

RESEARCH ARTICLE



A Result on probabilistic 2-metric space using strong semi compatible mappings

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 OPEN ACCESS

Received: 19.08.2021

Accepted: 27.11.2021

Published: 27.12.2021

Citation: Srinivas V, Satyanna K (2021) A Result on probabilistic 2-metric space using strong semi compatible mappings. Indian Journal of Science and Technology 14(47): 3467-3474. <https://doi.org/10.17485/IJST/V14i47.1513>

* **Corresponding author.**

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Funding: None

Competing Interests: None

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Published By Indian Society for Education and Environment ([iSee](#))

ISSN

Print: 0974-6846

Electronic: 0974-5645

Abstract

Objective/Aim: To generate a fixed point theorem in probabilistic 2-metric space. **Method:** By employing strong semi compatible mappings and sub sequentially continuous mappings. **Findings:** Generated unique common fixed point theorem and substantiated with appropriate example. **Novelty/Improvement:** The concepts of strong semi compatible mappings and sub sequentially continuous mappings are weaker than existing conditions like weakly compatible mappings which generalizes the theorem of V. K. Gupta, Arihant Jain and Rajesh kumar.

Keywords: Strong semi compatible; sub sequentially continuous; Probabilistic 2metric space; conditional semi compatible; conditional compatible

1 Introduction

The analysis is main branch of mathematics, one of its main ingredient is to give the solution of problems in all fields which is based on fixed points theory. This turned way to extraction of fixed point theorems with minimal effort. Menger⁽¹⁾ made ground stone to generate probabilistic concept for the distance, resulting several theorems. After words many generalizations of commuting mappings came into the fixed point theory⁽²⁾. The major achievement was the introduction of compatibility concept in Menger space by Mishra⁽³⁾, resulting the flood of many fixed point theorems established. Some more results were obtained on this area by Martinez et.al⁽⁴⁾. In this connection searching of fixed points theorems led to the arrival of the concepts of continuity, reciprocal continuity, sub sequential continuity, semi compatibility, conditional semi compatibility and strong semi compatibility in Menger space like in⁽⁵⁾. These concepts have been extended to generate some more results like in⁽⁶⁾. Chauhan⁽⁷⁾ introduced the notion of occasionally weakly compatible maps in megner space. In this context many researchers are focusing on the existence of fixed point theorems and their applications in different ways⁽⁸⁾. Mukesh Kumar Jain and Mohammad Saeed Khan⁽⁹⁾ coined the new concept strong semi compatible mappings and obtained some results in metric space. Ravindra K. Bisht and Naseer Shahazad⁽¹⁰⁾ used the concept of faintly compatible and extracted some more fixed point theorems in metric space. V.K. Gupta, Arihant jain and Rajesh kumar⁽¹¹⁾ presented some results on probabilistic 2- metric space by applying weakly compatible mappings. Recently some results in probabilistic 2- metric were obtained by using faintly compatible and reciprocally

continuous, sub sequentially continuous and semi compatible mappings in Menger space⁽¹²⁻¹⁴⁾. Further some more theorems witnessed like⁽¹⁵⁾ on C-class functions on an intuitionistic menger space and also functional spaces by A. M. Zidan et.al.⁽¹⁶⁾.

2 Materials and Methods

2.1 Definition⁽¹¹⁾

$F : R \rightarrow R^+$ is a distribution function if

1. Non- decreasing
2. Continuous from left
3. $\text{Inf} \{F(t) : t \in R\} = 0$
4. $\text{Sup} \{F(t) : t \in R\} = 1$.

Representing the collection of all distributive functions as D_f .

2.1.1 Example

Special distributive function H is defined by

$$H(x) = \begin{cases} 1, x > 1 \\ 0, x \leq 1. \end{cases}$$

2.2 Definition⁽¹¹⁾

An ordered pair (X, F) is known as probabilistic 2-metric space (2-PM space) where $X \neq \emptyset$ is an arbitrary set and F is mapping with $F : X \times X \times X \rightarrow D_f$ here D_f is the set of all distribution functions, where the function F takes the value at $(a, b, c) \in X \times X \times X$ is denoted by $F_{a,b,c}$ satisfying

$$\begin{aligned} F_{a,b,c}(0) &= 0 \\ \forall b, c \in X, b \neq c, \exists a \in X \text{ with } F_{a,b,c}(t_\phi) &< 1 \text{ such that } t_\phi > 0 \\ F_{a,b,c}(t_\phi) &= 1 \forall t_\phi > 0 \text{ when } a = b \text{ or } b = c \text{ or } c = a \\ F_{a,b,c}(t_\phi) &= F_{b,c,a}(t_\phi) = F_{c,a,b}(t_\phi) \\ F_{a,b,c}(t_u) = F_{b,c,a}(t_v) = F_{c,a,b}(t_w) &= 1 \text{ then } F_{a,b,c}(t_u + t_v + t_w) = 1 \\ \forall a, b, c \in X \text{ and } t_u, t_v, t_w &\geq 0. \end{aligned}$$

2.3 Definition⁽¹¹⁾

The mapping $t: [0, 1]^3 \rightarrow [0, 1]$ is called t-norm if

$$\begin{aligned} t(a, 1, 1) &= a, t(0, 0, 0) = 0 \\ t(a, b, c) &= t(b, c, a) = t(c, a, b) \\ t(a_1, b_1, c_1) &\geq t(a_2, b_2, c_2) \text{ for } a_1 \geq a_2, b_1 \geq b_2, c_1 \geq c_2 \\ t(t(a, b, w), p, q) &= t(a, t(b, w, p), q) = t(a, b, t(w, p, q)) \\ \forall a, b, c, p, q, w, a_i, b_i, c_i &\in X \text{ for } i = 1, 2. \end{aligned}$$

2.3.1 Example

Let $t(x, y, z) = \min\{x, y, z\}$ defines a t-norm.

2.4 Definition⁽¹¹⁾

A 2-Menger space is (X, F, t) formed by (X, F) and t where (X, F) is a 2-PM space and 't' is a t-norm having:

$$F_{a,b,c}(t_u + t_v + t_w) \geq t(F_{a,b,d}(t_u), F_{a,d,c}(t_v), F_{d,b,c}(t_w)) \forall a, b, c, d \in X \text{ and } t_u, t_v, t_w \geq 0.$$

2.4.1 Example

Consider (X, d) be any metric space and t_1 is a t-norm for each $t_1 \in [0, 1]$, define

$$F_{\alpha,\beta,a}(t_1) = \begin{cases} \frac{t_1}{t_1+d(\alpha,\beta)}, t_1 > 0 \\ 0, t_1 = 1 \end{cases}$$

$\forall \alpha, \beta$, in X and fixed $a, t_1 > 0$.

Then (X, F, t) forms a 2-PM space.

2.5 Definition (12)

A sequence in $\langle a_n \rangle$ in (X, F, t) **converges** to β if $\lim_{n \rightarrow \infty} F_{\alpha_n, \beta, \alpha}(t) = 1$ for all $t > 0$.

2.6 Definition (12)

A sequence in $\langle a_n \rangle$ in (X, F, t) is **cauchy** if $\lim_{n \rightarrow \infty} F_{a_n, a_m, a}(t) = 1$ for all $t > 0$.

2.7 Definition (9)

A 2-Menger space is (X, F, t) is **complete** if every cauchy sequence converges in X .

2.8 Definition (4,11)

Self-mappings P, S in (X, F, t) is termed as **compatible** if $\lim_{n \rightarrow \infty} F_{PSx_n, SPx_n, a}(\beta) = 1 \forall a \in X$ and $\beta > 0$.

2.9 Definition (11)

Self-mappings P, S in (X, F, t) are **weakly compatible** if they are commuting at every coincidence point.

2.10 Definition (9)

“ The self -mappings P and S are **conditionally semi compatible** if whenever the sequences $\langle x_n \rangle$ satisfying $\{ \langle x_n \rangle : \lim_{n \rightarrow \infty} Px_n = \lim_{n \rightarrow \infty} Sx_n \} \neq \emptyset$, then there exists another sequence $\langle t_n \rangle \in X$ with $\lim_{n \rightarrow \infty} Pt_n = \lim_{n \rightarrow \infty} St_n = u$ for some $u \in X$ such that $\lim_{n \rightarrow \infty} F_{PS_t_n, su, a}(\beta) = 1$ and $\lim_{n \rightarrow \infty} F_{SP_t_n, pu, a}(\beta) = 1 \quad \forall a \in X \forall a \in X$ and $\beta > 0$.”

2.11 Definition (9)

“ The self -mappings P and S are **strongly semi compatible** if P and S are conditionally semi compatible and P and S commute on a nonempty subset of the set of coincidence points, whenever the set of coincidence points is non-empty. ”

2.11.1 Example

Consider $X = \mathbb{R}$, d is usual metric in X and define $F_{\alpha,\beta,a}(t_1) = \begin{cases} \frac{t_1}{t_1+d(\alpha,\beta)}, t_1 > 0 \\ 0, t_1 = 1 \end{cases}$

$\forall \alpha, \beta$ in X and fixed $a, t_1 > 0$.

Define mappings $C, D : X \rightarrow X$ as $C(x) = e^x \quad \forall x \in \mathbb{R}$ and $D(x) = e^{x^2} \forall x \in \mathbb{R}$

Choose a sequence $\langle t_m \rangle = 1 - \frac{11}{m} \quad \forall m \geq 1$. Then

$$C(t_m) = C\left(1 - \frac{11}{m}\right) = e^{1 - \frac{11}{m}} \rightarrow e \text{ and}$$

$$D(t_m) = D\left(1 - \frac{11}{m}\right) = e^{\left(1 - \frac{11}{m}\right)^2} \rightarrow e \text{ as } m \rightarrow \infty.$$

Implies $\{ \langle t_m \rangle : \lim_{m \rightarrow \infty} Ct_m = \lim_{m \rightarrow \infty} Dt_m \} \neq \emptyset$.

Then there exists another sequence $\langle t_m \rangle = \frac{3}{5m} \quad \forall m \geq 1$ Then

$$C(t_m) = C\left(\frac{3}{5m}\right) = e^{\frac{3}{5m}} \rightarrow 1$$

$$D(t_m) = D\left(\frac{3}{5m}\right) = e^{\left(\frac{3}{5m}\right)^2} \rightarrow 1 \text{ as } m \rightarrow \infty$$

Further

$$CD(t_m) = C\left(e^{\left(\frac{3}{5m}\right)^2}\right) = e^{e^{\left(\frac{3}{5m}\right)^2}} \rightarrow e \text{ and}$$

$$DC(t_m) = D\left(e^{\frac{3}{5m}}\right) = e^{\left(e^{\frac{3}{5m}}\right)^2} \rightarrow e \text{ and also}$$

$$CD(t_m) \rightarrow e = D(1), DC(t_m) \rightarrow e = C(1) \text{ as } m \rightarrow \infty.$$

$$\text{Therefore } \lim_{n \rightarrow \infty} F_{CDt_n, D(1), a}(\beta) = 1 \text{ and } \lim_{n \rightarrow \infty} F_{DCt_n, C(1), a}(\beta) = 1$$

$\forall a \in X$ and $\beta > 0$.

As a result the pair (C, D) is conditional semi compatible more over at $t = 0, C(0) = D(0) = 1, CD(0) = C(1) = e = D(1) = DC(0)$

implies that the pair (C, D) is strong semi compatible.

2.11.2 Example

Strong semi compatible pair (C, D) of self-mappings do not have to be weakly compatible. Take (X, R) and d denotes the distance on X and for each $t_1 \in [0, 1]$

$$F_{\alpha, \beta, a}(t_1) = \begin{cases} \frac{t_1}{t_1 + d(\alpha, \beta)}, t_1 > 0 \\ 0, t_1 = 1 \end{cases}$$

$\forall \alpha, \beta$ in X and fixed $a, t_1 > 0$.

define mapping $C, D : X \rightarrow X$ as

$$C(x) = e^{2x} \quad \forall x \in R \text{ and } D(x) = e^{2x^2} \quad \forall x \in R.$$

choose a sequence $(t_m) = 1 - \frac{5}{m} \quad \forall m \geq 1$.

$$\text{Then } C(t_m) = C\left(1 - \frac{5}{m}\right) = e^{2\left(1 - \frac{5}{m}\right)} \rightarrow e^2$$

$$\text{and } D(t_m) = D\left(1 - \frac{5}{m}\right) = e^{2\left(1 - \frac{5}{m}\right)^2} \rightarrow e^2, \text{ as } m \rightarrow \infty.$$

Implies $\{\langle t_m \rangle : \lim_{m \rightarrow \infty} Ct_m = \lim_{m \rightarrow \infty} Dt_m\} \neq \emptyset$.

Then there exists another sequence $\langle t_m \rangle = \frac{2}{m} \quad \forall m \geq 1$ Then

$$C(t_m) = C\left(\frac{2}{m}\right) = e^{2\left(\frac{2}{m}\right)} \rightarrow 1,$$

$$D(t_m) = D\left(\frac{2}{m}\right) = e^{2\left(\frac{2}{m}\right)^2} \rightarrow 1 \text{ as } m \rightarrow \infty.$$

And also

$$CD(t_m) = C\left(e^{\frac{8}{m^2}}\right) = e^{2e^{\frac{8}{m^2}}} \rightarrow e^2 \text{ and}$$

$$DC(t_m) = D\left(e^{\frac{4}{m}}\right) = e^{2\left(e^{\frac{4}{m}}\right)^2} \rightarrow e^2.$$

$$CD(t_m) \rightarrow e^2 = D(1),$$

$$DC(t_m) \rightarrow e^2 = C(1) \text{ as } m \rightarrow \infty.$$

$$\text{This gives } \lim_{n \rightarrow \infty} F_{CDt_n, D(1), \alpha}(\beta) = 1 \text{ and } \lim_{n \rightarrow \infty} F_{DCt_n, C(1), a}(\beta) = 1$$

$\forall a \in X$ and $\beta > 0$.

As a result the pair (C, D) is conditional semi compatible more over at $t = 0, C(0) = D(0) = 1, CD(0) = C(1) = e^2 = D(1) = DC(0)$

implies pair (C, D) is strong semi compatible.

Further at coincidence point $t = 1, C(1) = e^2 = D(1)$ and $CD(1) = C(e^2) = e^{2e^2}, DC(1) = D(e^2) = e^{2e^4}$.

This gives $CD(1) \neq DC(1)$.

Resulting that the pair (C, D) is strong semi compatible but not weakly compatible.

2.12 Definition⁽¹²⁾

Self-mappings C, D in (X, F, t) are called **sub sequentially continuous** if there exists a sequence $\langle t_n \rangle \in X$ in order for $Ct_n, Dt_n \rightarrow w^1$ as $n \rightarrow \infty$ for certain $w^1 \in X$ and satisfying

$$\lim_{n \rightarrow \infty} F_{CDt_n, Dw^1, a}(\beta) = 1 \text{ and } \lim_{n \rightarrow \infty} F_{DCt_n, Cw^1, a}(\beta) = 1 \quad \forall a \in X \text{ and } \beta > 0.$$

2.12.1 Example

Here we give the example in which the (C, D) pair is sub sequentially continuous but not weakly compatible

Let $X = R$ and d be usual distance on X and for each $t_1 \in [0, 1]$.

Define

$$F_{\alpha, \beta, a}(t_1) = \begin{cases} \frac{t_1}{t_1 + d(\alpha, \beta)}, t_1 > 0 \\ 0, t_1 = 1 \end{cases}$$

$\forall \alpha, \beta$ in X and fixed $a, t_1 > 0$.

Define mappings $C, D : X \rightarrow X$ as

$$C(t) = e^{t^2} \forall t \in R \text{ and } D(t) = e^{\frac{1}{2}} \forall t \in R.$$

Choose a sequence $\langle t_m \rangle = \frac{2}{m^2} \quad \forall m \geq 1$. Then

$$C(t_m) = C\left(\frac{2}{m^2}\right) = e^{\left(\frac{2}{m^2}\right)^2} \rightarrow 1 \quad \text{and}$$

$$D(t_m) = D\left(\frac{2}{m^2}\right) = e^{\frac{1}{2}} \left(\frac{2}{m^2}\right) \rightarrow 1 \text{ as } m \rightarrow \infty.$$

Moreover

$$CD(t_m) = C\left(e^{\frac{1}{m^2}}\right) = e^{\left(e^{\frac{1}{m^2}}\right)^2} \rightarrow e = C(1) \Rightarrow CD(t_m) \rightarrow C(1)$$

$$DC(t_m) = D\left(e^{\frac{4}{m^4}}\right) = e^{\frac{1}{2}} e^{\frac{4}{m^4}} \rightarrow e^{\frac{1}{2}} = D(1) \Rightarrow DC(t_m) \rightarrow D(1) \text{ as } m \rightarrow \infty.$$

This gives $\lim_{n \rightarrow \infty} F_{CDt_n, C(1), a}(\beta) = 1$ and $\lim_{n \rightarrow \infty} F_{DCt_n, D(1), a}(\beta) = 1$
 $\forall a \in X$ and $\beta > 0$

Moreover $C(0) = D(0) = 1 \Rightarrow CD(0) \neq DC(0)$ since

$$CD(0) = C(1) = e, e^{\frac{1}{2}} = D(1) = DC(0)$$

Implies the (C, D) pair is sub sequentially continuous but not weakly compatible.

The following theorem was proved by V. K. Gupta et.al.

Theorem (A)⁽¹¹⁾ “ Let A, B, S and T be self-mappings on a complete probabilistic 2-metric space (X, F, t) satisfying:

(A₁) $A(X) \subseteq T(X), B(X) \subseteq S(X)$

(A₂) one of $A(X), B(X), T(X)$ or $S(X)$ is closed

(A₃) pairs (A, S) and (B, T) are weakly compatible (A₄) $F_{Ax, By, a}(t) \geq rF_{Sx, Ty, a}(t)$ for all x, y and $t > 0$

where $r: [0, 1] \rightarrow [0, 1]$ is some continuous function such that $r(t) > t$ for each $0 < t < 1$

then A, B, S and T have unique common fixed point in X . ”

The above Theorem (A) can be further generalized as under.

3 Results and Discussion

3.1 Theorem

Let A, B, S and T be self -mappings on a complete probabilistic 2-metric space (X, F, t) satisfying

(3.1.1) $A(X) \subseteq T(X), B(X) \subseteq S(X)$

(3.1.2) pair (A, S) is strongly semi compatible and sub sequentially continuous and

the pair (B, T) is strongly semi compatible

(3.1.3) $F_{Ax, By, a}(t) \geq r(F_{Sx, Ty, a}(t)), t > 0$ and for all values of x, y in X

where r defined on $[0, 1]$ to it -self be continuous function with $r(t) > t$ for each $0 < t < 1$.

Then A, B, S and T have unique common fixed point in X .

Proof: Take $x_0 \in X$ and $Ax_0 \in A(X) \subseteq T(X)$ then $\exists x_1 \in X$ with $Ax_0 = Tx_1$ having $Bx_1 \in B(X) \subseteq S(X)$ then $\exists x_2 \in X$ with $Bx_1 = Sx_2$. Recursively obtain the sequences $\langle y_n \rangle$ and $\langle x_n \rangle$ for $n \geq 1$ such that $\langle y_{2n} \rangle = Ax_{2n} = Tx_{2n+1}$ and $\langle y_{2n+1} \rangle = Bx_{2n+1} = Sx_{2n+2}$. Now our claim is to show that $\langle y_n \rangle$ be a Cauchy sequence.

For this take $x = x_{2n}, y = x_{2n+1}$ in (3.1.3) we get

$$F_{Ax_{2n}, Bx_{2n+1}, a}(t) \geq rF_{Sx_{2n}, Tx_{2n+1}, a}(t) \text{ this gives}$$

$$F_{y_{2n}, y_{2n+1}, a}(t) \geq rF_{y_{2n-1}, y_{2n}, a}(t) > F_{y_{2n-1}, y_{2n}, a}(t).$$

Similarly

$$F_{y_{2n+1}, y_{2n+2}, a}(t) > F_{y_{2n}, y_{2n+1}, a}(t).$$

In general $F_{y_{n+1}, y_n, a}(t) > F_{y_n, y_{n-1}, a}(t)$ for all $n \geq 0$.

Then $\langle F_{y_{n+1}, y_n, a}(t) \rangle$ for all $n \geq 0$ is an increasing sequence bounded above by 1.

Therefore it must converge to the limit $L \leq 1$ as $n \rightarrow \infty$

In case of $L < 1$ we have $F_{y_{n+1}, y_n, a}(t) = L > r(1) > 1$ this is absurd. Hence $L = 1$.

Hence for all n and p we have $F_{y_{n+p}, y_n, a}(t) = 1$.

As a result, the Cauchy sequence $\langle y_n \rangle$ in X must converge to point $z \in X$ because X is a complete space.

Consequently each sub sequence also has the same limit z .

That is $Ax_{2n}, Sx_{2n} \rightarrow z$ and $Bx_{2n+1}, Tx_{2n+1} \rightarrow z$ as $n \rightarrow \infty$.

By strongly semi compatibility of the pair (A, S) whenever

$Ax_{2n}, Sx_{2n} \rightarrow z$ implies there exists sequence $\langle t_n \rangle \in X$ such that

$At_n, St_n \rightarrow \bar{u}, \bar{u} \in X$ with

$$\lim_{n \rightarrow \infty} F_{ASt_n, \bar{u}, a}(t) = 1 \text{ and } \lim_{n \rightarrow \infty} F_{SA\bar{u}, At_n, a}(t) = 1 \text{ for all } t, a > 0 \dots (3.1)$$

Furthermore, the fact that sub sequentially continuous pair (A, S) implies $\lim_{n \rightarrow \infty} F_{ASt_n, A\bar{u}, a}(t) = 1$ and $\lim_{n \rightarrow \infty} F_{SA\bar{u}, S\bar{u}, a}(t) = 1$ for all $t, a > 0 \dots a$ (3.2)

Using (3.1) and (3.2) we obtain $F_{A\bar{u}, S\bar{u}, a}(t) = 1$ this implies $A\bar{u} = S\bar{u}$

From (3.1.1) $A(X) \subseteq T(X), S\bar{u} = A\bar{u} \in A(X) \subseteq T(X)$ implies

$\exists v_1 \in X$ such that $S\bar{u} = A\bar{u} = Tv_1$ for some $v_1 \in X$.

Claim $Bv_1 = Tv_1$.

Put $x = \bar{u}, y = v_1$ in (3.1.3) we have get

$$F_{A\bar{u}, Bv_1, a}(t) \geq r(F_{S\bar{u}, Tv_1, a}(t)) \text{ using } S\bar{u} = Tv_1.$$

$$F_{A\bar{u}, Bv_1, a}(t) \geq r(F_{S\bar{u}, S\bar{u}, a}(t)) = r(1) = 1$$

Hence $A\bar{u} = Bv_1$ results $A\bar{u} = S\bar{u} = Tv_1 = Bv_1 = P^1$ for some P^1 in X .

However, because the pair (A, S) is strongly semi compatible implies $A\bar{u} = S\bar{u} \Rightarrow AS\bar{u} = SA\bar{u} \Rightarrow AP^1 = SP^1$.

Furthermore, the pair (A, S) is strongly semi compatible, resulting in

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$$F_{AP^1, BP^1, a}(t) \geq r(F_{SP^1, TP^1, a}(t)) \text{ using } AP^1 = SP^1 \text{ and } BP^1 = TP^1 = P^1.$$

This gives $F_{AP^1, P^1, a}(t) \geq r(F_{\sigma P^1, P^1, a}(t))$.

If $P^1 \neq AP^1$ implies $F_{AP^1, P^1, a}(t) \geq r(F_{AP^1, P^1, a}(t)) > F_{AP^1, P^1, a}(t)$.

This gives $F_{AP^1, P^1, a}(t) \geq F_{AP^1, P^1, a}(t)$ which is not possible.

Therefore $AP^1 = SP^1 = BP^1 = TP^1 = P^1$.

Uniqueness:

Let p_1 be the another fixed point then

$$AP^1 = SP^1 = BP^1 = TP^1 = P^1 \text{ and } Ap_1 = Bp_1 = Sp_1 = Tp_1 = p_1.$$

Now assume $P^1 \neq p_1$.

By taking $x = P^1$ and $y = p_1$ in (3.1.3) we've got

$$F_{Ap^1 \cdot Bp_1} a(t) \geq r(F_{Sp^1 \cdot Tp_1} a(t))$$

$$F_{p^1 \cdot p_1} a(t) \geq r(F_{p^1 \cdot p_1} a(t)) > F_{p^1 \cdot p_1} a(t).$$

$$F_{p^1 \cdot p_1} a(t) > F_{p^1 \cdot p_1} a(t) \text{ which is not possible hence } P^1 = p_1.$$

Now we provide a supporting illustration to justify the theorem.

3.2 Example

Consider $X = [0, 1]$ d is general distance in X and each $t_1 \in [0, 1]$ take the projection, define

$$F_{\alpha, \beta, a}(t_1) = \begin{cases} \frac{t_1}{t_1 + d(\alpha, \beta)}, t_1 > 0 \\ 0, t_1 = 1 \end{cases}$$

$$A(x) = B(x) = \begin{cases} \frac{1}{2}, x = 0 \\ 1 - 7x, 0 < x \leq \frac{1}{9} \\ x^2, \frac{1}{9} < x \leq 1 \end{cases}$$

$$A(x) = B(x) = \begin{cases} \frac{1}{3}, x = 0 \\ 2x, 0 < x \leq \frac{1}{9} \\ x^4, \frac{1}{9} < x \leq 1 \end{cases}$$

Now $A(X) = (\frac{1}{81}, 1] = B(X)$ and $S(X) = (0, 1] = T(X)$ so that $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$.

Clearly $1/9$ and 1 are coincidence points for the mappings A, S .

$$\text{At } x = \frac{1}{9}, A(\frac{1}{9}) = S(\frac{1}{9}) = \frac{2}{9} \text{ and } AS(\frac{1}{9}) = A(\frac{2}{9}) = \frac{4}{81}.$$

$$SA(\frac{1}{9}) = S(\frac{2}{9}) = (\frac{2}{9})^4 \text{ implies } AS(\frac{1}{9}) \neq SA(\frac{1}{9}).$$

Consider a sequence $\langle x_n \rangle = \frac{1}{9} - \frac{3}{n}$ for all $n \geq 1$. Then

$$Ax_n = A\left(\frac{1}{9} - \frac{3}{n}\right) = 1 - 7\left(\frac{1}{9} - \frac{3}{n}\right) \rightarrow \frac{2}{9}.$$

$$Sx_n = S\left(\frac{1}{9} - \frac{3}{n}\right) = 2\left(\frac{1}{9} - \frac{3}{n}\right) \rightarrow \frac{2}{9} \text{ as } n \rightarrow \infty.$$

Implies $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n$ is nonempty.

There exist another sequence $\langle x_n \rangle = 1 - \frac{1}{n^3}$ for all $n \geq 1$. Then

$$ASx_n = AS\left(1 - \frac{1}{n^3}\right) = A\left(1 - \frac{1}{n^3}\right)^4 = \left(1 - \frac{1}{n^3}\right)^8 \rightarrow 1 = S(1),$$

$$SAx_n = SA\left(1 - \frac{1}{n^3}\right) = S\left(1 - \frac{1}{n^3}\right)^4 = \left(1 - \frac{1}{n^3}\right)^8 \rightarrow 1 \rightarrow 1 = A(1) \text{ as } n \rightarrow \infty.$$

This implies $\lim_{n \rightarrow \infty} F_{ASx_n, s(1), a(\beta)} = 1$ and $\lim_{n \rightarrow \infty} F_{SAx_n, A(1), a(\beta)} = 1$ $\forall a \in X$ and $\beta > 0$.

As a result, Pair (A, S) is strong semi-compatible.

Further for the sequence $\langle x_n \rangle = \left(1 - \frac{1}{n^2}\right)$ for all $n \geq 1$. Then we have

$$ASx_n = AS\left(1 - \frac{1}{n^2}\right) = A\left(1 - \frac{1}{n^3}\right)^4 = \left(1 - \frac{1}{n^3}\right)^8 \rightarrow 1 = A(1)$$

$$SAx_n = SA\left(1 - \frac{1}{n^2}\right) = S\left(1 - \frac{1}{n^3}\right)^4 = \left(1 - \frac{1}{n^3}\right)^8 \rightarrow 1 = S(1) \text{ as } n \rightarrow \infty$$

This gives $\lim_{n \rightarrow \infty} F_{ASx_n, A(1), a(\beta)} = 1$ and $\lim_{n \rightarrow \infty} F_{SAx_n, s(1), a(\beta)} = 1 \forall a \in X$ and $\beta > 0$.

As a result, the pair (A, S) is strong semi compatible and sub sequentially continuous, while the pair (B, T) is strong semi compatible and at $x=1$, the pair (B, T) is strong semi compatible and

$$A(1) = S(1) = B(1) = T(1) = 1$$

As a result, mappings A, S, B and T met all of the conditions in Theorem (3.1) while also contained the single common fixed point at $x=1$.

3.2.1 Corollary

In the theorem (3.3) take $T=S$ we get.

Consider A, B and T are self -mappings on a complete probabilistic 2-metric space (X, F, t) having

$$(3.1.2.1) A(X) \subseteq T(X), B(X) \subseteq T(X)$$

(3.1.2.2) pair (A, T) is strong semi compatible sub sequentially continuous mappings and the pair of mappings (B, T) is faintly compatible

$$(3.1.2.3) F_{Ax,By,a}(t) \geq rF_{Tx,Ty,a}(t) \text{ for all values of } x, y \text{ in } X \text{ and } t > 0$$

where some continuous function $r:[0, 1] \rightarrow [0, 1]$ in order for $r(t) > t$ for every $t \in (0, 1)$ then A, B and T have unique common fixed point in X .

3.2.2 Corollary

In Theorem (3.3) take $T=S$ and $B=A$ we get.

Let A, T are self -mappings on a complete probabilistic 2-metric space (X, F, t) satisfying

$$(3.1.3.1) A(X) \subseteq T(X)$$

(3.1.3.2) Pair (A, T) is strong semi compatible sub sequentially continuous

$$(3.1.3.3) F_{Ax,Ay,a}(t) \geq r(F_{Tx,Ty,a}(t)) \forall x, y \text{ and } t > 0$$

where some continuous function $r:[0, 1] \rightarrow [0, 1]$ in order for $r(t) > t$ for every $t \in (0, 1)$ then A, T have unique common fixed point in X .

4 Conclusion

We improved the result in Theorem (A) in this article by applying the concepts of strong semi compatible, sub sequentially continuous mappings and none of the mappings are assumed as closed range spaces. Further we supported our result by discussing a suitable example.

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