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Mathematical Analysis of Pluviculture in the Frame of Caputo Fractional Derivative

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Abstract

Objectives: Climate change is a major issue the mankind is facing in the present scenario. This is a consequence of global warming that result in the change in temperature and rainfall over a long period of time. In this work, we analyze the fractional mathematical model projecting pluviculture. The Caputo fractional derivative is incorporated for better analysis of this event. Also, we discuss the boundedness, existence and uniqueness of the solutions of the proposed system. Sufficient conditions required for existence of asymptotic stability are discussed. **Method:** We have used the generalized Adams-Bashforth-Moulton predictor-corrector technique to solve the pluviculture model. It is the linear multistep implicit method used to solve the system of equations. **Findings:** It is established that, the intensity of precipitation is enhanced by introducing the aerosols to the water vapor. The incorporation of the fractional derivatives strengthens the model in a more realistic way. The influence of some parameters and fractional derivative on the rain making process are numerically analyzed. **Novelty:** The incorporation of Caputo fractional derivative to the pluviculture model along with the two kinds of aerosols is the novel in the model.

Keywords: Pluviculture; Aerosol; Caputo fractional derivative; AdamsBashforthMoulton technique Mathematics Subject Classification: 26A33; 92D30

1 Introduction

Rainmaking, also known as pluviculture, is the method of inducing precipitation artificially. This is done to stave off drought and to save Earth from wider global warming. This can be achieved using rockets or airplanes to sow to the clouds along with the catalysts like silver iodide,[(1)] dry ice and salt powder, to increase precipitation and mitigate farmland drought. The concept of "pluviculture," or the artificial rain creations, has long history, both ancient and modern. Especially during the 19th and 20th centuries, it has excited the mind's eye and brought forth enormous rainmaking techniques. Rain making is a prime and complex phenomenon in the environment. Several physicochemical processes such as agglomeration, nucleation, condensation and so on are involved in this process.[(2,3)] Firstly, water vapors are converted to cloud droplets of small size by condensation and nucleation processes. Then, the small droplets randomly move inside the cloud, stick and collide to each other and forms large size cloud drops. Further, the cloud droplets of large size are converted into rain droplets by the process of agglomeration and nucleation. Aerosols has the vital role in this process which in turn enhances the precipitation. Aerosols refer to the tiny specks which are in solid or liquid state having the diameters in the range 10^{-9} to 10^{-4} m. They originate from anthropogenic springs and also created in the nature by transformation of gas into particles due to several chemical reactions. The key components of aerosols are sodium chloride, gritty particles, nitrate, crustal elements and biogenic organic molecules like spores, fragments of the plant and pollen.

Some theoretical and as laboratory experiments are carried out and seen that aerosols play the role of Cloud Condensation Nuclei (CCN), surrounding to which the cloud droplets generates.[(4–8)] Some studies show the analysis of the deletion of dusty particles of gaseous state from environment by bearing the synergy of these particles with the cloud drops. It is notable from the above studies that, the rain and wind play the major role in deletion of pollutants present in the local regional environment.[(9–11)] Some studies show the role of aerosols in the regional environment for inducing artificial rain.[(12–14)] In particular, Misra et al.[(15)] proposed the mathematical model for the artificial precipitation by presuming that two types of aerosols are instigated to environment where water vapors generates in an uninterrupted manner. Aerosols of first kind when associated with water vapors converts to cloud drops. The aerosols of second kind collide with the cloud drops and transform them to raindrops which finally leads to precipitation.

The development and use of fractional calculus has proven to be an useful tool. The Caputo, Gr*ü* nwald Letnikov, Riemann-Liouville, Atangana-Baleanu, Jumarie, Caputo-Fabrizio are few among the fractional derivatives that are gaining their importance among researchers to replicate real world problems. Over the years, theories of these derivatives have been developed to a great extent.[(16,17)] Applications of Caputo-Fabrizio fractional and Atangana-Baleanu derivative can be found in.[(18–23)] Many phenomena of mathematical biology[(24–26)] and their interdisciplinary fields[(27–30)] have been studied in a better way by using these fractional derivatives.

From above, it may be pointed out that chemical and physical properties of the natural atmospheric aerosols are much studied but the study of the use of aerosols in making precipitation is given a little attention. Especially the combination of two kinds of drops (big and small) and two kinds of aerosols is given even less attention. Also, the pluviculture model with the fractional order is nowhere discussed. Hence, keeping all these in view, in the present work, we have proposed a mathematical model including the two kinds of aerosols in the atmosphere leading to the precipitation incorporating the Caputo fractional derivative which is the novel in the model. In the modeling procedure, we have assumed that the water vapors are formed naturally in the air. But, in order to result in the precipitation, they are not condensed in the required proportion so as to form the cloud droplets which is necessary for rainfall. By introducing the combination of two kinds of aerosols in the atmosphere, the cloud droplets are formed which are of different sizes from the water vapor through the processes of agglomeration, nucleation, and condensation and so on, consequently these changes to water drops which finally leads to precipitation.

2 Some Essential Theorems

In the present work, we have used the Caputo fractional derivatives as it supports the integer order initial condition. We have also gone over certain theorems which is utilised to ascertain the stability of equilibrium locations. The Caputo fractional derivative is denoted by a capital letter with an upper-left index ${}^{C}D$.

Definition 2.1[(16)] (Caputo Fractional Derivative) Suppose $g(t)$ is k times continuously differentiable function and $g^{(k)}(t)$ is integrable in $[t_0,T]$. For $g(t)$, The fractional derivative of the order α established by Caputo sense is

$$ {}_{t_0}^{C}D_t^{\alpha} g(t) = \frac{1}{\Gamma(k-\alpha)} \int_{t_0}^{t} \frac{g^{(k)}(\tau)}{(t-\tau)^{\alpha+1-k}} d\tau $$

where $\Gamma(.)$ refers to Gamma function, $t > a$ and k is the positive integer with the property that $k-1 < \alpha < k$.

Lemma 1[(31)] Consider the system

$$ {}_{t_0}^{C}D_t^{\alpha} v(t) = g(t,v), t > t_0 \tag{1} $$

choosing the initial condition as $v(t_0)$, where $0 < \alpha \leq 1$ and $g : (t_0, \infty) \times \Omega \to R^n, \Omega \in R^n$. When $g(t,v)$ holds the locally Lipchitz conditions concerning to v, Eq. 1 has a unique solution on $(t_0, \infty) \times \Omega$.

Lemma 2[(29)] We assume that $g(t)$ is the continuous function on (t_0, ∞) satisfying

${}_{t_0}^{C}D_t^{\alpha} g(t) \leq -\lambda g(t) + \xi, g(t_0) = f_{t_0}$

here $0 < \alpha \leq 1,\ (\lambda, \xi) \in R^2$ and $\lambda \neq 0$. Consider $t_0 \geq 0$ as the initial time. Now,

$$g(t) \leq \left(g(t_0) - \frac{\xi}{\lambda}\right) E_\alpha - [\lambda (t-t_0)^\alpha] + \frac{\xi}{\lambda}$$

3 Model Formulation

Motivated by the mathematical models concerning the artificial precipitation process, here, we have analyzed the fractional pluviculture model with the aid of the Caputo fractional order derivative. The pluviculture model is ruled by six interacting variables viz. densities of water vapor (C_V), densities of small size cloud droplets (C_{SD}), densities of large size cloud droplets (C_{LD}), densities of rain drops (C_R) and concentrations of the first and second type aerosols C_1 and C_2 respectively.

$$\begin{aligned}
&{}^{C}{t_0}D_t^\alpha C_V = J_V - \beta_0 C_V - \beta_1 C_V C_1 \\
&{}^{C}_{t_0}D_t^\alpha C_{SD} = \delta_S C_V - \delta_{S_0} C_{SD} + \psi \beta_1 C_V C_1 - \delta_{S_1} C_{SD} C_2 \\
&{}^{C}_{t_0}D_t^\alpha C_{LD} = \delta_L C_{SD} - \delta_{L_0} C_{LD} + \delta_{S_1} C_{SD} C_2 - \delta_{L_1} C_{LD} C_2 \\
&{}^{c}_{t_0}D_t^\alpha C_R = \gamma C_{LD} - \gamma_0 C_R + \xi \delta_{L_1} C_{LD} C_2 \\
&{}^{0}_{t_0}D_t^\alpha C_1 = J_1 - \mu_1 C_1 - \beta_1 C_V C_1 \\
&{}^{C}_{t_0}D_t^\alpha C_2 = J_2 - \mu_2 C_2 - \delta_{S_1} C_{SD} C_2 - \delta_{L_1} C_{LD} C_2 - \delta_R C_R C_2
\end{aligned} \tag{2}$$

with initial condition $C_V(t_0) > 0,\ C_{SD}(t_0) > 0,\ C_{LD}(t_0) > 0,\ C_R(t_0) > 0,\ C_1(t_0) > 0,\ C_2(t_0) > 0$ where t_0 is the initial time. All the parameters $J_V,\ \beta_0,\ \beta_1,\ \delta_S, \delta_{S_0},\ \psi,\ \delta_{S_1},\ \delta_L,\ \delta_{L_0},\ \delta_{L_1},\ \gamma,\ \gamma_0,\ \xi,\ J_1,\ J_2,\ \mu_1,\ \mu_2,\ \delta_R$ are non-negative.

In the projected model 2, the phase of the water vapors are generated in the continuous basis at the rate J_V (the net rate of change in the density of the water vapor is presumed to be a constant). The conductive aerosols of the first kind and the second kind are introduced continuously to the environment at the rates J_1 and J_2 respectively. The constant terms $\beta_0,\ \delta_{S_0},\ \delta_{L_0},\ \gamma_0,\ \mu_1$ and μ_2 denote the coefficients of natural reduction rate of $C_V,\ C_{SD},\ C_{LD},\ C_R, C_1$ and C_2 respectively.

The coefficients $\delta_S \geq 0,\ \delta_L > 0$ and $r > 0$ denote the rate of natural genesis of small size cloud drops from the water vapor, cloud drops of large size from cloud drops of small size and then to rain drops from the cloud drops of large size. Hence, clearly it can be noted that $\delta_S \leq \beta_0$, $\delta_L < \delta_{S_0}$ and $r < \delta_{L_0}$. The terms ψ and ξ are the positive proportionality constants. The coefficients $\beta_1,\ \delta_{S_1},\ \delta_{L_1}$ and δ_R denote the rate of conversion between the water vapor phase and first kind aerosol, cloud drops of small size and second kind aerosol, cloud drops of large size and second kind aerosol and rain drops and the second kind aerosol respectively.

4 Existence of the solutions

The existence of the solutions for the pluviculture model 2 is demonstrated using the Fixed-Point Theorem in this Section. There are no precise algorithms or approaches for evaluating the exact solutions since the model is complex and non-local. However, the existence is guaranteed if certain conditions are met. The system 2 can be rewritten as:

$$\begin{aligned}
&{}^{C}_{t_0}D_t^\alpha [C_V(t)] = \mathrm{P}_1(t, C_V) \\
&{}^{C}_{t_0}D_t^\alpha [C_{SD}(t)] = \mathrm{P}_2(t, C_{SD}) \\
&{}^{C}_{t_0}D_t^\alpha [C_{LD}(t)] = P_3(t, C_{LD}) \\
&{}^{C}_{t_0}D_t^\alpha [C_R(t)] = \mathrm{P}_4(t, C_R) \\
&{}^{C}_{t_0}D_t^\alpha [C_1(t)] = \mathrm{P}_5(t, C_1) \\
&{}^{C}_{t_0}D_t^\alpha [C_2(t)] = \mathrm{P}_6(t, C_2)
\end{aligned} \tag{3}$$

he above system can be transformed into Volterra type integral equation as:

$$C_V(t) - C_V(0) = \frac{1}{\Gamma(\alpha)} \int_0^t P_1(\tau, C_V(\tau))(t-\tau)^{\alpha-1} d\tau$$

$$C_{SD}(t) - C_{SD}(0) = \frac{1}{\Gamma(\alpha)} \int_0^t P_2(\tau, C_{SD}(\tau))(t-\tau)^{\alpha-1} d\tau \tag{4}$$

$$C_{LD}(t) - C_{LD}(0) = \frac{1}{\Gamma(\alpha)}\int_0^t P_3(\tau, C_{LD}(\tau))(t-\tau)^{\alpha-1}d\tau$$

$$C_R(t) - C_R(0) = \frac{1}{\Gamma(\alpha)}\int_0^t P_4(\tau, C_R(\tau))(t-\tau)^{\alpha-1}d\tau$$

$$C_1(t) - C_1(0) = \frac{1}{\Gamma(\alpha)}\int_0^t P_5(\tau, C_1(\tau))(t-\tau)^{\alpha-1}d\tau$$

$$C_2(t) - C_2(0) = \frac{1}{\Gamma(\alpha)}\int_0^t P_6(\tau, C_2(\tau))(t-\tau)^{\alpha-1}d\tau$$

Theorem 4. 1 In the region $\eta \times [t_0, T]$, where
$\eta = \{(C_V, C_{SD}, C_{LD}, C_R, C_1, C_2) \in \mathbb{R}^6 : \max\{|\mathrm{C_V}|, |\mathrm{C_{SD}}|, |\mathrm{C_{LD}}|, |\mathrm{C_R}|, |\mathrm{C_1}|, |\mathrm{C_2}|\} \le P\}$
and $T < +\infty$, the Lipschitz condition holds good and contraction occurs by the kernel if P_1 if $0 \le \beta_0 + \beta_1 P < 1$.

Proof: We consider the two functions $C_V(t)$ and $\bar{C}_V(t)$ such as:

$$||\mathrm{P}_1(t, C_V) - \mathrm{P}_1(t, \bar{C}_V)|| = ||J_V - \beta_0 C_V - \beta_1 C_1 C_V - J_V + \beta_0 \bar{C}_V + \beta_1 \bar{C}_1 \bar{C}_V|| \tag{5}$$

$$\le (\beta_0 + \beta_1 P)||C_V(t) - \bar{C}_V(t)||$$

$$\left\|\mathrm{P}_1(t, C_V) - \mathrm{P}_1(t, \bar{C}_V)\right\| = \chi_1 ||C_V(t) - \bar{C}_V(t)||$$

where $\chi_1 = \beta_0 + \beta_1 P$ implies that,

The Lipschitz condition is met for P_1 and if $0 \le \chi_1 < 1$, then P_1 follows contraction. Similarly, it can be shown and illustrated in the other equations as follows:

$$\begin{aligned}
&\left\|\mathrm{P}_2(t, C_{SD}) - \mathrm{P}_2(t, \bar{C}_{SD})\right\| \le \chi_2 ||C_{SD}(t) - \bar{C}_{SD}(t)|| \\
&\left\|\mathrm{P}_3(t, C_{LD}) - \mathrm{P}_3(t, \bar{C}_{LD})\right\| \le \chi_3 ||C_{LD}(t) - \bar{C}_{LD}(t)|| \\
&\left\|\mathrm{P}_4(t, C_R) - \mathrm{P}_4(t, \bar{C}_R)\right\| \le \chi_4 ||C_R(t) - \bar{C}_R(t)|| \\
&\left\|\mathrm{P}_5(t, C_1) - \mathrm{P}_5(t, \bar{C}_1)\right\| \le \chi_5 ||C_1(t) - \bar{C}_1(t)|| \\
&\left\|\mathrm{P}_6(t, C_2) - \mathrm{P}_6(t, \bar{C}_2)\right\| \le \chi_6 ||C_2(t) - \bar{C}_2(t)||
\end{aligned} \tag{6}$$

Where $\chi_2 = (\delta_{S_0} + \delta_{S_1} P)$, $\chi_3 = (\delta_{L_0} + \delta_{L_1} P)$, $\chi_4 = \gamma_0$, $\chi_5 = (\mu_1 + \beta_1 P)$ and $\chi_6 = \mu_2 + (\delta_{S_1} + \delta_{L_1} + \delta_R)P$. P_i, $i = 2,3,4,5,6$ are the contraction if $0 < \chi_i < 1$, $i = 2,3,4,5,6$. Using system 4, the recursive form can now be written as:

$$\kappa_{1,n}(t) = C_{V_n}(t) - C_{V_{n-1}}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \left(P_1(\tau, C_{V_{n-1}}) - P_1(\tau, C_{V_{n-2}})\right)(t-\tau)^{\alpha-1}d\tau$$

$$\kappa_{2,n}(t) = C_{SD_n}(t) - C_{SD_{n-1}}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \left(P_2(\tau, C_{SD_{n-1}}) - P_2(\tau, C_{SD_{n-2}})\right)(t-\tau)^{\alpha-1}d\tau$$

$$\kappa_{3,n}(t) = C_{LD_n}(t) - C_{LD_{n-1}}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \left(P_3(\tau, C_{LD_{n-1}}) - P_3(\tau, C_{LD_{n-2}})\right)(t-\tau)^{\alpha-1}d\tau$$

$$\kappa_{4,n}(t) = C_{R_n}(t) - C_{R_{n-1}}(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \left(P_4(\tau, C_{R_{n-1}}) - P_4(\tau, C_{R_{n-2}})\right)(t-\tau)^{\alpha-1}d\tau$$

$$\kappa_{5,n}(t) = C_{1_n}(t) - C_{1_{n-1}}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(P_5\left(\tau, C_{1_{n-1}}\right) - P_5\left(\tau,\, C_{1_{n-2}}\right)\right)(t-\tau)^{\alpha-1} d\tau$$

$$\kappa_{6,n}(t) = C_{2_n}(t) - C_{2_{n-1}}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \left(P_6\left(\tau, C_{2_{n-1}}\right) - P_6\left(\tau,\, C_{2_{n-2}}\right)\right)(t-\tau)^{\alpha-1} d\tau \tag{7}$$

The prerequisites are: $C_{V0}(t) = C_V(0)$, $C_{SD0}(t) = C_{SD}(0)$, $C_{LD0}(t) = C_{LD}(0)$, $C_{R0}(t) = C_R(0)$, $C_{10}(t) = C_1(0)$, $C_{20}(t) = C_2(0)$.

By applying the norm to the first equation 7, we get

i

$$\begin{aligned} ||\kappa_{1,n}(t)|| = ||C_{V_n}(t) - C_{V_{n-1}}(t)|| = ||\frac{1}{\Gamma(\alpha)} \int_0^t \left(P_1\left(\tau, C_{V_{n-1}}\right) - P_1\left(\tau, C_{V_{n-2}}\right)\right)(t-\tau)^{\alpha-1} d\tau|| \\ \leq \frac{1}{\Gamma(\alpha)} \int_0^t \left\|\left(P_1\left(\tau, C_{V_{n-1}}\right) - P_1\left(\tau, C_{V_{n-2}}\right)\right)(t-\tau)^{\alpha-1} d\tau\right\| \end{aligned} \tag{8}$$

Using Lipchitz condition 5, we obtain

$$||\kappa_{1,n}(t)|| \leq \frac{1}{\Gamma(\alpha)} \chi_1 \int_0^t ||\kappa_{1,n-1}(\tau) d\tau|| \tag{9}$$

Similarly,

$$\begin{aligned} ||\kappa_{2,n}(t)|| &\leq \frac{1}{\Gamma(\alpha)} \chi_2 \int_0^t ||\kappa_{2,n-1}(\tau) d\tau|| \\ ||\kappa_{3,n}(t)|| &\leq \frac{1}{\Gamma(\alpha)} \chi_3 \int_0^t ||\kappa_{3,n-1}(\tau) d\tau|| \\ ||\kappa_{4,n}(t)|| &\leq \frac{1}{\Gamma(\alpha)} \chi_4 \int_0^t ||\kappa_{4,n-1}(\tau) d\tau|| \\ ||\kappa_{5,n}(t)|| &\leq \frac{1}{\Gamma(\alpha)} \chi_5 \int_0^t ||\kappa_{5,n-1}(\tau) d\tau|| \\ ||\kappa_{6,n}(t)|| &\leq \frac{1}{\Gamma(\alpha)} \chi_6 \int_0^t ||\kappa_{6,n-1}(\tau) d\tau|| \end{aligned} \tag{10}$$

Now, it can be written as

$C_{V\,n}(t) = \sum_{i=1}^{n} \kappa_{1,i}$, $C_{SDn}(t) = \sum_{i=1}^{n} \kappa_{2,i}$, $C_{LDn}(t) = \sum_{i=1}^{n} \kappa_{3,i}$.

$C_{Rn}(t) = \sum_{i=1}^{n} \kappa_{4,i}$, $C_{1n}(t) = \sum_{i=1}^{n} \kappa_{5,i}$, $C_{2n}(t) = \sum_{i=1}^{n} \kappa_{6,i}$.

The above Theorem will be used to illustrate the proceeding Theorem.

Theorem 4. 2 The solution of the pluviculture model 2 exists and will be unique, if we acquire some t_α such that

$\frac{1}{\Gamma(\alpha)} \chi_i t_\alpha < 1$,

for i=1,2,3,4,5,6.

Proof: Applying equation 9 and 10 recursively, we have

$$\begin{aligned} \|\kappa_{1,i}(t)\| &\leq ||C_{V_n}(t_0)|| \left(\frac{1}{\Gamma(\alpha)} \chi_1 \mathrm{t}\right)^{\mathrm{n}} \\ \|\kappa_{2,i}(t)\| &\leq \|C_{SD_n}(t_0)\| \left(\frac{1}{\Gamma(\alpha)} \chi_2 \mathrm{t}\right)^{\mathrm{n}} \\ \|\kappa_{3,i}(t)\| &\leq \|C_{LD_n}(t_0)\| \left(\frac{1}{\Gamma(\alpha)} \chi_3 \mathrm{t}\right)^{\mathrm{n}} \\ \|\kappa_{4,i}(t)\| &\leq \|C_{R_n}(t_0)\| \left(\frac{1}{\Gamma(\alpha)} \chi_4 \mathrm{t}\right)^{\mathrm{n}} \\ \left\|\kappa_{5,i}(t)\right\| &\leq \|C_{1_n}(t_0)\| \left(\frac{1}{\Gamma(\alpha)} \chi_5 \mathrm{t}\right)^{\mathrm{n}} \\ \left\|\kappa_{6,i}(t)\right\| &\leq \|C_{2_n}(t_0)\| \left(\frac{1}{\Gamma(\alpha)} \chi_6 \mathrm{t}\right)^{\mathrm{n}} \end{aligned} \tag{11}$$

As a result, the existence and continuity are established. To illustrate that the relation 11 formulate the solution for 2, we assume the following:

$C_V(t) - C_V(t_0) = C_{Vn}(t) - \varpi_{1n}(t)$
$C_{SD}(t) - C_{SD}(t_0) = C_{SDn}(t) - \varpi_{2n}(t)$
$C_{LD}(t) - C_{LD}(t_0) = C_{LDn}(t) - \varpi_{3n}(t)$
$C_R(t) - C_R(t_0) = C_{Rn}(t) - \varpi_{4n}(t)$
$C_1(t) - C_1(t_0) = C_{1n}(t) - \varpi_{5n}(t)$
$C_2(t) - C_2(t_0) = C_{2n}(t) - \varpi_{6n}(t)$

In order to achieve the desired outcomes, we set

$||\varpi_{1n}(t)|| = ||\frac{1}{\Gamma(\alpha)} \int_0^t \left(\mathrm{P}_1(\tau, C_V) - \mathrm{P}_1\left(\tau, C_{V_{n-1}}\right)\right) d\tau||$

This yields

$$||\varpi_{1n}(t)|| \leq \frac{1}{\Gamma(\alpha)} \chi_1 ||C_V - C_{V_{n-1}}|| t. \tag{12}$$

Continuing the same procedure recursively, we get

$$||\varpi_{1n}(t)|| \leq \left(\frac{1}{\Gamma(\alpha)} \chi_1 \mathrm{t}\right)^{\mathrm{n+1}} P. \tag{13}$$

At t_α, we have

$||\varpi_{1n}(t)|| \leq \left(\frac{1}{\Gamma(\alpha)} \chi_1 \mathrm{t}\right)^{\mathrm{n+1}} P.$

From equation 13, it results that as n tends to ∞, $||\varpi_{1n}(t)||$ tends to 0. In the same way, it may be demonstrated that $||\varpi_{2n}(t)||, ||\varpi_{3n}(t)||, ||\varpi_{4n}(t)||, ||\varpi_{5n}(t)||, ||\varpi_{6n}(t)||$ tends to 0. Hence the proof.

We will now demonstrate the uniqueness of the solution of the system 2. Suppose that there is a different set of solution of the system 2, namely $\widehat{C}_V$, $\widehat{C}_{SD}$, $\widehat{C}_{LD}$, $\widehat{C}_R$, $\widehat{C}_1$, $\widehat{C}_2$. Then, from the first equation of 4 we write

$C_V(t) - \widehat{C}_V(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (P_1(t, C_V) - P_1(t, \widehat{C}_V)) d\tau$

Using the norm, the equation above becomes:

$$||C_V(t) - \hat{C}_V(t)|| = \frac{(1-\alpha)}{\Gamma(\alpha)} \int_0^t \left\| \left(\mathrm{P}_1(t, C_V) - \mathrm{P}_1\left(t, \hat{C}_V\right)\right) d\tau \right\| \tag{14}$$

By applying the Lipschitz condition, we get

$||C_V(t) - \hat{C}_V(t)|| \leq \frac{1}{\Gamma(\alpha)} \chi t ||C_V - \hat{C}_V||$

This results in,

$||C_V(t) - \hat{C}_V(t)|| \left(1 - \frac{1}{\Gamma(\alpha)} \chi t\right) \leq 0$

Since $(1 - \frac{1}{\Gamma(\alpha)} \chi_1 t) > 0$, we must have $||C_V(t) - \hat{C}_V(t)|| = 0$.. This implies $C_V(t) = \widehat{C}_V(t)$.

5 Boundedness

Here, we establish the boundedness of the solutions of the system 2.

Theorem 5. 1 The solutions of the pluviculture model 2 are uniformly bounded.

Proof. Let, $C(t) = C_V(t) + C_{SD}(t) + C_{LD}(t) + C_R(t) + C_1(t) + C_2(t)$

Taking the fractional Caputo derivative, we obtain

$${}_{t_0}^{C}D_t^{\alpha} C(t) + \mu_1 C(t) = {}_{t_0}^{C}D_t^{\alpha}[C_V(t) + C_{SD}(t) + C_{LD}(t) + C_R(t) + C_1(t) + C_2(t)]$$
$$+\mu_1[C_V(t) + C_{SD}(t) + C_{LD}(t) + C_R(t) + C_1(t) + C_2(t)]$$
$$= J_V - \beta_0 C_V - \beta_1 C_V C_1 + \delta_S C_V - \delta_{S_0} C_{SD} + \psi \beta_1 C_V C_1 - \delta_{S_1} C_{SD} C_2 \quad (15)$$
$$+\delta_L C_{SD} - \delta_{L_0} C_{LD} + \delta_{S_1} C_{SD} C_2 - \delta_{L_1} C_{LD} C_2 + \gamma C_{LD} - \gamma_0 C_R + \xi \delta_{L_1} C_{LD} C_2$$
$$+J_1 - \mu_1 C_1 - \beta_1 C_V C_1 + J_2 - \mu_2 C_2 - \delta_{S_1} C_{SD} C_2 - \delta_{L_1} C_{LD} C_2 - \delta_R C_R C_2$$
$$+\mu_1[C_V(t) + C_{SD}(t) + C_{LD}(t) + C_R(t) + C_1(t) + C_2(t)]$$
$$\leq J_V + \delta_S C_V + \psi \beta_1 C_V C_1 + \delta_L C_{SD} + \gamma C_{LD} + \xi \delta_{L_1} C_{LD} C_2$$
$$+J_1 + J_2 + \mu_1[C_V(t) + C_{SD}(t) + C_{LD}(t) + C_R(t) + C_2(t)]$$

The solution exists and is unique in

$$\eta = \{(C_V,\, C_{SD},\, C_{LD},\, C_R,\, C_1,\, C_2) : max\{(C_V|\,,(C_{SD}|\,,(C_{LD}|\,,\,(C_R|\,,\,(C_1|\,,\,(C_2|\} \leq P\}$$

The above inequality yields,
${}_{t_0}^{C}D_t^{\alpha} C(t) + \mu_1 C(t) \leq J_V + J_1 + J_2 + P(\delta_S + \psi\beta_1 P + \delta_L + \gamma + \xi\,\delta_{L_1} + 5\mu_1)$
By the lemma 2, we get

$${}_{t_0}^{C}D_t^{\alpha} C\,(t) \leq \left(C\,(t_0) - \frac{1}{\mu_1}\,(J_V + J_1 + J_2 + P\,(\delta_S + \psi\beta_1 P + \delta_L + \gamma + \xi\,\delta_{L_1} + 5\mu_1))\right) E_\alpha\left(-\theta(t-t_0)^\alpha\right)$$

$+\frac{1}{\mu_1}(J_V + J_1 + J_2 + P(\delta_S + \psi\beta_1 P + \delta_L + \gamma + \xi\,\delta_{L_1} + 5\mu_1)) \to J_V + J_1 + J_2 + P(\delta_S + \psi\beta_1 P + \delta_L + \gamma + \xi\,\delta_{L_1} + 5\mu_1),\, t \to \infty$
Therefore, all the solution of the system 2 that initiates in η remained bounded in

$$\Xi = ((C_V,\, C_{SD},\, C_{LD},\, C_R,\, C_1,\, C_2) \in \eta\,|\,C\,(t) \leq J_V + J_1 + J_2 + P\,(\delta_S + \psi\beta_1 P + \delta_L + \gamma + \xi\,\delta_{L_1} + 5\,\mu_1) + \varepsilon,\, \varepsilon > 0\}$$

6 Existence of the Points of Equilibrium and the Stability

The system 2 has the following set of interesting points of equilibrium. The criteria for the stability of the set of equilibrium points have been discussed here. The Jacobian matrix of the pluviculture model 2 is

$$J\,(C_V, C_{SD}, C_{LD}, C_R, C_1, C_2) = \begin{pmatrix} J_{11} & J_{12} & J_{13} & J_{14} & J_{15} & J_{16} \\ J_{21} & J_{22} & J_{23} & J_{24} & J_{25} & J_{26} \\ J_{31} & J_{32} & J_{33} & J_{34} & J_{35} & J_{36} \\ J_{41} & J_{42} & J_{43} & J_{44} & J_{45} & J_{46} \\ J_{51} & J_{52} & J_{53} & J_{54} & J_{55} & J_{56} \\ J_{61} & J_{62} & J_{63} & J_{64} & J_{65} & J_{66} \end{pmatrix}$$

where

$$J_{11} = -\beta_0 - C_1\beta_1, \quad J_{12} = 0, \quad J_{13} = 0, \quad J_{14} = 0, \quad J_{15} = C_V\beta_1, \quad J_{16} = 0,$$

$$J_{21} = C_1\psi\beta_1 + \delta_S, \quad J_{22} = -\delta_{S_0} - C_2\delta_{S_1}, \quad J_{23} = 0, \quad J_{24} = 0, \quad J_{25} = C_V\psi\beta_1, \quad J_{26} = -C_{SD}\delta_{S_1},$$

$$J_{31} = 0, \quad J_{32} = \delta_L + C_2\delta_{S_1}, \quad J_{33} = -\delta_{L_0} - C_2\delta_{L_1}, \quad J_{34} = 0, \quad J_{35} = 0, \quad J_{36} = -C_{LD}\delta_{L_1} + C_{SD}\delta_{S_1},$$

$$J_{41} = 0, \quad J_{42} = 0, \quad J_{43} = \gamma + C_2\xi\,\delta_{L_1}, \quad J_{44} = -\gamma_0, \quad J_{45} = 0, \quad J_{46} = C_{LD}\xi\,\delta_{L_1},$$

$$J_{51} = -C_1\beta_1, \quad J_{52} = 0, \quad J_{53} = 0, \quad J_{54} = 0, \quad J_{55} = -C_V\beta_1 - \mu_1, \quad J_{56} = 0,$$

$$J_{61} = 0, \quad J_{62} = -C_2\delta_{S_1}, \quad J_{63} = -C_2\delta_{L_1}, \quad J_{64} = -C_2\delta_R, \quad J_{65} = 0,$$

$J_{66} = -C_{LD}\delta_{L_1} - C_R\delta_R - C_{SD}\delta_{S_1} - \mu_2.$
Equilibrium points of the pluviculture model 2 is discussed below:

1. Axial equilibrium point is $\mathfrak{E} = \left(\frac{J_V}{\beta_0}, 0, 0, 0, 0, 0\right)$.

Theorem 6.1 Axial equilibrium point E always exists and is always stable.
Proof. At E, Jacobian matrix J of system 2 has the eigenvalues:

$$\lambda_{11}=-\beta_0,\quad \lambda_{12}=-\delta_{L_0},\quad \lambda_{13}=-\delta_{S_0},\quad \lambda_{14}=-\gamma_0,\quad \lambda_{15}=-(\frac{J_V\beta_1+\beta_0\mu_1}{\beta_0}),\quad \lambda_{16}=-\mu_2.$$

Since all the above eigen values are negative, the axial equilibrium point E is always stable.

2. Aerosol free equilibrium point is \widetilde $\mathfrak{E}=\left(\tilde{C}_V,\tilde{C}_{SD},\tilde{C}_{LD},\tilde{C}_R,0,0\right)$

Theorem 6.2 Aerosol free equilibrium point $\widetilde{E}$ always exists and is always stable.
Proof. At $\widetilde{E}$, Jacobian matrix J of system 2 has the eigenvalues:

$$\lambda_{11}=-\beta_0,\quad \lambda_{12}=-\delta_{L_0},\quad \lambda_{13}=-\delta_{S_0},\quad \lambda_{14}=-\gamma_0,\quad \lambda_{15}=-\left(\frac{J_V\beta_1+\beta_0\mu_1}{\beta_0}\right),$$

$$\lambda_{16}=-(\frac{J_V\delta_S(\gamma\delta_L\delta_R+\gamma_0(\delta_L\delta_{L_1}+\delta_{L_0}\delta_{S_1}))+\beta_0\delta_{L_0}\delta_{S_0}\gamma_0\mu_2}{\beta_0\delta_{L_0}\delta_{S_0}\gamma_0}).$$

Since all the above eigen values are negative, the aerosol free equilibrium point $\widetilde{E}$ is always stable.

7 Numerical Method

In the present section, we have presented the generalized Adams-Bashforth-Moulton technique[32] to solve the pluviculture model 2.

Consider,
${}^{C}D_t^{\alpha}x(t)=\phi(t,x(t)),\quad 0\le t\le T x^{(m)}(0)=x_0^{(m)},\quad m=0,1,2,3,....,\nu..,\nu=\lceil\alpha\rceil.$
The corresponding Volterra integral equation may be written as

$$x(t)=\sum_{m=0}^{\nu-1}x_0^{(m)}\frac{t^m}{m!}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\phi\left(s,x(s)\right)ds. \tag{16}$$

In order to integrate 16, Adams-Bashforth Moultan method has been used by Diethelm et al.[33,34]. Set $h=\frac{T}{N}$, $t_n=nh$, $n=0,1,2,....,N\in Z$. Now, the system 2 can be written as:

$$\begin{aligned}
&C_{V\,n+1}=C_{V\,0}+\tfrac{h^\alpha}{\Gamma(\alpha+2)}[J_V-\beta_0C_{V\,n+1}^{\,p}-\beta_1C_{V\,n+1}^{\,p}C_{1\,n+1}^{\,p}]\\
&+\tfrac{h^\alpha}{\Gamma(\alpha+2)}\textstyle\sum_{i=0}^n a_{i,n+1}[J_V-\beta_0C_{V\,i}-\beta_1C_{V\,i}C_{1\,i}],\\
&C_{SD\,n+1}=C_{SD\,0}+\tfrac{h^\alpha}{\Gamma(\alpha+2)}[\delta_SC_{V\,n+1}^{\,p}-\delta_{S_0}C_{SD\,n+1}^{\,p}+\psi\beta_1C_{V\,n+1}^{\,p}C_{1\,n+1}^{\,p}-\delta_{S_1}C_{SD\,n+1}^{\,p}C_{2\,n+1}^{\,p}]\\
&+\tfrac{h^\alpha}{\Gamma(\alpha+2)}\textstyle\sum_{i=0}^n a_{i,n+1}[\delta_SC_{V\,i}-\delta_{S_0}C_{SD\,i}+\psi\beta_1C_{V\,i}C_{1\,i}-\delta_{S_1}C_{SD\,i}C_{2\,i}],\\
&C_{LD\,n+1}=C_{LD\,0}+\tfrac{h^\alpha}{\Gamma(\alpha+2)}[\delta_LC_{SD\,n+1}^{\,p}-\delta_{L_0}C_{LD\,n+1}^{\,p}+\delta_{S_1}C_{SD\,n+1}^{\,p}C_{2\,n+1}^{\,p}-\delta_{L_1}C_{LD\,n+1}^{\,p}C_{2\,n+1}^{\,p}]\\
&+\tfrac{h^\alpha}{\Gamma(\alpha+2)}\textstyle\sum_{i=0}^n a_{i,n+1}[\delta_LC_{SD\,i}-\delta_{L_0}C_{LD\,i}+\delta_{S_1}C_{SD\,i}C_{2\,i}-\delta_{L_1}C_{LD\,i}C_{2\,i}],\\
&C_{R\,n+1}=C_{R\,0}+\tfrac{h^\alpha}{\Gamma(\alpha+2)}[\gamma C_{LD\,n+1}^{\,p}-\gamma_0C_{R\,n+1}^{\,p}+\xi\delta_{L_1}C_{LD\,n+1}^{\,p}C_{2\,n+1}^{\,p}]\\
&+\tfrac{h^\alpha}{\Gamma(\alpha+2)}\textstyle\sum_{i=0}^n a_{i,n+1}[\gamma C_{LD\,i}-\gamma_0C_{R\,i}+\xi\delta_{L_1}C_{LD\,i}C_{2\,i}],\\
&C_{1\,n+1}=C_{1\,0}+\tfrac{h^\alpha}{\Gamma(\alpha+2)}[J_1-\mu_1C_{1\,n+1}^{\,p}-\beta_1C_{V\,n+1}^{\,p}C_{1\,n+1}^{\,p}]\\
&+\tfrac{h^\alpha}{\Gamma(\alpha+2)}\textstyle\sum_{i=0}^n a_{i,n+1}[J_1-\mu_1C_{1\,i}-\beta_1C_{V\,i}C_{1\,i}],\\
&C_{2\,n+1}=C_{2\,0}+\tfrac{h^\alpha}{\Gamma(\alpha+2)}[J_2-\mu_2C_{2\,n+1}^{\,p}-\delta_{S_1}C_{SD\,n+1}^{\,p}C_{2\,n+1}^{\,p}-\delta_{L_1}C_{LD\,n+1}^{\,p}C_{2\,n+1}^{\,p}-\delta_RC_{R\,n+1}^{\,p}C_{2\,n+1}^{\,p}]
\end{aligned}$$

$$+\frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{i=0}^{n}a_{i,n+1}[J_2-\mu_2C_{2\,i}-\delta_{S_1}C_{SD\,i}C_{2\,i}-\delta_{L_1}C_{LD\,i}C_{2\,i}-\delta_RC_{R\,i}C_{2\,i}], \tag{17}$$

where

$$
\begin{aligned}
C^{p}_{V^{p}_{n+1}} &= C_{V_0} + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{i=0}^{n} b_{i,n+1} \left[J_V - \beta_0 C_{V_i} - \beta_1 C_{V_i} C_{1_i}\right] \\
C_{SD^{p}_{n+1}} &= C_{SD_0} + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{i=0}^{n} b_{i,n+1} \left[\delta_S C_{V_i} - \delta_{S_0} C_{SD_i} + \psi \beta_1 C_{V_i} C_{1_i} - \delta_{S_1} C_{SD_i} C_{2_i}\right] \\
C_{LD^{p}_{n+1}} &= C_{LD_0} + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{i=0}^{n} b_{i,n+1} \left[\delta_L C_{SD_i} - \delta_{L_0} C_{LD_i} + \delta_{S_1} C_{SD_i} C_{2_i} - \delta_{L_1} C_{LD_i} C_{2_i}\right] \\
C_{R^{p}_{n+1}} &= C_{R_0} + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{i=0}^{n} b_{i,n+1} \left[\gamma C_{LD_i} - \gamma_0 C_{R_i} + \xi \delta_{L_1} C_{LD_i} C_{2i}\right] \\
C^{p}_{1} n_{n+1} &= C_{1_0} + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{i=0}^{n} b_{i,n+1} \left[J_1 - \mu_1 C_{1_i} - \beta_1 C_{V_i} C_{1_i}\right] \\
C^{p}_{2^{p}_{n+1}} &= C_{2_0} + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{i=0}^{n} b_{i,n+1} \left[J_2 - \mu_2 C_{2_i} - \delta_{S_1} C_{SD_i} C_{2_i} - \delta_{L_1} C_{LD_i} C_{2_i} - \delta_R C_{R_i} C_{2_i}\right],
\end{aligned}
\tag{18}
$$

in which

$$
a_{i,n+1} = \begin{cases} n^{\alpha_j+1} - (n-\alpha_j)(n+1)^{\alpha_j}, & i = 0, \\ (n-i+2)^{\alpha_j+1} + (n-i)^{\alpha_j+1} - 2(n-i+1)^{\alpha_j+1}, & 1 \le i \le n \\ 1, & i = n+1 \end{cases}
$$

and

$$
b_{i,n+1} = \frac{h^{\alpha_j}}{\alpha_j} \left((n-i+1)^{\alpha_j} - (n-i)^{\alpha_j}\right), \; 0 \le i \le n,
$$

with $j = 1,2,3,4,5,6$.

8 Numerical Simulation

In order to discuss the dynamics of the projected fractional pluviculture model 2, we have used the generalized Adams-Bashforth-Moulton technique by considering the parametric values as $J_V \in (0,3),\ \beta_0 = 1,\ \beta_1 = 0.5,\ \delta_S = 0.1,\ \delta_{S_0} = 1,\ \psi = 1,\ \delta_{S_1} = 0.,\ \delta_L = 0.1,\ \delta_{L_0} = 1,\ \delta_{L_1} = 0.3,\ \gamma = 0.1,\ \gamma_0 = 0.02,\ \xi = 1,\ J_1 \in (0,3),\ J_2 \in (0,3),\ \mu_1 = 1,\ \mu_2 = 1$ and $\delta_R = 0.01$ with the initial conditions $C_V = 0.7545,\ C_{SD} = 0.2422,\ C_{LD} = 0.0862,\ C_R = 0.4919,\ C_1 = 0.6507$ and $C_2 = 0.8534$. For $J_V = 0,\ 1,\ 2,\ 3$, we can observe the variation in $C_V,\ C_{SD},\ C_{LD}$ and C_R in Figures 1, 2 and 3 for $\alpha = 1,\ 0.8,\ 0.5$ respectively. As the natural formation of water vapor (J_V) increases, $C_V,\ C_{SD},\ C_{LD}$ and C_R also increase. As the fractional value are incorporated, the increasing trend is retained whereas the net value slightly decrease.

For $J_1 = 0,\ 1,\ 2,\ 3$, we can observe the variation in $C_V,\ C_{SD},\ C_{LD}$ and C_R in Figures 4, 5 and 6 for $\alpha = 1,\ 0.8,\ 0.5$ respectively. As the rate of entry of first type of aerosols (J_1) into the environment increases, the value of C_V starts decreasing since natural formation of water vapors will be less than that of due to first kind aerosol. Whereas, the values of $C_{SD},\ C_{LD}$ and C_R shows the increasing trend and hence the rainfall is stimulated by this action.

For $J_2 = 0,\ 1,\ 2,\ 3$, we can observe the variation in $C_{SD},\ C_{LD}$ and C_R in Figures 7, 8 and 9 for $\alpha = 1,\ 0.8,\ 0.5$ respectively. As the rate of entry of second type of aerosols (J_2) into the environment increases, the value of C_{SD} starts decreasing since the conversion of small drops into large drops and then to rain drops is more than that of the small drops formation. Hence the values of C_{LD} and C_R is increased and induces more rainfall in the environment than the previous.

Figures 10, 11 and 12 depicts the global stability of C_1 - C_V, C_2 - C_{SD} and C_2 - C_{LD} space for $\alpha = 1,\ 0.8$ respectively. Solution paths of system 2 starting in the domain of attraction are attaining equilibrium values herein.

9 Results and Discussion

By considering the previous results and values as that of in[15] along with other predicted values for the new parameters used in our present model, the dynamics of the model is studied by using generalized Adams-Bashforth-Moulton technique. In the above reference, only one kind of aerosol is used whereas, we have proposed our model by the aid of two kinds of aerosols

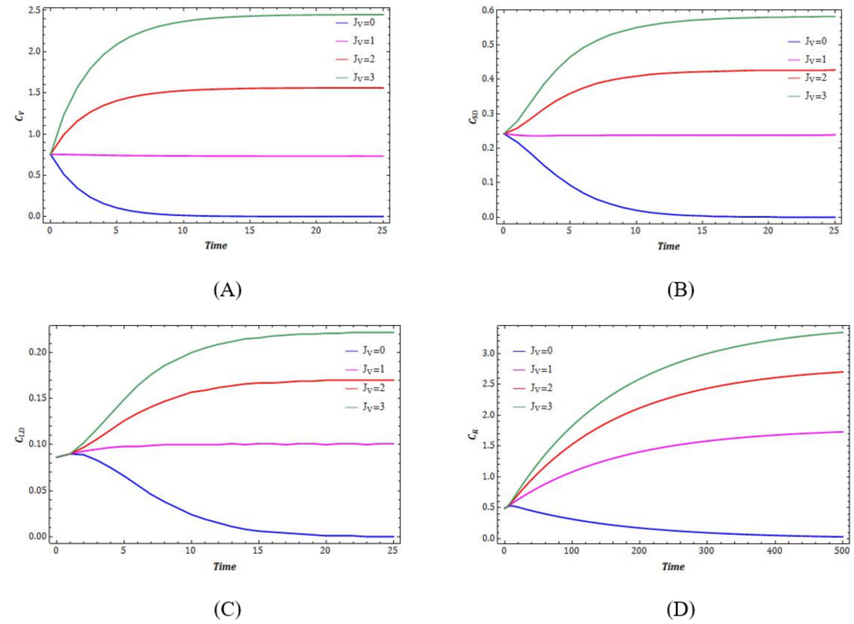

Fig 1. Variation in the (A)C_V, (B) C_{SD}, (C) C_{LD} and (D) C_R for different values of J_V for $\alpha = 1$

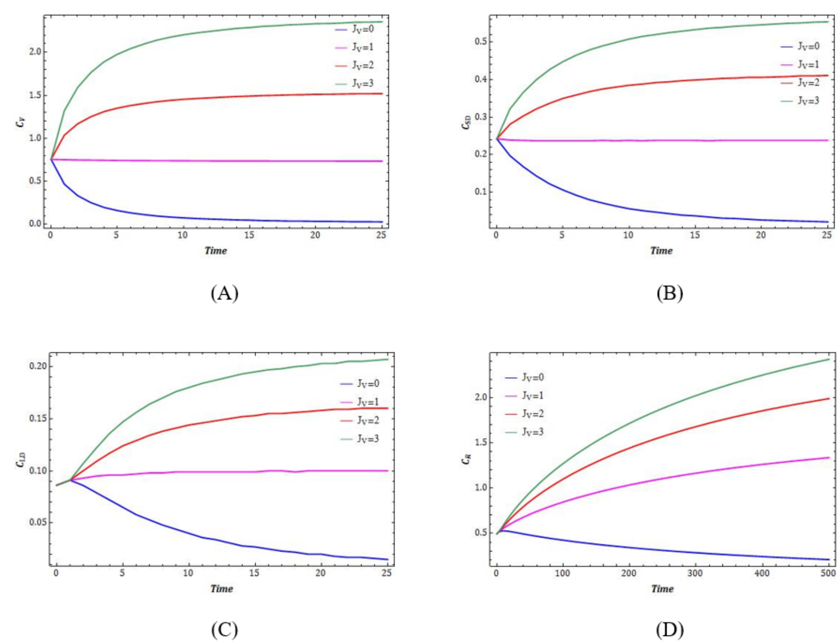

Fig 2. Variation in the (A)C_V, (B) C_{SD}, (C) C_{LD} and (D) C_R for different values of J_V for $\alpha = 0.8$

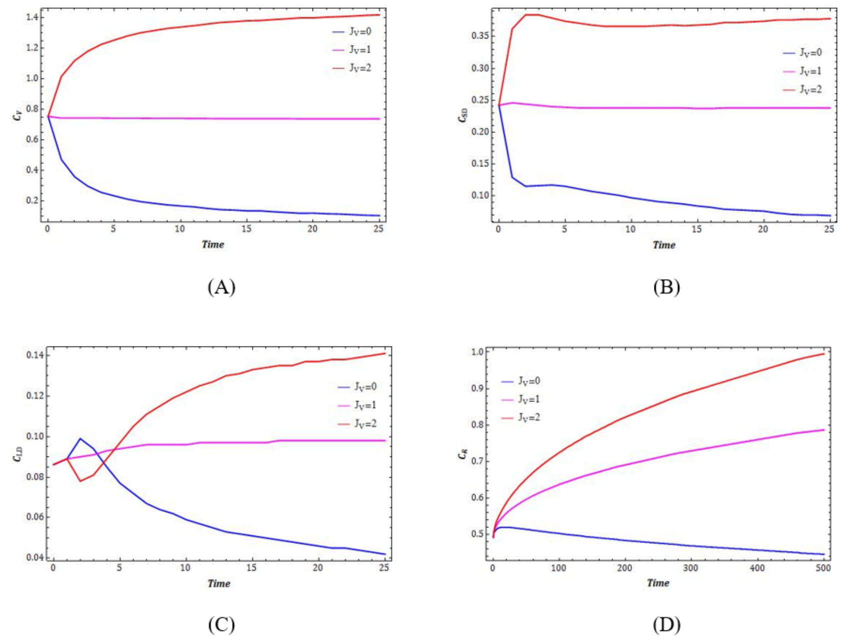

Fig 3. Variation in the (A)C_V, (B) C_{SD}, (C) C_{LD} and (D) C_R for different values of J_V for $\alpha = 0.5$

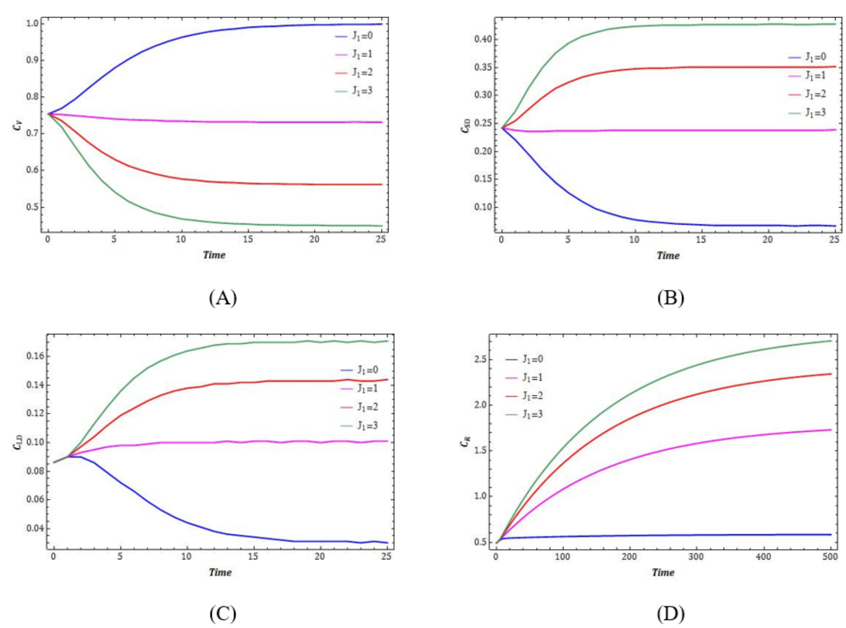

Fig 4. Variation in the (A)C_V, (B) C_{SD}, (C) C_{LD} and (D) C_R for different values of J_1 for $\alpha = 1$

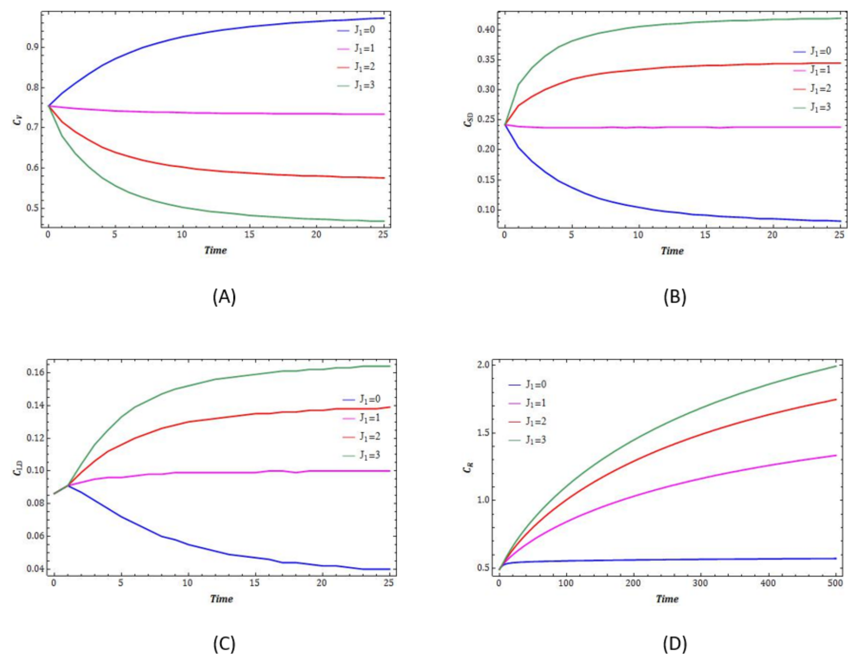

Fig 5. Variation in the (A)C_V, (B) C_{SD}, (C) C_{LD} and (D) C_R for different values of J_1 for $\alpha = 0.8$

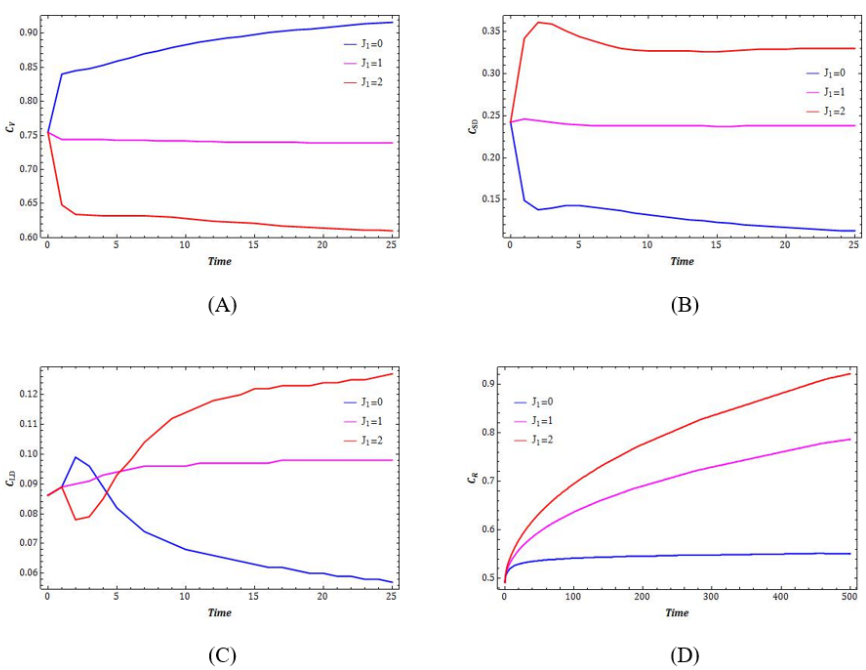

Fig 6. Variation in the (A)C_V, (B) C_{SD}, (C) C_{LD} and (D) C_R for different values of J_1 for $\alpha = 0.5$.

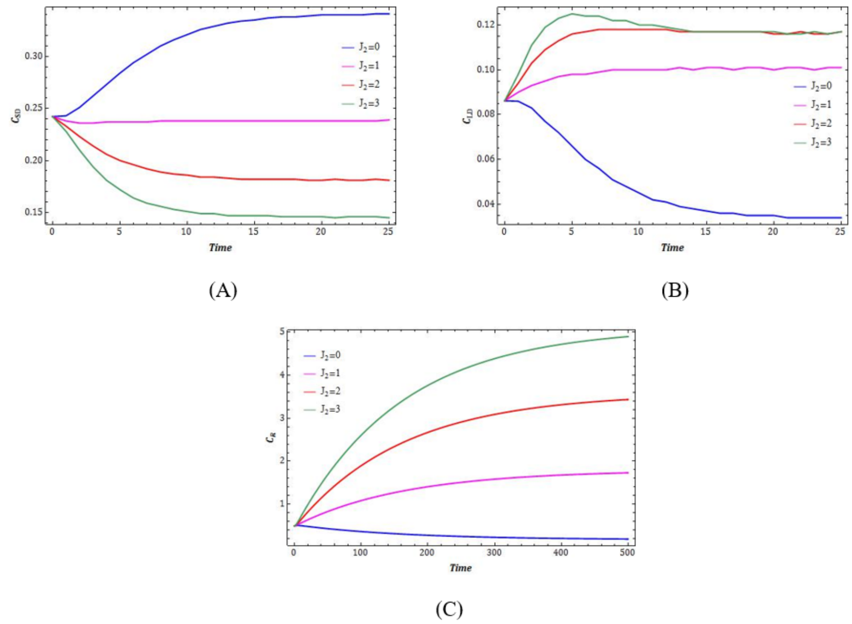

Fig 7. Variation in the (A)C_{SD}, (B) C_{LD} and (C) C_R for different values of J_2 for $\alpha = 1$.

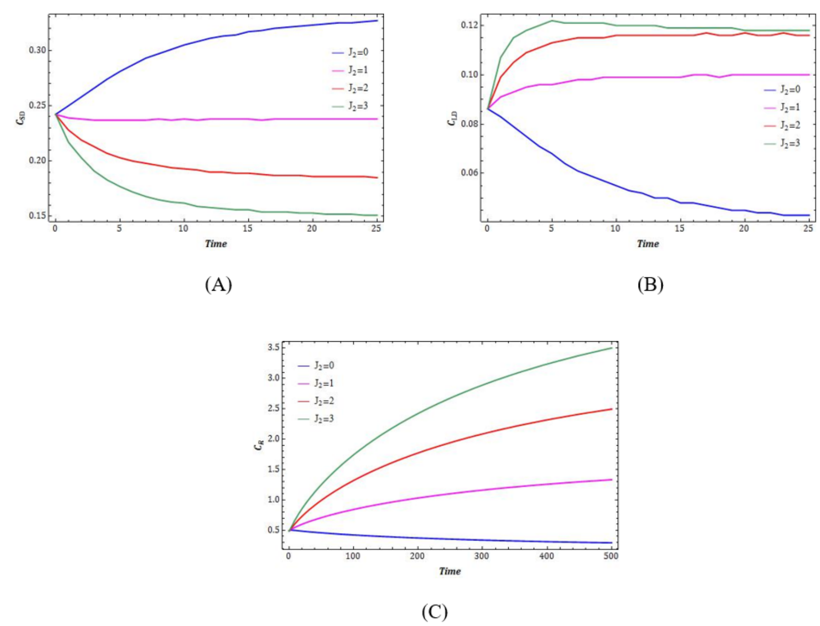

Fig 8. Variation in the (A)C_{SD}, (B) C_{LD} and (C) C_R for different values of J_2 for $\alpha = 0.8$

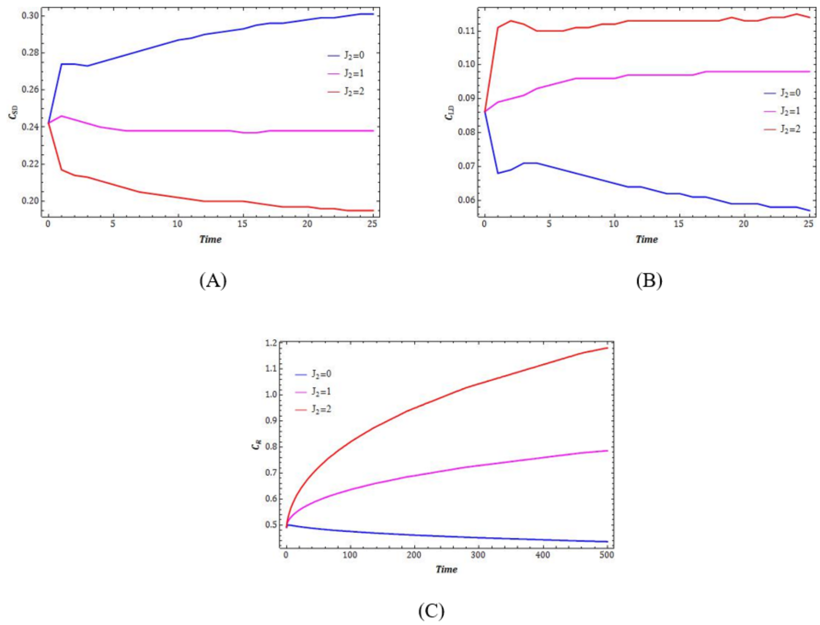

Fig 9. Variation in the (A)C_{SD}, (B) C_{LD} and (C) C_R for different values of J_2 for $\alpha = 0.5$

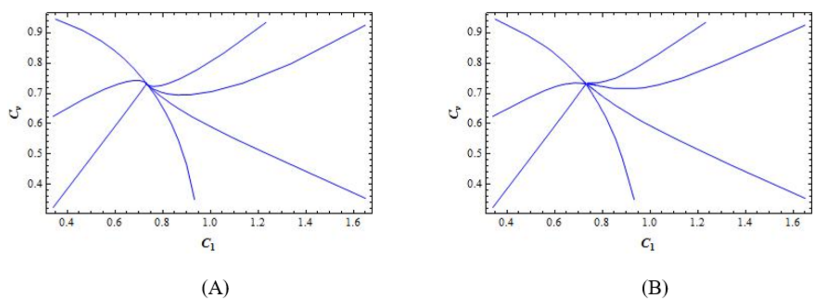

Fig 10. Global stability inC_1 - C_V space for (A) $\alpha = 1$ and (B) $\alpha = 0.8$.

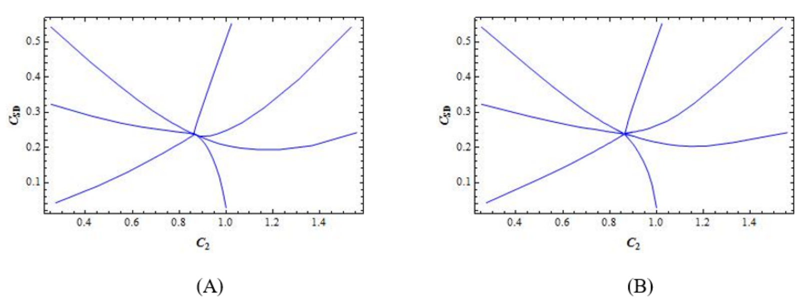

Fig 11. Global stability inC_2 - C_{SD} space for (A) $\alpha = 1$ and (B) $\alpha = 0.8$.

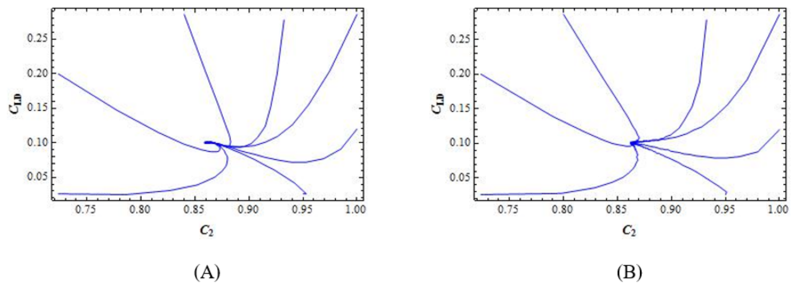

Fig 12. Global stability inC_2 - C_{LD} space for (A) $\alpha = 1$ and (B) $\alpha = 0.8$.

along with the Caputo fractional derivative. We see that, the results are in good agreement with the previous ones and also the results are more realistic when the fractional values are incorporated which is discussed in the Section (8). We have also analyzed the global stability numerically for different fractional values which is not done in the previous works so far. Overall, the pluviculture model incorporating the fractional derivatives is the novel, and we see the effect of different parameters with different fractional values which gives the better and realistic representation of our proposed model.

10 Conclusion

A non-linear mathematical approach for pluviculture by introducing particles into the atmosphere is suggested and analyzed in this study. The effect of fractional order derivative is numerically analyzed and it is established that numerical simulation strengthens the analytical results of the model. According to the model, rainfall occurs only when the water vapor are concentrated on the nature, not on a continual basis. In order for water vapor molecules to develop in the atmosphere, they must be continually generated. According to the study, rainfall is enhanced when the accumulative concentration of favorable aerosol particles rises. Pluviculture model with the fractional derivatives along with the two kinds of aerosols is the novel of the model which provides the more realistic representation of the model. The results obtained are more convincing and holds good with the real world phenomena that can be predicted.

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