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A Study on Plithogenic Product Fuzzy Graphs with Application in Social network

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Abstract

Objectives: The main objective of this study is to define Plithogenic product fuzzy graphs (PPFGs), and introduce its properties. **Method:** PPFGs is newly introduced as a new graphical model where P-vertices are characterized by four or more attributes and the attribute values of P-edges are computed using the product operator. **Findings:** Theoretical discussions and results related to PPFGs, and subgraphs, paths, cycles, trees, bridge and cut vertex in PPFGs are demonstrated with examples. A social network model based on the notion of PPFGs has been presented and analyzed to show the utility and the advantage of Plithogenic product fuzzy graph model. **Novelty:** Strong and weak P-vertices, and highly strong, strong and weak P-edges are identified to analyze the strength of connectivity between different units. P-order, P-size, P-vertex range, P-edge range, degree, total degree, and average P-weight of P-vertices are computed to examine proximity, significance and centrality of units.

Keywords: Plithogenic fuzzy sets; Fuzzy graphs; Plithogenic fuzzy graphs; Plithogenic product fuzzy graphs; Social networks

1 Introduction

Fuzzy sets and fuzzy relations were formulated by L.A. Zadeh in 1965[1]. The introduction to Fuzzy graphs (FGs) was done by Kaufmann in 1973[2]. It was further developed by Azriel Rosenfeld in 1975[3]. Since the single values of membership degree given to vertices and edges from [0,1] in FGs provide limited knowledge and perception regarding any real-life problem under consideration, Intuitionistic fuzzy sets and fuzzy logic, and Intuitionistic fuzzy graphs (IFGs) were introduced by Atanassov in 1986[4], and Karunambigai and Parvathi in 2006[5] respectively. In IFGs, membership and non-membership degrees from [0,1] are given to each element with the condition that the sum of membership and non-membership degrees is less than or equal to one. But then there are many real-life situations where the concept of neutrality degree is existent. So, IFGs are insufficient to handle such scenarios.

Cen Zuo et al. in 2019[6] introduced Picture fuzzy graphs which is an extension of FGs and IFGs. Picture fuzzy graphs allows the idea of membership, neutral membership and non-membership degrees of an element from [0,1] with the condition that their sum must be less than or equal to one. There are also situations where the models of Product

vague graphs and Neutrosphic graphs are used to study any vague and inconsistent circumstances[7–9]. Neutrosophic sets and Neutrosophic graphs were introduced by Smarandache in 2006 and 2015 respectively[10–12]. In Neutrosophic fuzzy graphs membership, indeterminacy and non-membership values, which are within the real standard or nonstandard unit interval]⁻0,1⁺[are allotted to vertices and corresponding edges. In order to apply neutrosophic logic in real-life situations the concept of Single valued neutrosophic graphs (SVNGs) was introduced in 2016 by Broumi et al.[13]. In a Single valued neutrosophic graph, membership, indeterminacy and non-membership values of an element are independent entities from [0,1] unlike FGs, IFGs, and other extensions of FGs.

The interesting development in the field of applied mathematics is the emergence of Plithogenic logic and Plithogenic sets by Smarandache in 2017[14]. The elements belonging to a Plithogenic set are characterized by one or more attributes with the corresponding attribute values. Plithogenic graphs (PGs) were introduced by F. Smarandache et al. in 2020[15]. PGs is a generalization of Crisp graphs, FGs, IFGs and SVNGs. One of the types of Plithogenic fuzzy graphs (PFGs) is PPFGs which is stated as Type III PFGs in the examples given for PFGs by Smarandache et al.[15]. In PPFGs, vertices and edges are characterized by four or more attributes which have their corresponding attribute values from [0,1]. The computation of the attribute values of an edge in a PPFG is done using the usual product operator.

Being motivated by the discussions on Plithogenic fuzzy graphs and their applications in[15], the study on PPFGs is undertaken hereby to investigate for newer results and proofs. Since the study on PPFGs is still blank in the literature, we have newly defined PPFGs and introduced some interesting graphical terms in PPFGs with properties, results and examples in section 3, after some relevant definitions in the preliminaries in section 2. Subgraphs, paths, cycles, trees, bridge and cut vertex in PPFGs are discussed in section 4. Since parameters such as degrees, nature of edges, order and size are interesting and important concepts in fuzzy graphs[16–21], we are motivated to define and discuss about P-order, P-size, P-vertex range, P-edge range, degree of a P-vertex, total degree of a P-vertex, minimum degree, maximum degree and nature of P-edges in PPFGs in sections 5 and 6.

As the number of attributes for a PPFG is greater than or equal to four, it is more reliable and realistic representation of vagueness in certain real-life situations. It is also more efficient in obtaining higher precision in our perception of uncertain scenarios in the decision-making process. PPFG model can be utilized to represent and analyze social network of persons, medical diagnosis of patients, students' performance, resource/computer networking, psychological and sociological issues, weather forecast, etc. where the units involved are to be characterized with many attributes, and the relations between them need to be analyzed depending on the corresponding attribute values allotted to them. Since social networks play a vital role in human life, in section 7 of this paper, we have applied the PPFG model to analyze a small-sized social network.

2 Preliminaries

In this section we mainly recall definitions of FGs, degrees of a vertex, order and size of a fuzzy graph, Product fuzzy graphs, Plithogenic fuzzy sets (PFSs) and PFGs which are relevant to the study on PPFGs.

2.1. Definition[2] A fuzzy graph $G : (\alpha, \beta)$ is a pair of functions $\alpha: V \to [0,1]$ and $\beta : V \times V \to [0,1]$, such that $\beta(u,v) \leq \alpha(u) \wedge \alpha(v)$ for all $u, v \in V$ where V is the vertex set.

2.2. Definition[16,17] Let $G : (V, \alpha, \beta)$ be a fuzzy graph. Then the degree of a vertex v is defined by $d(v) = \sum_{u \neq v} \beta(v, u)$. The minimum degree of G is defined by $\delta(G) = \wedge \{d(v) / v \in V\}$. The maximum degree of G is defined by $\triangle(G) = \vee \{d(v) / v \in V\}$. The total degree of $v \in V$ is defined by $td(v) = \sum_{v \neq u} \beta(v, u) + \alpha(v) = d(v) + \alpha(v)$.

2.3. Definition[17] Let $G : (V, \alpha, \beta)$ be a fuzzy graph. Then the order of G is defined by $O(G) = \sum_{v \in V} \alpha(v)$. The size of G is defined by $S(G) = \sum_{u \neq v} \beta(u, v)$.

2.4 . Definition[22] Let G*: (V, E) be a graph. γ be a fuzzy subset of V, and τ be a fuzzy subset of $V \times V$. We call $G : (\gamma, \tau)$ a product fuzzy graph if $\tau(x,y) \leq \gamma(x) \times \gamma(y)$ for all $x, y \in V$.

2.5. Definition[15] A Plithogenic fuzzy set U is defined as a set where each element is characterized by many attribute values. That is if $c_1, c_2, \ldots\ldots\ldots, c_n$ are n attributes and $X_1, X_2, \ldots, X_t \in U$, then X_1's fuzzy degrees of these attributes is $X_1(d_{11}, d_{12}\ldots, d_{1n})$ where $d_{1i} \in [0,1]; 1 \leq i \leq n$. Similarly, X_2's fuzzy degrees of these attributes is $X_2(d_{21}, d_{22}\ldots, d_{2n})$, where $d_{2i} \in [0,1]; 1 \leq i \leq n$. Similary fuzzy degree values for any $X_k; 1 \leq k \leq t$ are assigned. Such a set is called a Plithogenic fuzzy set (PFS). Note that a PFS, in general, may have elements characterized by four or more attributes, which have the corresponding attribute values from [0,1].

2.6. Definition[15] If U is the universal set of discourse and $c_1, c_2, \ldots, c_n$ are the n attributes and if $X_1, X_2, \ldots, X_m \in U$, then the PFS is given by $X_1(d_{11}, d_{12}\ldots, d_{1n}), X_2(d_{21}, d_{22}\ldots, d_{2n})$, and $X_m(d_{m1}, d_{m2}\ldots, d_{mn})$ where $d_{ij} \in [0,1]; 1 \leq i \leq m$ and $1 \leq j \leq n$. Thus we can have a graph with vertex set $V_i = X_i(d_{i1}, d_{i2}, \ldots, d_{in}), 1 \leq i \leq m$. Depending on the problem and the experts opinion, the vertices in $V = \{V_1, V_2, \ldots, V_m\}$ can be adjacent or not. We call such graphs as PFGs. The concerned edges

are given values as per the choice of the operator by the researcher. Note that PFGs are distinctly different from FGs. Throughout this paper, in general, we denote a Plithogenic fuzzy graph (PFG) by $G_F = (V_F, E_F)$, where V_F and E_F are PFSs of vertices and relevant edges respectively.

3 Methodology: Plithogenic product fuzzy graphs

PPFGs has been defined in this section. Basic concepts and properties of PPFGs are newly introduced with examples and results.

3.1. Definition Let $G_F = (V_F, E_F)$ be a PFG, where $V_F = \{V_1, V_2, \ldots, V_n\}$ and $E_F = \{E_1, E_2, \ldots, E_m\}$ are PFSs of vertices and relevant edges respectively characterized by k attributes. G_F is said to be a Plithogenic product fuzzy graph (PPFG) denoted by $G_p = (V_p, E_p)$, if for any two adjacent $V_i, V_j \in V_F$, $i \neq j$, with the corresponding edge $E_c \in E_F$,

$V_i = X_i (s_1, s_2, \ldots, s_k)$
$V_j = X_j(t_1, t_2, \ldots, t_k)$
$E_c = Y_c(c_1, c_2, \ldots, c_k)$ where for any $c_d \in Y_c$, $c_d = s_a t_b$ with $s_a \in X_i$ and $t_b \in X_j$; $d = a = b$; $1 \leq d, a, b \leq k$; $c_d, s_a, t_b \in [0, 1]$. Here X_i, X_j and Y_c are sets of attribute values from [0,1] characterizing V_i, V_j and E_c respectively in G_F.

3.1.1. Note In this study, PPFGs which are undirected and simple (no loop or multiple edges) are considered. Since PFSs, in general, have elements characterized by four or more attributes which have the corresponding attribute values from [0,1], throughout this paper we consider $k \geq 4$ where k is the number of attributes which is the same for any $V \in V_P$ and $E \in E_P$ in $G_P = (V_P, E_P)$, where $V_P = \{V_1, V_2, \ldots, V_n\}$ and $E_P = \{E_1, E_2, \ldots, E_m\}$ are PFSs of vertices and relevant edges respectively.

3.1.2. Note In definition 3.1. if s_a and t_b are from $(0, 1)$ with k-decimal places and l-decimal places respectively, then c_d is with $(k + l)$ decimal places from $(0, 1)$.
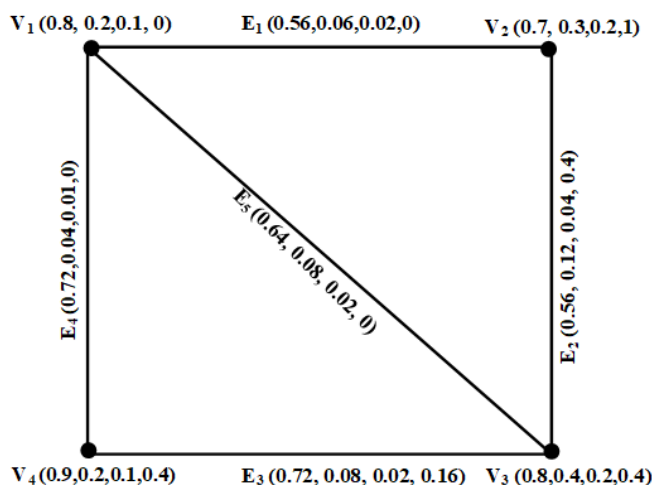

Fig 1. Plithogenic product fuzzy graph

3.2. Definition A vertex of G_P is referred as a P-vertex. An edge of G_P is referred as a P-edge. A P-vertex in G_P is called a P-isolated vertex if no P-edge is incident to it. The V_P-cardinality of G_P is the number of P-vertices present in it. It is denoted by $|V_p(G_p)|$. The E_P-cardinality of G_P is the number of P-edges present in it. It is denoted by $|E_p(G_p)|$. The sum of the attribute values of any $V \in V_P$ is known as its P-weight. It is denoted by $P_{VW}(V)$. The sum of the attribute values of any $E \in E_P$ is known as its P-weight. It is denoted by $P_{EW}(E)$. Note that $0 \leq P_{VW}(V) \leq k$ and $0 \leq P_{EW}(E) \leq k$. Any two P-vertices V_i and V_j are said to be neighbours if they are adjacent and $P_{EW}(V_iV_j) > 0$. A P-vertex in a PPFG having exactly one neighbour is called a pendant P-vertex. Otherwise, it is called non-pendent P-vertex. A P-edge incident to a pendant P-vertex is called a pendant P-edge. Otherwise, it is said to be non-pendent P-edge. A P-vertex adjacent to a pendant P-vertex is called an associate of the pendant P-edge.

3.2. Example Let us consider PPFG, $G_P = (V_P, E_P)$ as in Figure 1 . By computation we have the V_P-cardinality of G_P, E_P-cardinality of G_P, P-weight of each P-vertex and P-weight of each P-edge as follows:

$|V_P(G_P)| = 4$; $|E_P(G_P)| = 5$
$P_{VW}(V_1) = 1.1$; $P_{VW}(V_2) = 2.2$; $P_{VW}(V_3) = 1.8$; $P_{VW}(V_4) = 1.6$

$$P_{EW}(E_1) = 0.64 ; P_{EW}(E_2) = 1.12 ; P_{EW}(E_3) = 0.98 ; P_{EW}(E_4) = 0.77 ; P_{EW}(E_5) = 0.74$$

3.3. Proposition In a Plithogenic product fuzzy graph, an attribute value of any P-edge is zero if either of the corresponding attribute values of its adjacent P-vertices or both are zero.

Proof: Let us consider $E_c \in E_P$ of a PPFG, $G_P = (V_P, E_P)$ with attribute values $c_1, c_2, \ldots, c_k = 0$. Let $V_i, V_j \in V_P$ be the adjacent P-vertices to E_c with attribute values $s_1, s_2, \ldots, s_k$ and $t_1, t_2, \ldots, t_k$ respectively. i.e., $s_1, s_2, \ldots, s_k \in V_i$, $t_1, t_2, \ldots, t_k \in V_j$ and $c_1, c_2, \ldots, c_k \in E_c$

Since $c_d = s_a t_b$ for any $c_d \in E_c$ with $s_a \in V_i$ and $t_b \in V_j$; $1 \le d, a, b \le k$; $d = a = b$; $c_d, s_a, t_b \in [0, 1]$, it follows that for $c_d = 0$, either $s_a = 0$ or $t_b = 0$, or both $s_a = t_b = 0$. Hence the result.

3.4. Corollary In a Plithogenic product fuzzy graph, $G_P = (V_P, E_P)$ for any adjacent $V_i, V_j \in V_P$ if $s_1, s_2, \ldots, s_k = 1 \in V_i$ and $t_1, t_2, \ldots, t_k = 1 \in V_j$, then $c_1, c_2, \ldots, c_k = 1 \in E_c$ where $E_c \in E_P$ is the P-edge between to V_i and V_j.

Proof: Let us consider any two adjacent P-vertices $V_i, V_j \in V_P$ of a PPFG, $G_P = (V_P, E_P)$ with attribute values $s_1, s_2, \ldots, s_k = 1$ and $t_1, t_2, \ldots, t_k = 1$ respectively. Let $E_c \in E_P$ be the P-edge between V_i and V_j with attribute values $c_1, c_2, \ldots, c_k$.

Since $c_d = s_a t_b$ for any $c_d \in E_c$ with $s_a \in V_i$ and $t_b \in V_j$; $1 \le d, a, b \le k$; $d = a = b$; $c_d, s_a, t_b \in [0, 1]$, it follows that $c_1, c_2, \ldots, c_k = 1 \in E_c$ since $s_1, s_2, \ldots, s_k = 1 \in V_i$ and $t_1, t_2, \ldots, t_k = 1 \in V_j$. Hence the result.

3.5. Proposition In any Plithogenic product fuzzy graph the P-weight of any P-edge is less than or equal to the minimum of the P-weights of the corresponding adjacent P-vertices.

Proof: Let $G_P = (V_P, E_P)$ be a PPFG. For any P-edge $E_c \in E_P$ with the attribute values $c_1, c_2, \ldots, c_k$, let us consider the corresponding adjacent P-vertices $V_i, V_j \in V_P$ with attribute values $s_1, s_2, \ldots, s_k$ and $t_1, t_2, \ldots, t_k$ respectively.

(i.e.) $s_1, s_2, \ldots, s_k \in V_i$, $t_1, t_2, \ldots, t_k \in V_j$ and $c_1, c_2, \ldots, c_k \in E_c$ where $c_d = s_a t_b$ for any $c_d \in E_c$ with $s_a \in V_i$ and $t_b \in V_j$; $d = a = b$; $1 \le d, a, b \le k$; $c_d, s_a, t_b \in [0, 1]$.

We know that $P_{VW}(V_i) = s_1 + s_2 + \cdots + s_k$; $P_{VW}(V_j) = t_1 + t_2 + \cdots + t_k$; and $P_{EW}(E_c) = c_1 + c_2 + \cdots + c_k$. Then the following cases hold true:

Case (i): When $s_1, s_2, \ldots, s_k \in (0, 1)$ and $t_1, t_2, \ldots, t_k \in (0, 1)$,
$c_1 + c_2 + \cdots + c_k < min\{s_1 + s_2 + \cdots + s_k, t_1 + t_2 + \cdots + t_k\}$

Case (ii): When either $s_1, s_2, \ldots, s_k = 0$ or $t_1, t_2, \ldots, t_k = 0$, then $c_1 + c_2 \cdots + c_k = 0$ which is the $min\{s_1 + s_2 + \cdots + s_k, t_1 + t_2 + \cdots + t_k\}$.

Case (iii): when $s_1, s_2, \ldots, s_k = 1$ and $t_1, t_2, \ldots, t_k = 1$, then $c_1, c_2, \ldots, c_k = 1$. (i.e.) if $s_1 + s_2 + \cdots + s_k = 1 + 1 + \ldots k$ times and $t_1 + t_2 + \cdots + t_k = 1 + 1 + \ldots k$ times, then $c_1 + c_2, \cdots + c_k = 1 + 1 + \ldots k$ times which is the $min\{s_1 + s_2 + \cdots + s_k, t_1 + t_2 + \cdots + t_k\}$

Hence the proposition is proved.

3.6 . Theorem Every Plithogenic product fuzzy graph is a Plithogenic fuzzy graph, but the converse need not be true.

Proof: Let us assume that $G_P = (V_P, E_P)$ be a PPFG where V_P and E_P are PFSs of P-vertices and relevant P-edges respectively in G_P. We need to prove that, G_P is a PFG. For any P-edge $E_c \in E_P$ with attribute values $c_1, c_2, \ldots, c_k$, let $V_i, V_j \in V_P$ be its adjacent P-vertices with attribute values $s_1, s_2, \ldots, s_k$ and $t_1, t_2, \ldots, t_k$ respectively. This follows that for any $c_d \in E_c$, $c_d = s_a t_b$, where $s_a \in V_i$ and $t_b \in V_j$; $1 \le d, a, b \le k$; $d = a = b$; $c_d, s_a, t_b \in [0, 1]$. Since the attribute values $c_1, c_2, \ldots, c_k$ of any P-edge $E_c \in E_P$ are obtained using the usual product operator with the condition that for any $c_d \in E_c$, $c_d = s_a t_b$, where $s_a \in V_i$ and $t_b \in V_j$; $1 \le d, a, b \le k$; $d = a = b$; $c_d, s_a, t_b \in [0, 1]$, this implies that $E_c \in E_F$ and $V_i, V_j \in V_F$ of PFG, $G_F = (V_F, E_F)$. i.e., $G_P = (V_P, E_P)$ equals $G_F = (V_F, E_F)$. Hence G_P is a PFG.

Conversely, let $G_F = (V_F, E_F)$ be a PFG where V_F and E_F are PFSs of vertices and relevant edges respectively. For any edge $E_c \in E_F$ with attribute values $c_1, c_2, \ldots, c_k$, let its adjacent vertices be $V_i, V_j \in V_F$ with attribute values $s_1, s_2, \ldots, s_k$ and $t_1, t_2, \ldots, t_k$ respectively. This follows that for any $c_d \in E_c$ either $c_d = s_a t_b$ or $c_d \ne s_a t_b$, where $s_a \in V_i$ and $t_b \in V_j$; $1 \le d, a, b \le k$; $d = a = b$; $c_d, s_a, t_b \in [0, 1]$.

If $c_d = s_a t_b$ for any $c_d \in E_c$ with $s_a \in V_i$ and $t_b \in V_j$; $1 \le d, a, b \le k$; $d = a = b$, then $E_c \in E_P$ and its adjacent vertices $V_i, V_j \in V_P$ of $G_P = (V_P, E_P)$. i.e., $G_F = (V_F, E_F)$ equals $G_P = (V_P, E_P)$. Hence G_F is a PPFG. If $c_d \ne s_a t_b$ for any $c_d \in E_c$ with $s_a \in V_i$ and $t_b \in V_j$; $1 \le d, a, b \le k$; $d = a = b$, then $E_c \notin E_P$ and its adjacent vertices $V_i, V_j \notin V_P$ of $G_P = (V_P, E_P)$. i.e., $G_F = (V_F, E_F)$ is not equal to $G_P = (V_P, E_P)$. Therefore G_F is not a PPFG. Hence the proof.

3.7. Proposition The P-weight of any P-vertex in a Plithogenic product fuzzy graph is always less than or equal to k which is the number of attributes.

Proof: Let $G_P = (V_P, E_P)$ be a PPFG where V_P and E_P are PFSs of P-vertices and relevant P-edges respectively. For any $V \in V_P$ with attribute values $s_1, s_2, \ldots, s_k$, the following two cases hold true:

Case (i): When $s_1, s_2, \ldots, s_k \in [0,1)$, then $P_{VW}(V) < k$

Case (ii): When $s_1, s_2, \ldots, s_k = 1$, then $P_{VW}(V) = k$. Hence the result.

3.8. Definition Let $G_P = (V_P, E_P)$ be a PPFG, where $V_P = \{V_1, V_2, \ldots, V_n\}$ and $E_P = \{E_1, E_2, \ldots, E_m\}$ are the PFSs of P-vertices and relevant P-edges respectively. The Supremum (S_A) of any P-vertex $V \in V_P$ is the highest attribute value associated

with V and it is denoted by $S_A(V)$. The Supremum (S_A) of any P-edge $E \in E_P$ is the highest attribute value of associated with E and it is denoted by $S_A(E)$. The Infimum (I_A) of any P-vertex $V \in V_P$ is the least attribute value associated with V and it is denoted by $I_A(V)$. The Infimum (I_A) of any P-edge $E \in E_P$ is the least attribute value of associated with E and it is denoted by $I_A(E)$. Any $V \in V_P$ is a central P-vertex of G_P, if it is adjacent to the maximum number of P-vertices in G_P. It is denoted by $C_{PV}(G_P)$. Note that it is possible to have more than one central P-vertices in a PPFG.

3.8. Example Let us consider a PPFG, $G_P = (V_P, E_P)$ where $V_P = \{V_1, V_2, V_3, V_4\}$ and $E_P = \{E_1, E_2, E_3, E_4\}$
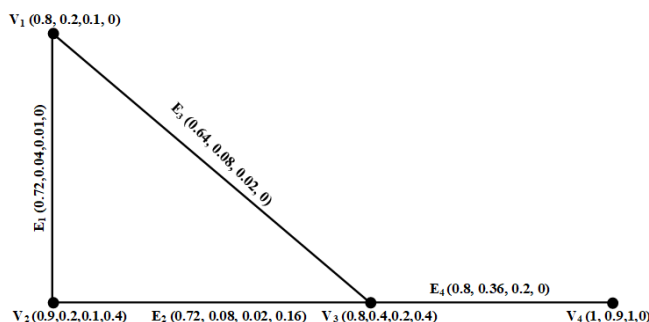
Fig 2. Supremum and Infimum of each P-vertex and each P-edge and Central P-vertex

We have the supremum and the infimum of each P-vertex, the supremum and the infimum of each P-edge, and the central P-vertex of G_P as follows:

$$S_A(V_1) = 0.8 , \ I_A(V_1) = 0 ; \ S_A(V_2) = 0.9 , \ I_A(V_2) = 0.1; \ S_A(V_3) = 0.8 , \ I_A(V_3) = 0.2 ;$$

$$S_A(V_4) = 1 , \ I_A(V_4) = 0 ; \ S_A(E_1) = 0.72, \ I_A(E_1) = 0; \ S_A(E_2) = 0.72, \ I_A(E_2) = 0.02 ;$$

$$S_A(E_3) = 0.64, \ I_A(E_3) = 0; \ S_A(E_4) = 0.8, \ I_A(E_4) = 0 ; \ C_{PV}(G_P) = V_3$$

4 Subgraphs, Path, Cycle, tree, bridge and cut vertex in PPFGs

4.1. Definition Let $G_P=(V_P, E_P)$ be a PPFG. $H_P = (R_P, S_P)$ is said to be a P-vertex subgraph of G_P, if it is induced by the elimination of a P-vertex or P-vertices in G_P. Here $R_P \subset V_P$ and $S_P \subset E_P$. $H_P = (R_P, S_P)$ is said to be a P-spanning subgraph (P-edge subgraph) of G_P, if it is induced by the elimination of a P-edge or P-edges in G_P. Here $R_P = V_P$ and $S_P \subset E_P$. $H_P = (R_P, S_P)$ is said to be a P-vertex-edge subgraph of G_P, if it is induced by the removal of P-vertex/P-vertices and P-edge/P-edges in G_P. Here $R_P \subset V_P$ and $S_P \subset E_P$. It is to be noted that a subgraph of PPFG need not be always a connected PPFG.

4.1. Example Consider G_P in Fig. 3 as the original PPFG. The P-vertex subgraph of G_P in Fig. 4 is obtained by removing V_1, V_4, V_7, V_8 and V_9. The P-edge subgraph of G_P in Fig. 5 is obtained by the removal of the P-edges E_6, E_7, E_{11} and E_{12}. Similarly the P-vertex-edge subgraph in Fig. 6 is obtained by the elimination of V_7 and E_1, E_2, E_5, E_{12}.

Table 1. P-weights of P-vertices of G_P

P-vertices	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8	V_9
P_{VW}	2	1	1.4	2.7	1.4	1.3	1.7	2.3	1.7

Table 2. P-weights of P-edges of Gp

P-edges	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}	E_{11}	E_{12}
P_{EW}	0.25	0.22	1.53	0.36	0.47	0.89	0.45	1.31	0.93	0.51	0.78	1.06

4.2. Theorem Every subgraph of a Plithogenic product fuzzy graph is a Plithogenic product fuzzy graph.

Proof : Let $G_P = (V_P, E_P)$ be a PPFG, where V_P and E_P are PFSs of P-vertices and relevant P-edges respectively. Consider $H_P = (R_P, S_P)$ to be a subgraph of G_P, where R_P and S_P are PFSs of P-vertices and relevant P-edges respectively in H_P.
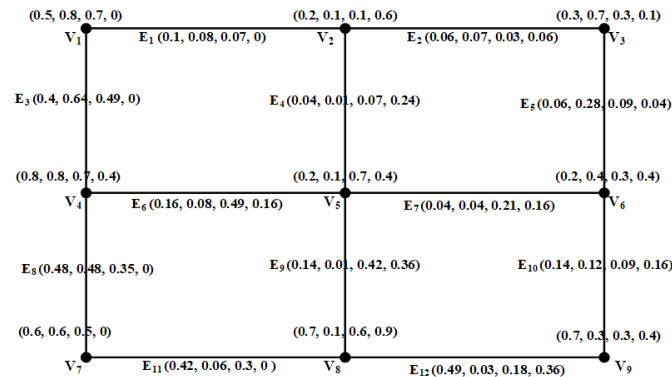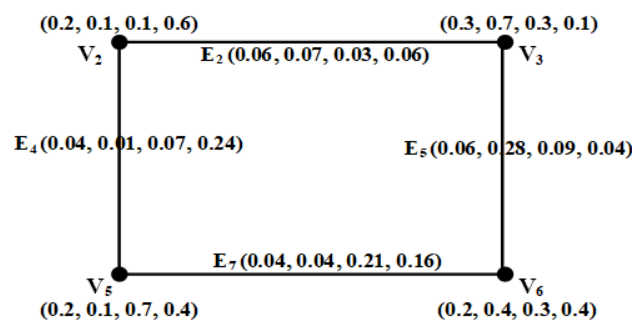
Fig 3. Plithogenic product fuzzy graph, G_p
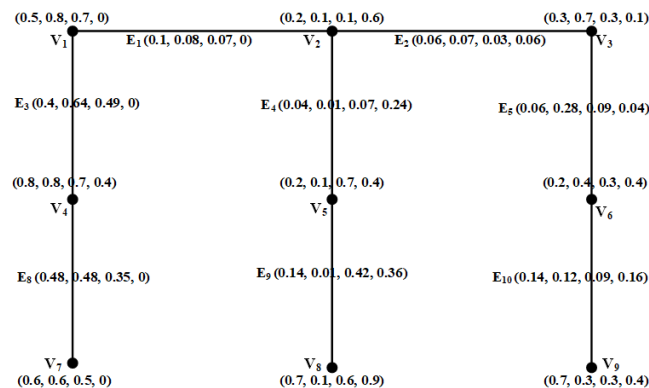

Fig 4. P- vertex subgraph of Gp

Fig 5. P-edge subgraph of Gp

Case (i): If $H_P = (R_P, S_P)$ is a P-vertex subgraph, then $R_P \subset V_P$ and $S_P \subset E_P$. This implies that every $V \in R_P$ is also an element of V_P with the same attribute values and, every $E \in S_P$ is also an element of E_P with the same attribute values. This proves that by definition 3.1. $H_P = (R_P, S_P)$ is a PPFG.

Case (ii): If $H_P = (R_P, S_P)$ is a P-edge subgraph, then $R_P = V_P$ and $S_P \subset E_P$. This implies that every V in R_P is also in V_P with the same attribute values and, every $E \in S_P$ is also an element of E_P with the same attribute values. This proves that by definition 3.1. $H_P = (R_P, S_P)$ is a PPFG.

Case (iii): If $H_P = (R_P, S_P)$ is a P-vertex-edge subgraph, then $R_P \subset V_P$ and $S_P \subset E_P$. This implies that every V in R_P is also an element of V_P with the same attribute values. Similarly, every $E \in S_P$ is also an element of E_P with the same attribute values. This follows that by definition 3.1., $H_P = (R_P, S_P)$ is a PPFG. Hence the proof.
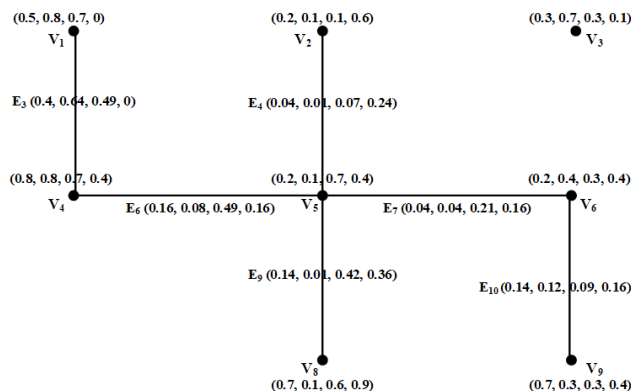
Fig 6. P-vertex-edge subgraph of Gp

4.3. Proposition Every P-vertex subgraph of a connected Plithogenic product fuzzy graph is also P-vertex-edge subgraph.

Proof: Consider $G_P= (V_P, E_P)$ to be a connected PPFG, where V_P and E_P are PFSs of P-vertices and relevant P-edges respectively in G_P. Let $H_P = (R_P, S_P)$ a P-vertex subgraph of G_P, where R_P and S_P are PFSs of P-vertices and relevant P-edges respectively in H_P. This implies that $R_P \subset V_P$ and $S_P \subset E_P$. By definition 4.1., H_P is obtained by the elimination of at least one $V \in V_P$ and its adjacent P-edges in G_P. This follows that H_P is also P-vertex-edge subgraph of G_P. Hence the proof.

4.4. Definition A Plithogenic product fuzzy path (PPFP) is a sequence $V_0, E_1, V_1, E_2, \ldots, V_{n-1}, E_n, V_n$ of distinct P-vertices and P-edges such that for any P-edge E_i with end points V_{i-1} and V_i, $P_{EW}(E_i) > 0$ where $1 \leq i \leq n$. Here n is the length of the PPFP. The strength of a PPFP is defined as $min\{P_{EW}(E_i): i = 1, 2, \ldots n\}$. In other words, the strength of a PPFP is the minimum of the P-weights of all the P-edges in it. A PPFP : $V = V_0, E_1, V_1, E_2, \ldots, V_{n-1}, E_n, V_n = W$ from V to W is said to be strong if all the P-vertices in it are strong P-vertices. A PPFP from V to W is called a Plithogenic product fuzzy cycle (PPFC) provided $V_0 = V_n$ and $n \geq 3$. A PPFC of odd length is said to be an odd PPFC. Similarly, a PPFC of even length is said to be an even PPFC. If there is no PPFC in a PPFG, then it is said to be acyclic. The strength of connectedness between any two P-vertices V_i and V_j in a PPFG is defined as the maximum of the strengths of all Plithogenic product fuzzy paths (PPFPs) between V_i and V_j and it is denoted by $S_{CONP}(V_i, V_j)$. A PPFP that contains $S_{CONP}(V_i, V_j)$ between any two P-vertices V_i and V_j in a PPFG is called as the strongest PPFP provided it is also a strong PPFP. It is to be noted that there can be more than one strongest PPFPs in a PPFG. A PPFG is said to be connected if every pair of P-vertices in it has at least one PPFP between them. Otherwise, it is said to be a disconnected PPFG. A PPFG which is connected and acyclic is called a Plithogenic product fuzzy tree (PPFT). A P-edge in a PPFG is said to be a bridge if its removal decreases the strength of connectedness between some pair of P-vertices in it. A P-vertex in a PPFG is said to be a cut vertex if its removal decreases the strength of connectedness between some pair of P-vertices in it.

4.4. Example In Fig. 3, $V_1, E_1, V_2, E_2, V_3, E_5, V_6$ is a PPFP of length 3. Since $P_{EV}(E_1) = 0.25$, $P_{EV}(E_2) = 0.22$ and $P_{EV}(E_5) = 0.47$, the strength of the path is 0.22. The other PPFPs between V_1 and V_6 are $V_1, E_1, V_2, E_4, V_5, E_7, V_6$ whose strength is 0.25; $V_1, E_3, V_4, E_6, V_5, E_7, V_6$ whose strength is 0.45; $V_1, E_3, V_4, E_8, V_7, E_{11}, V_8, E_{12}, V_9, E_{10}, V_6$ whose strength is 0.51 and $V_1, E_3, V_4, E_8, V_7, E_{11}, V_8, E_9, V_5, E_7, V_6$ whose strength is 0.45. Therefore, strength of connectedness between V_1 and V_6 i.e., $S_{CONP}(V_1, V_6)$ is 0.51 which is the maximum of all strengths of the PPFPs between V_1 and V_6 in Figure 3. V_1, E_3, V_4, E_8, V_7 and V_1, E_3, V_4 and V_4, E_8, V_7 and V_8, E_3, V_4 are strongest PPFPs in Fig. 3. E_{12} is a bridge, since its removal reduces the strength of connectedness between V_1 and V_6 i.e., when E_{12} is removed, then $S_{CONP}(V_1, V_6)= 0.45$. Similarly, V_9 is a cut vertex as its removal reduces the strength of connectedness between V_1 and V_6 Fig. 3. The PPFGs given in Fig. 5 and Fig. 6 are Plithogenic product fuzzy trees since there are no Plithogenic product fuzzy cycles in them, while the PPFG in Fig. 4 is a PPFC of length 4.

4.5. Proposition The strength and the strength of connectedness are identical for every Plithogenic product fuzzy path in a Plithogenic product fuzzy tree.

Proof : Let $G_P= (V_P, E_P)$ be a PPFT, where V_P and E_P are PFSs of P-vertices and relevant P-edges respectively in G_P. Let $V_i, V_j \in V_P$ be any two P-vertices in G_P. Suppose there exists more than one PPFP between V_i and V_j. Then there exists at least one PPFC in G_P which passes through either V_i or V_j, or both V_i and V_j, or some points between V_i and V_j, since G_P is connected. This contradicts that G_P is a PPFT. Therefore, between V_i and V_j there exists a unique path. This implies that strength and strength of connectedness between V_i and V_j are identical. Hence the proof.

4.6. Proposition The longest Plithogenic product fuzzy path in a Plithogenic product fuzzy cycle of length n is of length $n-1$.

Proof Let $G_P = (V_P, E_P)$ be a PPFC, where V_P and E_P are PFSs of P-vertices and relevant P-edges respectively in G_P of length n. By definition 4.4., $G_P - E$ is a PPFP, where $E \in E_P$ is any P-edge in G_P. Since the length of G_P is n, the length of $G_P - E$ is $n-1$. Since G_P is a PPFC, $G_P - E$ is a PPFP of maximum length, i.e., n-1, in G_P. Hence the proof.

4.7. Definition [15] A PPFG, $G_P = (V_P, E_P)$ is said to be a complete PPFG if every $V \in V_P$ is adjacent with the rest of the P-vertices in G_P. The PPFG given in Fig. 8 is a complete PPFG.

5 P-order, P-size, Degree, P-vertex range and P-edge range in Plithogenic product fuzzy graphs

In this section, P-order, P-size, P-vertex range, P-edge range, degree of a P-vertex, total degree of a P-vertex, the minimum degree and the maximum degree of a PPFG are defined and discussed with examples.

5.1. Definition The P-order of a PPFG, $G_P = (V_P, E_P)$ is the sum of the P-weights of all the P-vertices in it, and it is denoted by $P_O(G_P)$. The P-size of G_P is the sum of the P-weights of all the P-edges in it, and it is denoted by $P_S(G_P)$. Note that $P_S(G_P)$ is always less than $P_O(G_P)$ when the attribute values of all elements of V_P are from (0,1).

The difference between the maximum P-weight ($\triangle_{VW}$) and the minimum P-weight (δ_{VW}) of the P-vertices in G_P is said to be its P-vertex range and it is denoted by $P_{VR}(G_P)$. The difference between the maximum P-weight ($\triangle_{EW}$) and the minimum P-weight (δ_{EW}) of the P-edges in G_P is said to be its P-edge range and it is denoted by $P_{ER}(G_P)$.

5.2. Definition Let $G_P = (V_P, E_P)$ be a PPFG, where $V_P = \{V_1, V_2, \ldots, V_n\}$ and $E_P = \{E_1, E_2, \ldots, E_m\}$ are PFSs of P-vertices and P-edges respectively. The degree of any P-vertex $V \in V_P$ is the sum of the P-weights of the P-edges incident to V and it is denoted by $d_P(V)$. The total degree of any P-vertex $V \in V_P$ is the sum of the P-weights of the P-edges incident to V and $P_{VW}(V)$. It is denoted by $td_P(V)$. The minimum degree of $G_P = \wedge \{d_P(V_i) : V_i \in V_P ; i = 1, 2, \ldots, n$ and it is denoted by $\delta_P(G_P)$. The maximum degree of $G_P = \vee \{d_P(V_i) : V_i \in V_P ; i = 1, 2, \ldots, n$ and it is denoted by $\triangle_P(G_P)$.

5.2. Example Consider a PPFG, $G_P = (V_P, E_P)$ as in Fig. 2. By usual computation, we have P-order, P-size, P-vertex range, P-edge range, degree of each P-vertex, total degree of each P-vertex, the minimum degree and the maximum degree of G_P as follows:

$$P_{VW}(V_1) = 1.1 ; P_{VW}(V_2) = 1.6 ; P_{VW}(V_3) = 1.8; P_{VW}(V_4) = 2.9$$

$$P_{EW}(E_1) = 0.77 ; P_{EW}(E_2) = 0.98; P_{EW}(E_3) = 0.74 ; P_{EW}(E_4) = 1.36$$

$$P_O(G_P) = 7.4 , P_S(G_P) = 3.85 , P_{VR}(G_P) = \triangle_{VW} - \delta_{VW} = 1.8 ,$$

$$P_{ER}(G_P) = \triangle_{EW} - \delta_{EW} = 0.62$$

$$d_P(V_1) = 1.51 ; d_P(V_2) = 1.75 ; d_P(V_3) = 3.08 ; d_P(V_4) = 1.36$$

$$td_P(V_1) = 2.61; td_P(V_2) = 3.35 ; td_P(V_3) = 4.88 ; td_P(V_4) = 4.26$$

$$\delta_P(G_P) = 1.36 ; \triangle_P(G_P) = 3.08$$

5.2 Properties of P-vertex range and P-edge range

- For some $V \in V_P$ if $P_{VW}(V) = 0$, then $P_{VR}(G_P) = \triangle_{VW} - \delta_{VW} = \triangle_{VW} - 0 = \triangle_{VW}$
- If some $V_i, V_j \in V_P$ with attribute values $s_1, s_2, \ldots, s_k = 1$ and $t_1, t_2, \ldots, t_k = 0$ respectively, then $P_{VR}(G_P) = 1 + 1 + \ldots k$ times, which is the maximum possible $P_{VR}(G_P)$.

- For some $E \in E_P$ if $P_{EW}(E) = 0$ then $P_{ER}(G_P) = \Delta_{EW} - \delta_{EW} = \Delta_{EW} - 0 = \Delta_{EW}$
- For some E_i, $E_j \in E_P$ with attribute values $m_1, m_2, \ldots, m_k = 1$ and $n_1, n_2, \ldots, n_k = 0$ respectively, $P_{ER}(G_P) = 1 + 1 + \ldots k$ times, which is the maximum possible $P_{ER}(G_P)$.
- If $\triangle_{VW}(G_P) = \delta_{VW}(G_P)$, then $P_{VR}(G_P) = P_{ER}(G_P) = 0$. Such PPFGs are referred as Zero-range PPFGs.

6 Strong and weak P-vertices, highly strong, strong and weak P-edges in Plithogenic product fuzzy graphs

6.1. Definition Let $G_P = (V_P, E_P)$ be a PPFG. Any $V \in V_P$ with attribute values $s_1, s_2, \ldots, s_k$ is said to be a strong P-vertex, if $(s_1 + s_2 + \cdots + s_k) \geq A_{VW}(G_P)$, *where* $A_{VW}(G_P)$ is the average P-weight of P-vertices in G_P, i.e. $A_{VW}(G_P) = \frac{P_O(G_P)}{(V_P(G_P)|}$. Here $P_O(G_P)$ is the P-order of G_P and $(V_P(G_P)|$ is the V_P-cardinality of G_P. Otherwise, V is called a weak P-vertex.

 6.1. Example Consider a PPFG, $G_P = (V_P, E_P)$ such that $V_P = \{V_1, V_2, V_3, V_4, V_5\}$ and $E_P = \{E_1, E_2, E_3, E_4, E_5\}$
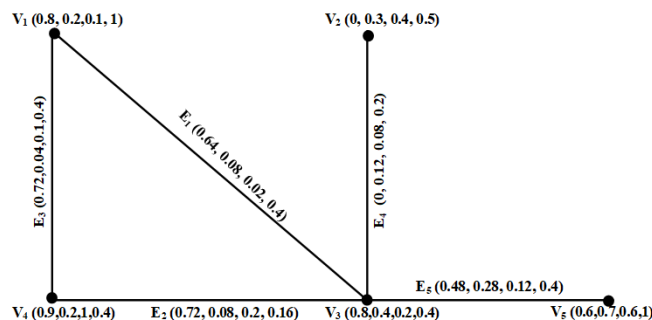

Fig 7. Strong P-vertices and Weak P- vertices

In the above PPFG, by computation we have the P-weight of each P-vertex as follows:
$P_{VW}(V_1) = 2.1$; $P_{VW}(V_2) = 1.2$; $P_{VW}(V_3) = 1.8$; $P_{VW}(V_4) = 2.5$; $P_{VW}(V_5) = 2.9$.
Here $A_{VW}(G_P) = \frac{P_O(G_P)}{|V_P(G_P)|} = \frac{10.5}{5} = 2.1$ Therefore, V_1, V_4 and V_5 are strong P-vertices. V_2 and V_3 are weak P-vertices.

 6.2. Definition Let $G_P = (V_P, E_P)$ be a PPFG. For any $E \in E_P$ if its adjacent P-vertices are strong P-vertices, then E is said to be a highly strong P-edge. If any one of the adjacent P-vertices of E is a strong P-vertex and the other is a weak P-vertex, then E is said to be a strong P-edge. If both the adjacent P-vertices of E are weak P-vertices, then E is said to be a weak P-edge.

 6.2. Example Consider the PPFG as in Fig. 7. Here $E_3 = V_1 V_4$ is a highly strong P-edge since both V_1 and V_4 are strong P-vertices. $E_1 = V_1 V_3$, $E_2 = V_3 V_4$ and $E_5 = V_3 V_5$ are strong P-edges since V_3 is a weak P-vertex, and V_1, V_4 and V_5 are strong P-vertices. $E_4 = V_2 V_3$ is a weak P-edge since both V_2 and V_3 are weak P-vertices.

7 Application of Plithogenic product fuzzy graph in social network

A network is a representation of different entities and their relations or connections. A social network is a representation of a social phenomenon of a group of people, organizations, structures, events, countries, places, animals, etc. In any social network all the units not necessarily have the same importance. The density of connections at every unit is not always the same. The strength of connection between every pair of units varies due to various reasons. Therefore, all these need to be measured to decide upon the stability of any social network[23,24]. Since the property or the quality of vertices and the strength of connectedness between them are of different levels, they can be represented by fuzzy values from [0,1].[25,26] Connectivity parameters such as number of edges present in the network, number of neighbours to each vertex, number of common neighbors, order, size, cut vertex, bridge, etc. are helpful to analyze the performance of a social network.[27–29]

 Since PPFG model has four or more attribute values associated with vertices, a better prediction about the quality of P-vertices possible in PPFG model. The computation of the attribute values for the P-edges in PPFG model reduces the error to the minimum in our prediction regarding strength of connectivity. Similarly the connectivity parameters discussed in this paper are better tools to investigate connectedness in the network. In this section a new mathematical model based on PPFGs has been utilized to analyze a social network of persons. P- vertices and P-edges in the network are characterized by four attributes which have the corresponding attribute values from [0,1]. Some of the other terms in PPFGs and their corresponding meanings in our example of social network are as follows:

Table 3. Terms of PPFGs and their meanings

Terms of PPFG	Meanings
$\left|V_P(G_P)\right|$	Total number of persons
$\left|E_P(G_P)\right|$	Total number of connections
P-weight of a P-vertex	Strength of sociability of a person
P-weight of a P-edge	Strength of connectivity between two persons
Strong P-vertex	Important or influential person
Weak P-vertex	Unimportant or uninfluential person
Highly strong P-edge	Highly cohesive connection
Strong P-edge	Cohesive connection
Weak P-edge	Weak connection
$C_{PV}(G_P)$	Central and significant persons of the network
$P_O(G_P)$	Total strength of sociability of persons in the network
$P_S(G_P)$	Total strength of connectivity in the network
$P_{VR}(G_P)$	The maximum difference in the strengths of sociability of persons
$P_{ER}(G_P)$	The maximum difference in the strengths of connectivity of persons
$d_P(V)$	Strength of direct connectivity of a person
$td_P(V)$	Strength of direct connectivity and sociability of a person
A_{VW}	Average strength of sociability in the network
Strength of a PPFP	Strength of indirect connectivity
S_{CONP}	Maximum strength of indirect connectivity

7.1 The objectives of this analysis of the social network of persons are as follows:

- To extract the overall impression regarding the cohesiveness of their connections.
- To find the types of relations among them in the network.
- To identify the most social, sociable, influential and central persons in the network.
- Any other relational information about the persons in the network

7.2 Results

Let us consider a PPFG, $G_P = (V_P, E_P)$ as in Fig. 8, where $V_P = \{V_1, V_2, V_3, V_4\}$ and $E_P = \{E_1, E_2, E_3, E_4, E_5, E_6\}$ are PFSs of P-vertices and relevant P-edges respectively. Here G_P represents a social network characterized by four attributes. Here V_1, V_2, V_3 and V_4 are siblings staying in different places, while E_1, E_2, E_3, E_4, E_5 and E_6 show the existence of direct connections between them. The following matrix shows if there is a direct connection between any two persons. 1 indicates that there is a direct connection, while 0 signifies that there is no connection between a particular pair of persons.

$$\begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

The attributes to the persons and their connections are (i) Number of visits made (h_1), (ii) Number of gifts sent (h_2), (iii) Number of phone calls (h_3), and (iv) Number of WhatsApp messages (h_4) during the period of a year. The formulae that are used to calculate the fuzzy attribute values of h_1, h_2, h_3 and h_4 for each P-vertex are as follows:

h_1's fuzzy attribute value $(g_1) = \frac{h_1}{\sum_{i=1}^{4} V_i(h_1)}$

h_2's fuzzy attribute value $(g_2) = \frac{h_2}{\sum_{i=1}^{4} V_i(h_2)}$

h_3's fuzzy attribute value $(g_3) = \frac{h_3}{\sum_{i=1}^{4} V_i(h_3)}$

h_4's fuzzy attribute value $(g_4) = \frac{h_4}{\sum_{i=1}^{4} V_i(h_4)}$

The corresponding PPFG, $G_P = (V_P, E_P)$ for the above matrix and other details in Tables 4 and 5 and Table 6 are shown in Figure 8.
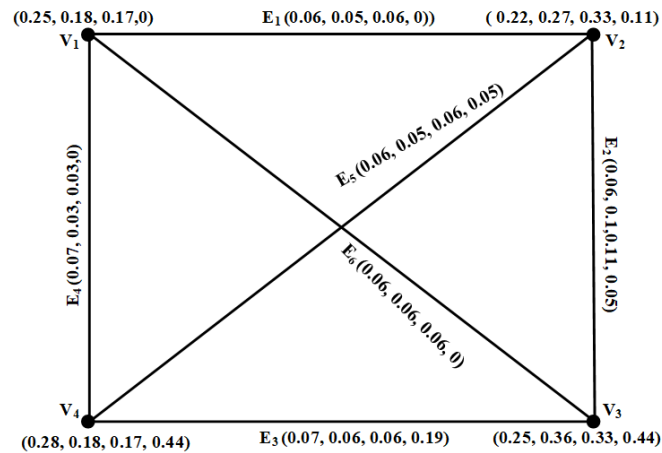

Fig 8. Social network of persons

Table 4 shows the computation of attribute values and the P-weights of P-vertices (P_{VW}). All decimal values are rounded off up to two decimal places for the sake of convenience in calculation.

Table 4. Attribute values and P-weights of P-vertices

Persons	h_1	h_2	h_3	h_4	g_1	g_2	g_3	g_4	P_{VW}
V_1	8	2	1	0	0.25	0.18	0.17	0	0.6
V_2	7	3	2	1	0.22	0.27	0.33	0.11	0.93
V_3	8	4	2	4	0.25	0.36	0.33	0.44	1.38
V_4	9	2	1	4	0.28	0.18	0.17	0.44	1.07
Total	32	11	6	9	1	0.99	1	0.99	3.98

The attribute values of P-edges are calculated using the usual product operator by the method shown in the definition 3.1. The attribute values of P-edges concerning h_1, h_2, h_3 and h_4, are referred by the variables f_1, f_2, f_3 and f_4 respectively. Table 5 shows the computation of attribute values and the P-weights of P-edges (P_{EW}).

Table 5. Attribute values and P-weights of P-edges

Connection	$E_1 = V_1V_2$	$E_6 = V_1V_3$	$E_4 = V_1V_4$	$E_2 = V_2V_3$	$E_5 = V_2V_4$	$E_3 = V_3V_4$
f_1	0.06	0.06	0.07	0.06	0.06	0.07
f_2	0.05	0.06	0.03	0.1	0.05	0.06
f_3	0.06	0.06	0.03	0.11	0.06	0.06
f_4	0	0	0	0.05	0.05	0.19
P_{EW}	0.17	0.18	0.13	0.32	0.22	0.38

Degree and total degree of P-vertices are computed and shown in Table 6.

$P_O(G_P) = 3.98 \quad P_S(G_P) = 1.4 \quad P_{VR}(G_P) = 0.78 \quad P_{ER}(G_P) = 0.25$

$$A_{VW}(G_P) = \frac{P_O(G_P)}{(V_P(G_P)|} = \frac{3.98}{4} = 0.995 = 1.00$$

Table 6. Degree and total degree of P-vertices

	V_1	V_2	V_3	V_4
d_P	0.48	0.71	0.88	0.73
td_P	1.08	1.64	2.26	1.8

7.3 Discussion

The strengths of sociability of V_3 and V_4 are greater than 1, which is the average sociability of persons in the network, while that of V_1 is the lowest, and that of V_2 is 0.93 which is slightly less than 1. This shows that V_3 and V_4 are more sociable persons in the network. Since the strength of sociability of V_1 is the lowest, he is considered to be the least sociable among the four persons. Also the strength of connectivity between V_3 and V_4 is the best in the network. This implies that there are significant transactions and commutations taking place between V_3 and V_4. On the other hand, V_1V_4 has the lowest strength of connectivity as its P-weight is the least in the network. This indicates that V_1's relationship with V_4 is the weakest in the network. However, the strengths of connectivity of V_1 with V_2 and V_3 are satisfactory.

The degree and the total degree of V_3 are higher than those of others in the network. Therefore, both in strength of direct connectivity as well as in strength of sociability V_3 is the best. Hence V_3 is more significant and influential for the overall interactions, flow of information and the smooth functioning of the network. V_3 is a leader inside this network [24].

The overall impression regarding the strength of sociability of persons in the network seems to be very good as three out of four persons have better strengths of sociability. Since the grading of strength of connectivity among them depends on their strengths of sociability, based on the data in Table 4 we conclude that E_3 is a highly cohesive connection, E_2, E_4, E_5 and E_6 are cohesive connections, and E_1 is a weak connection.

8 Conclusion

PPFGs has been newly defined and its properties are investigated with results and examples in this paper. It is proved that every PPFG is a PFG, but the converse need not be true. It is also verified that every subgraph of a PPFG is PPFG. Finally, the utility of PPFG and its application in a small-sized social network is demonstrated to understand the advantage of it in decision making process. As future work, A large-scale practical network can be modeled using PPFG to analyze a real-life problem under uncertain environment. Operations on PPFGs are also areas of scope for future research to understand better the nature of PPFGs.

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