_{3,0,2}^{(1)}$ are plotted against time t for different values of $\rho_1 = \{0.3, 0.6, 0.9\}$ From both the figures, it is concluded that due to their particular behavior, the probabilities $P_{3,0,2}^{(0)}$ and $P_{3,0,2}^{(1)}$ increase in the starting times and then start decrease for higher values of time. It is also observed that for bigger $\rho_1$values, the probabilities $P_{3,0,2}^{(0)}$ and $P_{3,0,2}^{(1)}$ has larger values for initial to moderate values of time but this trend reverse for higher values of time.

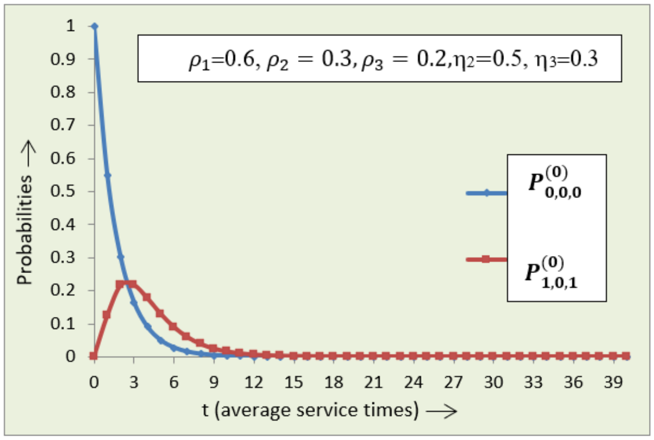

Fig 1. Probabilities $P^{(0)}_{0,0,0}$ and $P^{(0)}_{1,0,1}$ against time

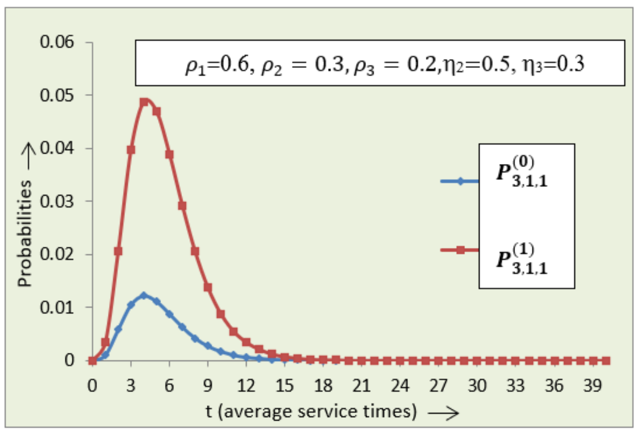

Fig 2. Probabilities $P^{(0)}_{3,1,1}$ and $P^{(1)}_{3,1,1}$ against time

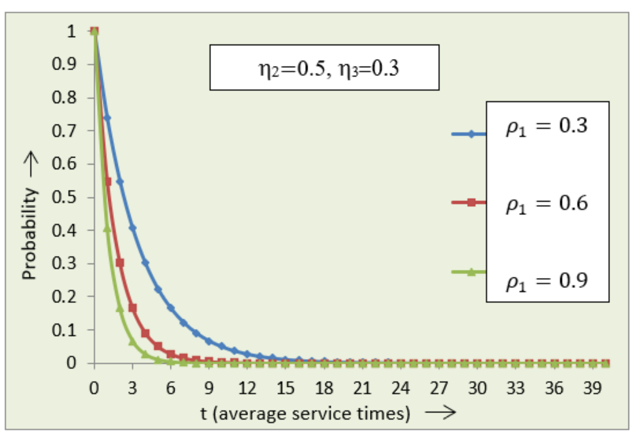

Fig 3. Effect of ρ_1 on against $P^{(0)}_{0,0,0}$ time t

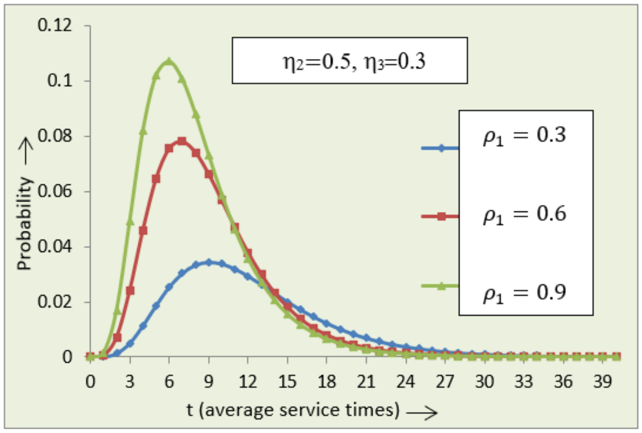

Fig 4. Effect of ρ_1 on $P^{(0)}_{3,0,2}$ against time t

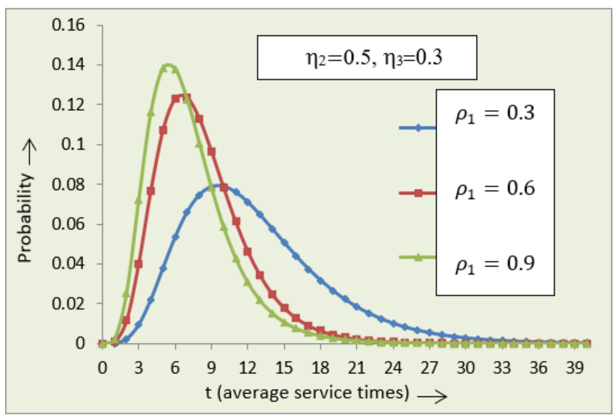

Fig 5. Effect of ρ_1 on $P^{(1)}_{3,0,2}$ against time t

5.2 Busy Period Probabilities

The busy period analysis plays a vital role in understanding various operations taking place in any queueing system. A busy period is initiated with the arrival of a customer who finds the system empty and ends when the next system becomes free. In this section, we discuss some numerical results about the busy period distribution of the server and busy period distribution of the system.

The probability when the server is busy is given by

P (Server is busy) = $\sum_{i>j\geq 0} P^{(1)}_{i,k,j}\ (t)$ **(5.1)**

The probability when the system is busy is given by

P (System is busy) = $\sum_{i>j\geq 0}\left(P^{(0)}_{i,k,j}\ (t)+P^{(1)}_{i,k,j}\ (t)\right)$ **(5.2)**

5.3. Numerical & Graphical representation of the busy period

The numerical results are generated using MATLAB programming. The probability when the system is busy and the probability when the server is busy are presented in Table 1 for different values of ρ_1 at $\eta_2 = 0.5,\ \eta_3 = 0.3$.

In Figure 6 the probability (system busy) and the probability (server busy) are plotted for the cases $\rho_1 = 0.9, \rho_2 = 0.5,\ \rho_3 = 0.3, \eta_2$=0.5, η_3= 0.3 against time t. It is observed that the two curves are increasing rapidly in starting moments and then start decreasing gradually for higher values of t. Figure also shows that the probability when system is busy remains higher than the

Table 1. Probability of system busy and server busy for different values ρ_1 $(\eta_2 = 0.5, \eta_3 = 0.3)$

	Probability (Server busy)			Probability (System busy)		
t	$\rho_1 = 0.3$	$\rho_1 = 0.5$	$\rho_1 = 0.7$	$\rho_1 = 0.3$	$\rho_1 = 0.5$	$\rho_1 = 0.7$
0	0	0	0	0	0	0
1	0.1720	0.3070	0.4129	0.2003	0.3624	0.4930
2	0.2319	0.3889	0.4903	0.3061	0.5278	0.6827
3	0.2614	0.4164	0.4924	0.3767	0.6262	0.7753
4	0.2786	0.4178	0.4608	0.4274	0.6820	0.8054
5	0.2877	0.4008	0.4129	0.4628	0.7013	0.7866
6	0.2898	0.3709	0.3591	0.4843	0.6886	0.7324
7	0.2856	0.3331	0.3057	0.4929	0.6501	0.6563

probability when server is busy, as desired.

The probability when the system is busy is plotted in Figure 7 and the probability when the server is busy is plotted in Figure 8 for different values of traffic intensity. From these figures, it is clearly visible that as the value of ρ_1 increases, both the probabilities achieved higher highest values for some t. So we interpret that more the arrivals, more is the probability that the system and the server remain busy.

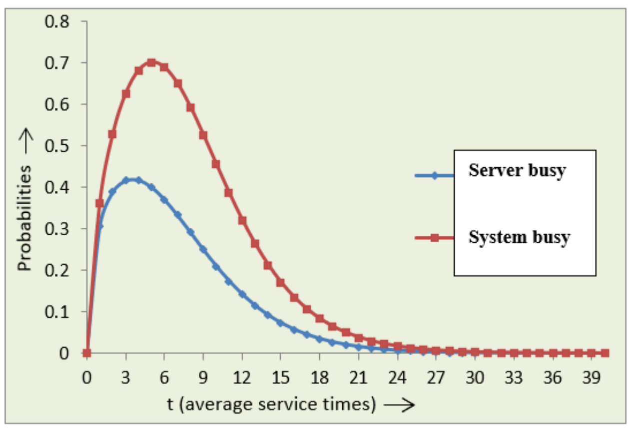

Fig 6. Probability(system busy) and Probability (server busy) against time

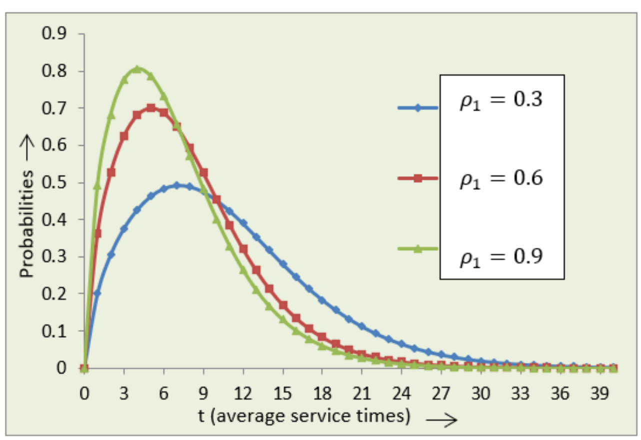

Fig 7. Effect of ρ_1 on Probability (system busy) against time t

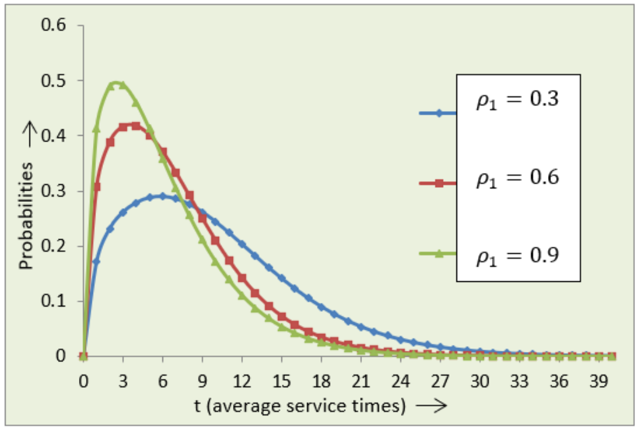

Fig 8. Effect of ρ_1 on Probability (server busy) against time

6 Conclusion

The factor "two-state" makes our model more realistic and quantified. Discussion in this article enriches modern queueing theory, and has wide use in practical questions. As in computer communication net, in order to offer multi-layer quality of service for different kind of customer, priority control is necessary. The presence of retrial customers can radically change the behavior of a queueing system. This model developed two features simultaneously including retrial and priority, which makes our results applicable to more versatile congestion situations encountered in computer and communication systems, production and manufacturing systems and distribution and service sectors.

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