Generalised neo-pseudo projective recurrent Finsler space

Indiwar Singh Chauhan1*, T S Chauhan2, Mohammad Gauhar3
1 Assistant Professor, Deptt. of Mathematics, Bareilly College, Bareilly (U.P.), India. Tel.: 09219269009
2 Associate Professor, Deptt. of Mathematics, Bareilly College, Bareilly (U.P.), India
3 Research Scholar, IFTM University, Moradabad (U.P.), India

Abstract

Objectives: The purpose of this paper is to obtain several results in the field of generalised neo-pseudo projective recurrent Finsler space. Methods: A generalization technique is employed to solve the resulting problem. We provide its application in the study of space-time. Findings: In section 1, we have defined and studied some of the basic and useful results for later work. Section 2 deals for the neo-pseudo projective recurrent curvature tensor. The notion of neo-pseudo projective recurrent space of second order has been delineated in the section 3. In the section 4 we have studied the generalised neo-pseudo projective recurrent space and established several new results. Novelty/Conclusion: In this paper we have studied some recurrent properties of neo-pseudo projective curvature tensor in a Finsler space. We have obtained several new results which are as follows:

- If the space $F^n$ admits a neo-pseudo projective curvature tensor $Q^{\alpha}_{\beta\gamma\delta}$ then $Q^{\alpha}_{\beta\gamma\delta}$ is skew-symmetric with regard to last two indices.
- If the neo-pseudo projective deviation tensor $Q^{\alpha}_{\beta}$ and pseudo deviation tensor field $T^{\alpha}_{\beta}$ coincides to each other for $q = 1$ then space is W-flat.
- If $F^n$ admits the projectively flat $Q$-recurrent space then the relation $\nabla_{e}Q^{\alpha}_{\beta\gamma} + \nabla_{\beta}Q^{\alpha}_{\gamma e} + \nabla_{\gamma}Q^{\alpha}_{\beta e} = 0$ holds good.
- If a Finsler space $F^n$ admits projectively flat $Q$-birecurrent space then the relation $K_{\epsilon\epsilon}Q^{\alpha}_{\beta\gamma} + K_{\epsilon\beta}Q^{\alpha}_{\gamma e} + K_{\epsilon\gamma}Q^{\alpha}_{\beta e} = 0$ holds good.
- If the space is $Q$-birecurrent then the generalised $Q$-recurrent space is $Q$-symmetric.
- For the projective flat generalised $Q$-recurrent space the relation $\nabla_{\rho} \nabla_{e}Q^{\alpha}_{\beta s} + \nabla_{\rho} \nabla_{\gamma}Q^{\alpha}_{s e} + \nabla_{\rho} \nabla_{s}Q^{\alpha}_{e \gamma} = 0$ holds good.

AMS Subject Classification: 58B20, 53C20, 53C60.
Keywords: Neo-pseudo; recurrent; projective; curvature tensor; bisymmetric; flat
1 Introduction

Let $\mathbb{F}^n$ be an $n$-dimensional Finsler space with a positive definite metric $g_{\alpha \beta}$, which admit a projective deviation tensor field $W^\alpha_{\beta}$ and pseudo deviation tensor field $T^\alpha_{\beta}$ satisfying

$$Q^\alpha_{\beta} = p W^\alpha_{\beta} + q T^\alpha_{\beta} \quad (1.1)$$

where in $p$ and $q$ are scalars which are positively homogenous of degree zero in $\dot{x}^\alpha$.

Prof. U.P. Singh and Prof. A.K. Singh while developing the theory of neo-pseudo projective curvature tensor, obtain two kinds of curvature tensor $Q^\alpha_{\beta\gamma}$ and $Q^\alpha_{\beta\gamma\delta}$ (1). With a view to defining the projective deviation tensor field and pseudo deviation tensor field, he constructed the quantities $Q^\alpha_{\beta}(x, \dot{x})$ which behave like neo-pseudo projective deviation tensor.

With the help of tensor $Q^\alpha_{\beta}(x, \dot{x})$ the absolute differential of concerning vector referred to the scalar function $Q(x, \dot{x})$ is defined as follows (1–3):

$$Q^\alpha_{\beta\gamma} = \frac{1}{3} \left( \dot{\gamma}^\beta Q^\alpha_{\gamma} - \dot{\gamma}^\gamma Q^\alpha_{\beta} \right) \quad (1.2)$$

and

$$Q^\alpha_{\beta\gamma\delta} = \dot{\delta}^\beta Q^\alpha_{\gamma\delta} \quad (1.3)$$

It is easy to verify that the neo-pseudo projective curvature tensor satisfies the following relations (1,4):

$$Q^\alpha_{\beta\gamma\delta} + Q^\alpha_{\gamma\delta\beta} + Q^\alpha_{\delta\beta\gamma} = 0 \quad (1.4)$$

and

$$Q^\alpha_{\beta\gamma} \dot{x}^\beta = Q^\alpha_{\gamma} \quad (1.5)$$

Moreover, these curvature tensor also satisfy the following identities

$$Q^\alpha_{\beta\gamma\delta} \dot{x}^\beta \dot{x}^\gamma = Q^\alpha_{\delta} \quad (1.7)$$

and

$$Q^\alpha_{\beta} \dot{x}^\beta = 0 \quad (1.8)$$

As it is well known, in the Finsler space a scalar function $Q(x, \dot{x})$ is given by

$$Q^\alpha_{\alpha} = (n - 1)qT \quad (1.9)$$

Let us consider a curvature tensor $W^\alpha_{\beta}$ in Finsler space, is termed as projective curvature tensor in the Finsler space and is defined as follows (2,5,6):

$$W^\alpha_{\beta} = H^\alpha_{\beta} + T^\alpha_{\beta} \quad (1.10)$$

Wherein $H^\alpha_{\beta}$ is positively homogeneous of degree one in $\dot{x}^\alpha$.

In analogy with the relation (1.1) the projective curvature tensors $W^\alpha_{\beta\gamma}$ and $W^\alpha_{\beta\gamma\delta}$ in the Finsler space with the condition $p = q = 1$ may be defined as follows (2,7):

$$W^\alpha_{\beta\gamma} = H^\alpha_{\beta\gamma} + Q^\alpha_{\beta\gamma} \quad (1.11)$$

$$W^\alpha_{\beta\gamma\delta} = H^\alpha_{\beta\gamma\delta} + Q^\alpha_{\beta\gamma\delta} \quad (1.12)$$

https://www.indjst.org/
and

\[ W^\alpha_{\beta\gamma\delta} = H^\alpha_{\beta\gamma\delta} + Q^\alpha_{\beta\gamma\delta} \]  

(1.13)

In view of above discussions, we have the following theorems:

**Theorem 1.1:**

For the neo-pseudo projective curvature tensor the relation

\[ Q^\alpha_{\beta\gamma} = -Q^\alpha_{\gamma\beta} \]  

(1.14)

holds good.

**Proof:**

Interchanging \( \beta \) and \( \gamma \) in equation (1.2) and adding this with new equation, we get the desired result.

**Theorem 1.2:**

If the space \( F^n \) admits a neo-pseudo projective curvature tensor \( Q^\alpha_{\beta\gamma\delta} \) then \( Q^\alpha_{\beta\gamma\delta} \) is skew-symmetric with regard to last two indices.

**Proof:**

Interchanging \( \gamma \) and \( \delta \) in equation (1.3) and adding the new equation to the equation (1.3), we obtain

\[ Q^\alpha_{\beta\gamma\delta} + Q^\alpha_{\beta\delta\gamma} = \partial_\beta Q^\alpha_{\gamma\delta} + \partial_\delta Q^\alpha_{\beta\gamma} \]  

(1.15)

From equations (1.14) and (1.15), we get

\[ Q^\alpha_{\beta\gamma\delta} = -Q^\alpha_{\beta\delta\gamma} \]  

(1.16)

**Theorem 1.3:**

If the neo-pseudo projective deviation tensor \( Q^\alpha_{\beta} \) coincides with geodesic deviation tensor field \( H^\alpha_{\beta} \) in the Finsler space \( F^n \) then projective deviation tensor field \( W^\alpha_{\beta} \) and the neo-pseudo projective deviation tensor \( Q^\alpha_{\beta} \) are identically equal to each other.

**Proof:**

If the neo-pseudo projective deviation tensor \( Q^\alpha_{\beta} \) coincides with geodesic deviation tensor field \( H^\alpha_{\beta} \). Consequently, from equation (1.11) follows

\[ W^\alpha_{\beta} = Q^\alpha_{\beta} \]  

(1.17)

Hence projective deviation tensor field \( W^\alpha_{\beta} \) and the neo-pseudo projective deviation tensor \( Q^\alpha_{\beta} \) are identically equal to each other.

**Theorem 1.4:**

If the neo-pseudo projective deviation tensor \( Q^\alpha_{\beta} \) and pseudo deviation tensor field \( T^\alpha_{\beta} \) coincides to each other for \( q = 1 \) then Finsler space admits the condition \( W^\alpha_{\beta} = 0 \) i.e. W-flat.

**Proof:**

Inseting \( q = 1 \) in equation (1.1), we obtain

\[ Q^\alpha_{\beta} = p W^\alpha_{\beta} + T^\alpha_{\beta} \]  

(1.18)

If the neo-pseudo projective deviation tensor \( Q^\alpha_{\beta} \) and pseudo deviation tensor field \( T^\alpha_{\beta} \) coincides to each other then from equation (1.18) we observe that

\[ W^\alpha_{\beta} = 0 \]  

(1.19)

This manifests that the space is W-flat.

**Theorem 1.5:**

If the neo-pseudo projective deviation tensor \( Q^\alpha_{\beta} \) and projective deviation tensor field \( W^\alpha_{\beta} \) coincides to each other then the geodesic deviation tensor field \( H^\alpha_{\beta} \) vanish identically i.e. H-flat.

**Proof:**
If the neo-pseudo projective deviation tensor $Q^a_{\beta}$ and projective deviation tensor field $W^a_{\beta}$ coincides to each other then from equation (1.11) follows the result

$$H^a_{\beta} = 0$$ (1.20)

Consequently, the space is H-flat.

**Theorem 1.6:**

If the projective deviation tensor field $W^a_{\beta}$ and geodesic deviation tensor field $H^a_{\beta}$ coincides to each other then the neo-pseudo projective deviation tensor $Q^a_{\beta}$ vanish identically i.e. Q-flat.

**Proof:**

If the projective deviation tensor field $W^a_{\beta}$ and geodesic deviation tensor field $H^a_{\beta}$ coincides to each other. Consequently, from equation (1.11) follows

$$Q^a_{\beta} = 0$$ (1.21)

Therefore the space is Q-flat.

### 2 Recurrent neo-pseudo projective curvature tensor in Finsler space

In view of the investigation of Prof. U.P. Singh and Prof. A.K. Singh (1) we observe that if the neo-pseudo projective deviation tensor $Q^a_{\beta}$ is necessarily recurrent then projective deviation tensor and pseudo deviation tensor are proportional to each other.

As a consequence of this follows the result

$$W^a_{\beta} = t \, T^a_{\beta}$$ (2.1)

wherein t is a scalar.

As a consequence of equations (1.1) and (2.1), we obtain

$$Q^a_{\beta} = s \, T^a_{\beta}$$ (2.2)

wherein $s = pt + q$ is any scalar and positively homogeneous of degree zero in $x^a$.

**Definition 2.1:**

A Finsler space whose curvature tensor is recurrent is called Q-recurrent Finsler space.

In view of the definition it follows that for a recurrent space, we have

$$\nabla_{\epsilon} Q^a_{\beta \gamma \delta} = R_{\epsilon} Q^a_{\beta \gamma \delta}$$ (2.3)

wherein $R_{\epsilon}$ is a non-zero vector termed as the recurrent vector field.

**Definition 2.2:**

An n-dimensional Finsler space $F^n$ is called Q-symmetric when the covariant derivative of curvature tensor is everywhere zero i.e.

$$\nabla_{\epsilon} Q^a_{\beta \gamma \delta} = 0$$ (2.4)

**Definition 2.3:**

A Finsler space $F^n$ is said to be Q-flat when its curvature tensor vanishes identically.

As a consequence of this definition follows the result:

$$Q^a_{\beta \gamma \delta} = 0$$ (2.5)

Contracting (2.3) with $x^\beta$ and use of equation (1.6), we obtain

$$\nabla_{\epsilon} Q^a_{\gamma \delta} = R_{\epsilon} Q^a_{\gamma \delta}$$ (2.6)

Again contracting (2.6) with $x^\gamma$ and making use of equation (1.5), we get

$$\nabla_{\epsilon} Q^a_{\delta} = R_{\epsilon} Q^a_{\delta}$$ (2.7)
Thus, we have now the following theorem:

**Theorem 2.1:**

For the recurrence vector space \( R_e \) in the Finsler space \( F^n \) there exists the relation

\[
(\nabla_\rho \nabla_e - \nabla_e \nabla_\rho) (\log T) = (\nabla_\rho R_e - \nabla_e R_\rho).
\]

**Proof:**

Differentiating (2.3) covariantly with regard to \( x^\rho \), we get

\[
\nabla_\rho \nabla_e Q_{\beta \gamma \delta}^a = (\nabla_\rho R_e + R_e R_\rho) Q_{\beta \gamma \delta}^a \tag{2.8}
\]

Interchanging \( e \) and \( d \) in equation (2.8) and subtracting the new equation from equation (2.8), we obtain

\[
\nabla_\rho \nabla_e Q_{\beta \gamma \delta}^a - \nabla_e \nabla_\rho Q_{\beta \gamma \delta}^a = (\nabla_\rho R_e - \nabla_e R_\rho) Q_{\beta \gamma \delta}^a \tag{2.9}
\]

Contracting (2.9) with \( \dot{x}^\beta \dot{x}^\gamma \) and use of equation (1.7), we get

\[
\nabla_\rho \nabla_e Q_{\beta \gamma}^a - \nabla_e \nabla_\rho Q_{\gamma \beta}^a = (\nabla_\rho R_e - \nabla_e R_\rho) Q_{\beta \gamma}^a \tag{2.10}
\]

Contracting \( a \) and \( d \) in equation (2.10) and use of equation (1.9), we obtain

\[
(\nabla_\rho \nabla_e - \nabla_e \nabla_\rho) (\log T) = (\nabla_\rho R_e - \nabla_e R_\rho) \tag{2.11}
\]

**Definition 2.4:**

If the neo-pseudo projective curvature tensor \( Q_{\beta \gamma \delta}^a \) in the Finsler space \( F^n \) satisfies the relation

\[
\nabla_e Q_{\beta \gamma \delta}^a = R_e Q_{\beta \gamma}^a \tag{2.12}
\]

then \( F^n \) is termed as \( Q \)-recurrent with recurrence vector field \( R_e \).

Consequently, we have a theorem:

**Theorem 2.2:**

If \( F^n \) admits the projectively flat \( Q \)-recurrent space then the relation

\[
\nabla_e Q_{\beta \gamma}^a + \nabla_\beta Q_{\gamma e}^a + \nabla_\gamma Q_{e \beta}^a = 0
\]

holds good.

**Proof:**

If the space is projectively flat then from equation (1.12), we have

\[
H_{\beta \gamma}^a + Q_{\beta \gamma}^a = 0 \tag{2.13}
\]

Differentiating (2.13) covariantly with respect to \( x^e \), we get

\[
\nabla_e H_{\beta \gamma}^a + \nabla_\gamma Q_{e \beta}^a = 0 \tag{2.14}
\]

Taking the cyclic permutation in \( \beta, \gamma, e \) and adding, we have

\[
(\nabla_e H_{\beta \gamma}^a + \nabla_\beta H_{\gamma e}^a + \nabla_\gamma H_{e \beta}^a) + (\nabla_e Q_{\beta \gamma}^a + \nabla_\beta Q_{\gamma e}^a + \nabla_\gamma Q_{e \beta}^a) = 0 \tag{2.15}
\]

The first part of equation (2.15) vanishes due to commutation formula \((8)\), equation (6.13), p.128), hence we obtain

\[
\nabla_e Q_{\beta \gamma}^a + \nabla_\beta Q_{\gamma e}^a + \nabla_\gamma Q_{e \beta}^a = 0 \tag{2.16}
\]

### 3 Neo-Pseudo Projective Recurrent Space:

**Definition 3.1:**

Neo-pseudo projective curvature tensor \( Q_{\beta \gamma \delta}^a \) of a Finsler space satisfies the relation

\[
\nabla_\rho \nabla_e Q_{\beta \gamma \delta}^a = K_{\epsilon \rho} Q_{\beta \gamma \delta}^a \tag{3.1}
\]

wherein \( K_{\epsilon \rho} \) is non-zero recurrent tensor, then it is called neo-pseudo projective recurrent space of second order or briefly a \( Q \)-birecurrent space \((3, 4)\).
Definition 3.2: If the covariant derivative of neo-pseudo projective curvature tensor $Q^a_{\beta\gamma\delta}$ vanishes identically then the space is termed as Q-bisymmetric.

As a consequence of above definition follows the result

$$\nabla_\rho \nabla_\epsilon Q^a_{\beta\gamma\delta} = 0 \quad (3.2)$$

In this regard we shall now establish the following theorem:

Theorem 3.1: The necessary and sufficient condition for Finsler space to be Q-bisymmetric space is that the neo-pseudo projective tensor vanishes identically.

Proof:

Since neo-pseudo projective tensor vanishes i.e. $Q^a_{\beta\gamma\delta} = 0$. Consequently from equation (3.1) it follows that $\nabla_\rho \nabla_\epsilon Q^a_{\beta\gamma\delta} = 0$. This manifests that the space to be Q-bisymmetric.

Conversely, if the space to be Q-bisymmetric then the converse of theorem is immediately proof.

Remark 3.1: It is noteworthy that every Q-recurrent is necessarily Q-birecurrent.

Theorem 3.2: In a Finsler space $F^n$, the recurrent tensor field $K_{\epsilon\rho}$ is not symmetric in general.

Proof:

Contracting $\alpha$ and $\delta$ in equation (3.1) yields

$$\nabla_\rho \nabla_\epsilon Q^a_{\beta\gamma} = K_{\epsilon\rho} Q^a_{\beta\gamma} \quad (3.3)$$

Interchanging $\epsilon$ and $\rho$ in equation (3.3) and subtracting the new equation from equation (3.3), we obtain

$$\nabla_\rho \nabla_\epsilon Q^a_{\beta\gamma} - \nabla_\epsilon \nabla_\rho Q^a_{\beta\gamma} = (K_{\epsilon\rho} - K_{\rho\epsilon}) Q^a_{\beta\gamma} \quad (3.4)$$

Contracting (3.4) with $\dot{x}^\beta \dot{x}^\gamma$ and use of equation (1.7), we get

$$\nabla_\rho \nabla_\epsilon Q - \nabla_\epsilon \nabla_\rho Q = (K_{\epsilon\rho} - K_{\rho\epsilon}) Q \quad (3.5)$$

Using commutation formula (5), equation (6.10), p.126 and equation (2.2), we obtain

$$(K_{\epsilon\rho} - K_{\rho\epsilon}) Q = (\partial_\sigma Q) H^\sigma_{\rho\epsilon} \quad (3.6)$$

From equations (1.9) and (3.6), consequently, follows

$$K_{\epsilon\rho} - K_{\rho\epsilon} = \partial_\sigma (\log T) H^\sigma_{\rho\epsilon} \quad (3.7)$$

Yields the result

$$K_{\epsilon\rho} \neq K_{\rho\epsilon} \quad (3.8)$$

Theorem 3.3: If a Finsler space $F^n$ admits projectively flat Q-birecurrent space then the relation $K_{\epsilon\rho} Q^a_{\beta\gamma} + K_{\beta\rho} Q^a_{\gamma\epsilon} + K_{\gamma\rho} Q^a_{\epsilon\beta} = 0$ holds good.

Proof:

Differentiating equation (2.16) covariantly with respect to $x^\rho$, we get

$$\nabla_\rho \nabla_\epsilon Q^a_{\beta\gamma} + \nabla_\rho \nabla_\beta Q^a_{\gamma\epsilon} + \nabla_\rho \nabla_\gamma Q^a_{\epsilon\beta} = 0 \quad (3.9)$$

Since the space is Q-birecurrent then equation (3.2) assumes the form

$$K_{\epsilon\rho} Q^a_{\beta\gamma} + K_{\beta\rho} Q^a_{\gamma\epsilon} + K_{\gamma\rho} Q^a_{\epsilon\beta} = 0 \quad (3.10)$$
4 Generalised Neo-Pseudo Projective Recurrent Space:

Let us consider the relation

$$\nabla_\rho \nabla_\epsilon Q^\alpha_{\beta\gamma\delta} = R_\rho \nabla_\epsilon Q^\alpha_{\beta\gamma\delta} + K_{\epsilon \rho} Q^\alpha_{\beta\gamma\delta} \quad (4.1)$$

wherein $R_\rho$ and $K_{\epsilon \rho}$ are recurrence vector and recurrence tensor fields respectively.

**Definition 4.1:**
The neo-pseudo projective curvature tensor $Q^\alpha_{\beta\gamma\delta}$ of Finsler space $F^n$ satisfying the condition (4.1) is called generalised neo-pseudo projective recurrent curvature tensor $^{(3,4)}$.

**Definition 4.2:**
Finsler space $F^n$ equipped with the generalised neo-pseudo projective recurrent curvature tensor $Q^\alpha_{\beta\gamma\delta}$ is called generalised neo-pseudo projective recurrent Finsler space $^{(3,9)}$.

In this regard, we have the following theorems:

**Theorem 4.1:**
The necessary and sufficient condition for Finsler space $F^n$ to be Q-symmetric is that the space has to be Q-birecurrent.

*Proof:*
If the space is to be Q-symmetric i.e. $\nabla_\epsilon Q^\alpha_{\beta\gamma\delta} = 0$,

Consequently, from equation (4.1) follows

$$\nabla_\rho \nabla_\epsilon Q^\alpha_{\beta\gamma\delta} = K_{\epsilon \rho} Q^\alpha_{\beta\gamma\delta} \quad (4.2)$$

which is the condition of Q-birecurrent.

Conversely, let us assume that the space be Q-birecurrent, follows the condition (4.2). Inserting equation (4.2) into equation (4.1), we obtain $\nabla_\epsilon Q^\alpha_{\beta\gamma\delta} = 0$. Hence the space is Q-symmetric.

**Theorem 4.2:**
If the space $F^n$ is Q-symmetric and Q-flat then its generalised neo-pseudo projective recurrent space vanishes identically.

*Proof:*
If the space be Q-symmetric i.e. $\nabla_\epsilon Q^\alpha_{\beta\gamma\delta} = 0$ and Q-flat i.e. $Q^\alpha_{\beta\gamma\delta} = 0$. Consequently, from equation (4.1) follows

$$\nabla_\rho \nabla_\epsilon Q^\alpha_{\beta\gamma\delta} = 0. \text{ This establishes the validity of the theorem.}$$

**Remark 4.1:**
It is noteworthy that if $F^n$ to be Q-symmetric and Q-flat follows that the generalised neo-pseudo projective recurrent space necessarily vanishes. Consequently, the space is simply generalised Q-symmetric one.

**Theorem 4.3:**
If space $F^n$ admits Q-symmetric and Q-flat then the space is a generalised Q-symmetric one.

*Proof:*
It follows immediately from theorem 4.2.

**Theorem 4.4:**
In Finsler space $F^n$, if the space is Q-birecurrent then the generalised Q-recurrent space is Q-symmetric.

*Proof:*
It is obvious from equations (2.4), (3.1) and (4.1).

**Theorem 4.5:**
For the recurrence vector $R_\rho$ the relation

$$R_\rho \nabla_\epsilon T - R_\epsilon \nabla_\rho T = 0 \quad (4.3)$$

holds good.

*Proof:*
Contracting (4.1) by $\alpha$ and $\delta$, we obtain

$$\nabla_\rho \nabla_\epsilon Q^\alpha_{\beta\gamma} = R_\rho \nabla_\epsilon Q^\alpha_{\beta\gamma} + K_{\epsilon \rho} Q^\alpha_{\beta\gamma} \quad (4.4)$$

Interchanging $\epsilon$ and $\rho$ in equation (4.4) and subtracting the new equation from equation (4.4), we get

$$\nabla_\rho \nabla_\epsilon Q^\alpha_{\beta\gamma} - \nabla_\epsilon \nabla_\rho Q^\alpha_{\beta\gamma} = R_\rho \nabla_\epsilon Q^\alpha_{\beta\gamma} - R_\epsilon \nabla_\rho Q^\alpha_{\beta\gamma} + (K_{\epsilon \rho} - K_{\rho \epsilon}) Q^\alpha_{\beta\gamma} \quad (4.5)$$
Contracting (4.5) with $\dot{x}^{\beta} x^{\gamma}$ and use of equation (1.7), we get

$$\nabla_{\rho} \nabla_{\epsilon} Q - \nabla_{\epsilon} \nabla_{\rho} Q = R_{\rho} \nabla_{\epsilon} Q = R_{\rho} \nabla_{\epsilon} Q + (K_{\epsilon \rho} - K_{\rho \epsilon}) Q$$

(4.6)

By virtue of equations (3.5) and (4.6), we get

$$R_{\rho} \nabla_{\epsilon} Q = 0$$

(4.7)

Inserting equation (1.9) in equation (4.7), we get the desired result.

**Theorem 4.6:**

For the projective flat generalised $Q$-recurrent space the relation

$$\nabla_{\rho} \nabla_{\epsilon} Q^{\alpha} + \nabla_{\rho} \nabla_{\gamma} Q^{\alpha} + \nabla_{\rho} \nabla_{\delta} Q^{\alpha} = 0$$

holds good.

**Proof:**

Contracting (4.1) with $\dot{x}^{\beta}$, we obtain

$$\nabla_{\rho} \nabla_{\epsilon} Q^{\alpha} = R_{\rho} \nabla_{\epsilon} Q^{\alpha} + K_{\epsilon \rho} Q^{\alpha}$$

(4.8)

Taking the cyclic permutation in $\epsilon, \gamma, \delta$ and on making use of equations (2.16) and (3.10), we observe that

$$\nabla_{\rho} \nabla_{\epsilon} Q^{\alpha} + \nabla_{\rho} \nabla_{\gamma} Q^{\alpha} + \nabla_{\rho} \nabla_{\delta} Q^{\alpha} = 0$$

(4.9)

**References**