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* **Corresponding author**.

Tel: 09219269009
indiwarsingh.chauhan@gmail.com

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Generalised neo-pseudo projective recurrent Finsler space

Indiwar Singh Chauhan[1]*, T S Chauhan[2], Mohammad Gauhar[3]

1 Assistant Professor, Deptt. of Mathematics, Bareilly College, Bareilly (U.P.), India. Tel.: 09219269009
2 Associate Professor, Deptt. of Mathematics, Bareilly College, Bareilly (U.P.), India
3 Research Scholar, IFTM University, Moradabad (U.P.), India

Abstract

Objectives: The purpose of this paper is to obtain several results in the field of generalised neo-pseudo projective recurrent Finsler space. **Methods:** A generalization technique is employed to solve the resulting problem. We provide its application in the study of space-time. **Findings:** In section 1, we have defined and studied some of the basic and useful results for later work. Section 2 deals for the neo-pseudo projective recurrent curvature tensor. The notion of neo-pseudo projective recurrent space of second order has been delineated in the section 3. In the section 4 we have studied the generalised neo-pseudo projective recurrent space and established several new results. **Novelty/Conclusion:** In this paper we have studied some recurrent properties of neo-pseudo projective curvature tensor in a Finsler space. We have obtained several new results which are as follows:

- If the space F^n admits a neo-pseudo projective curvature tensor $Q^{\alpha}_{\beta\gamma\delta}$ then $Q^{\alpha}_{\beta\gamma\delta}$ is skew-symmetric with regard to last two indices.
- If the neo-pseudo projective deviation tensor Q^{α}_{β} and pseudo deviation tensor field T^{α}_{β} coincides to each other for q = 1 then space is W-flat.
- If F^n admits the projectively flat Q-recurrent space then the relation $\nabla_{\varepsilon} Q^{\alpha}_{\beta\gamma} + \nabla_{\beta} Q^{\alpha}_{\gamma\varepsilon} + \nabla_{\gamma} Q^{\alpha}_{\varepsilon\beta} = 0$ holds good.
- If a Finsler space F^n admits projectively flat Q-birecurrent space then the relation $K_{\varepsilon\rho} Q^{\alpha}_{\beta\gamma} + K_{\beta\rho} Q^{\alpha}_{\gamma\varepsilon} + K_{\gamma\rho} Q^{\alpha}_{\varepsilon\beta} = 0$ holds good.
- If the space is Q-birecurrent then the generalised Q-recurrent space is Q-symmetric.
- For the projective flat generalised Q-recurrent space the relation $\nabla_{\rho}\nabla_{\varepsilon} Q^{\alpha}_{\gamma\delta} + \nabla_{\rho}\nabla_{\gamma} Q^{\alpha}_{\delta\varepsilon} + \nabla_{\rho}\nabla_{\delta} Q^{\alpha}_{\varepsilon\gamma} = 0$ holds good.

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1 Introduction

Let F^n be an n-dimensional Finsler space with a positive definite metric $g_{\alpha\beta}$, which admit a projective deviation tensor field W^α_β and pseudo deviation tensor field T^α_β satisfing

$$Q^\alpha_\beta = p\,W^\alpha_\beta + q\,T^\alpha_\beta \tag{1.1}$$

where in p and q are scalars which are positively homogenous of degree zero in $\dot{x}^\alpha$.

Prof. U.P. Singh and Prof. A.K. Singh while developing the theory of neo-pseudo projective curvature tensor, obtain two kinds of curvature tensor $Q^\alpha_{\beta\gamma}$ and $Q^\alpha_{\beta\gamma\delta}$ [1]. With a view to defining the projective deviation tensor field and pseudo deviation tensor field, he constructed the quantities $Q^\alpha_\beta(x,\dot{x})$ which behave like neo-pseudo projective deviation tensor.

With the help of tensor $Q^\alpha_\beta(x,\dot{x})$ the absolute differential of concerning vector referred to the scalar function $Q(x,\dot{x})$ is defined as follows [1–3]:

$$Q^\alpha_{\beta\gamma} = \frac{1}{3}(\dot{\partial}_\beta Q^\alpha_\gamma - \dot{\partial}_\gamma Q^\alpha_\beta) \tag{1.2}$$

and

$$Q^\alpha_{\beta\gamma\delta} = \dot{\partial}_\beta Q^\alpha_{\gamma\delta} \tag{1.3}$$

It is easy to verify that the neo-pseudo projective curvature tensor satisfies the following relations [1,4]:

$$Q^\alpha_{\beta\gamma\delta} + Q^\alpha_{\gamma\delta\beta} + Q^\alpha_{\delta\beta\gamma} = 0 \tag{1.4}$$

$$Q^\alpha_{\beta\gamma}\,\dot{x}^\beta = Q^\alpha_\gamma \tag{1.5}$$

and

$$Q^\alpha_{\beta\gamma\delta}\,\dot{x}^\beta = Q^\alpha_{\gamma\delta} \tag{1.6}$$

Moreover, these curvature tensor also satisfy the following identities

$$Q^\alpha_{\beta\gamma\delta}\,\dot{x}^\beta\,\dot{x}^\gamma = Q^\alpha_\delta \tag{1.7}$$

and

$$Q^\alpha_\beta\,\dot{x}^\beta = 0 \tag{1.8}$$

As it is well known, in the Finsler space a scalar function $Q(x,\dot{x})$ is given by

$$Q^\alpha_\alpha = (n-1)qT \tag{1.9}$$

Let us consider a curvature tensor W^α_β in Finsler space, is termed as projective curvature tensor in the Finsler space and is defined as follows [2,5,6]:

$$W^\alpha_\beta = H^\alpha_\beta + T^\alpha_\beta \tag{1.10}$$

Wherein H^α_β is positively homogeneous of degree one in $\dot{x}^\alpha$.

In analogy with the relation (1.1) the projective curvature tensors $W^\alpha_{\beta\gamma}$ and $W^\alpha_{\beta\gamma\delta}$ in the Finsler space with the condition p = q = 1 may be defined as follows [2,7]:

$$W^\alpha_\beta = H^\alpha_\beta + Q^\alpha_\beta \tag{1.11}$$

$$W^\alpha_{\beta\gamma} = H^\alpha_{\beta\gamma} + Q^\alpha_{\beta\gamma} \tag{1.12}$$

and

$$W^{\alpha}_{\beta\gamma\delta} = H^{\alpha}_{\beta\gamma\delta} + Q^{\alpha}_{\beta\gamma\delta} \tag{1.13}$$

In view of above discussions, we have the following theorems:

Theorem 1.1:

For the neo-pseudo projective curvature tensor the relation

$$Q^{\alpha}_{\beta\gamma} = -Q^{\alpha}_{\gamma\beta} \tag{1.14}$$

holds good.

Proof:

Interchanging β and γ in equation (1.2) and adding this with new equation, we get the desired result.

Theorem 1.2:

If the space F^n admits a neo-pseudo projective curvature tensor $Q^{\alpha}_{\beta\gamma\delta}$ then $Q^{\alpha}_{\beta\gamma\delta}$ is skew-symmetric with regard to last two indices.

Proof:

Interchanging γ and δ in equation (1.3) and adding the new equation to the equation (1.3), we obtain

$$Q^{\alpha}_{\beta\gamma\delta} + Q^{\alpha}_{\beta\delta\gamma} = \dot{\partial}_{\beta} Q^{\alpha}_{\gamma\delta} + \dot{\partial}_{\beta} Q^{\alpha}_{\delta\gamma} \tag{1.15}$$

From equations (1.14) and (1.15), we get

$$Q^{\alpha}_{\beta\gamma\delta} = -Q^{\alpha}_{\beta\delta\gamma} \tag{1.16}$$

Theorem 1.3:

If the neo-pseudo projective deviation tensor Q^{α}_{β} coincides with geodesic deviation tensor field H^{α}_{β} in the Finsler space F^n then projective deviation tensor field W^{α}_{β} and the neo-pseudo projective deviation tensor Q^{α}_{β} are identically equal to each other.

Proof:

If the neo-pseudo projective deviation tensor Q^{α}_{β} coincides with geodesic deviation tensor field H^{α}_{β}. Consequently, from equation (1.11) follows

$$W^{\alpha}_{\beta} = Q^{\alpha}_{\beta} \tag{1.17}$$

Hence projective deviation tensor field W^{α}_{β} and the neo-pseudo projective deviation tensor Q^{α}_{β} are identically equal to each other.

Theorem 1.4:

If the neo-pseudo projective deviation tensor Q^{α}_{β} and pseudo deviation tensor field T^{α}_{β} coincides to each other for q = 1 then Finsler space admits the condition $W^{\alpha}_{\beta} = 0$ i.e. W-flat.

Proof:

Inseting q = 1 in equation (1.1), we obtain

$$Q^{\alpha}_{\beta} = p\, W^{\alpha}_{\beta} + T^{\alpha}_{\beta} \tag{1.18}$$

If the neo-pseudo projective deviation tensor Q^{α}_{β} and pseudo deviation tensor field T^{α}_{β} coincides to each other then from equation (1.18) we observe that

$$W^{\alpha}_{\beta} = 0 \tag{1.19}$$

This manifests that the space is W-flat.

Theorem 1.5:

If the neo-pseudo projective deviation tensor Q^{α}_{β} and projective deviation tensor field W^{α}_{β} coincides to each other then the geodesic deviation tensor field H^{α}_{β} vanish identically i.e. H-flat.

Proof:

If the neo-pseudo projective deviation tensor Q^{α}_{β} and projective deviation tensor field W^{α}_{β} coincides to each other then from equation (1.11) follows the result

$$H^{\alpha}_{\beta} = 0 \quad (1.20)$$

Consequently, the space is H-flat.

Theorem 1.6:

If the projective deviation tensor field W^{α}_{β} and geodesic deviation tensor field H^{α}_{β} coincides to each other then the neo-pseudo projective deviation tensor Q^{α}_{β} vanish identically i.e. Q-flat.

Proof:

If the projective deviation tensor field W^{α}_{β} and geodesic deviation tensor field H^{α}_{β} coincides to each other. Consequently, from equation (1.11) follows

$$Q^{\alpha}_{\beta} = 0 \quad (1.21)$$

Therefore the space is Q-flat.

2 Recurrent neo-pseudo projective curvature tensor in Finsler space

In view of the investigation of Prof. U.P. Singh and Prof. A.K. Singh[1] we observe that if the neo-pseudo projective deviation tensor Q^{α}_{β} is necessarily recurrent then projective deviation tensor and pseudo deviation tensor are proportional to each other.

As a consequence of this follows the result

$$W^{\alpha}_{\beta} = t\, T^{\alpha}_{\beta} \quad (2.1)$$

wherein t is a scalar.

As a consequence of equations (1.1) and (2.1), we obtain

$$Q^{\alpha}_{\beta} = s\, T^{\alpha}_{\beta} \quad (2.2)$$

wherein s = pt + q is any scalar and positively homogeneous of degree zero in $\dot{x}^{\alpha}$.

Definition 2.1:

A Finsler space whose curvature tensor is recurrent is called Q-recurrent Finsler space.

In view of the definition it follows that for a recurrent space, we have

$$\nabla_{\varepsilon} Q^{\alpha}_{\beta\gamma\delta} = R_{\varepsilon}\, Q^{\alpha}_{\beta\gamma\delta} \quad (2.3)$$

wherein R_{ε} is a non-zero vector termed as the recurrent vector field.

Definition 2.2:

An n-dimensional Finsler space F^n is called Q-symmetric when the covariant derivative of curvature tensor is everywhere zero i.e.

$$\nabla_{\varepsilon} Q^{\alpha}_{\beta\gamma\delta} = 0 \quad (2.4)$$

Definition 2.3:

A Finsler space Fn is said to be Q-flat when its curvature tensor vanishes identically.

As a consequence of this definition follows the result:

$$Q^{\alpha}_{\beta\gamma\delta} = 0 \quad (2.5)$$

Contracting (2.3) with $\dot{x}^{\beta}$ and use of equation (1.6), we obtain

$$\nabla_{\varepsilon} Q^{\alpha}_{\gamma\delta} = R_{\varepsilon}\, Q^{\alpha}_{\gamma\delta} \quad (2.6)$$

Again contracting (2.6) with $\dot{x}^{\gamma}$ and making use of equation (1.5), we get

$$\nabla_{\varepsilon} Q^{\alpha}_{\delta} = R_{\varepsilon}\, Q^{\alpha}_{\delta} \quad (2.7)$$

Thus, we have now the following theorem:

Theorem 2.1:

For the recurrence vector space R_ε in the Finsler space F^n there exists the relation $\left(\nabla_\rho \nabla_\varepsilon - \nabla_\varepsilon \nabla_\rho\right)(logT) = (\nabla_\rho R_\varepsilon - \nabla_\varepsilon R_\rho)$.

Proof:

Differentiating (2.3) covariantly with regard to x^ρ, we get

$$\nabla_\rho \nabla_\varepsilon Q^\alpha_{\beta\gamma\delta} = (\nabla_\rho R_\varepsilon + R_\varepsilon R_\rho) Q^\alpha_{\beta\gamma\delta} \tag{2.8}$$

Interchanging ε and δ in equation (2.8) and subtracting the new equation from equation (2.8), we obtain

$$\nabla_\rho \nabla_\varepsilon Q^\alpha_{\beta\gamma\delta} - \nabla_\varepsilon \nabla_\rho Q^\alpha_{\beta\gamma\delta} = (\nabla_\rho R_\varepsilon - \nabla_\varepsilon R_\rho) Q^\alpha_{\beta\gamma\delta} \tag{2.9}$$

Contracting (2.9) with $\dot{x}^\beta\ \dot{x}^\gamma$ and use of equation (1.7), we get

$$\nabla_\rho \nabla_\varepsilon Q^\alpha_{\delta} - \nabla_\varepsilon \nabla_\rho Q^\alpha_{\delta} = (\nabla_\rho R_\varepsilon - \nabla_\varepsilon R_\rho) Q^\alpha_{\delta} \tag{2.10}$$

Contracting α and δ in equation (2.10) and use of equation (1.9), we obtain

$$\left(\nabla_\rho \nabla_\varepsilon - \nabla_\varepsilon \nabla_\rho\right)(logT) = (\nabla_\rho R_\varepsilon - \nabla_\varepsilon R_\rho) \tag{2.11}$$

Definition 2.4:

If the neo-pseudo projective curvature tensor $Q^\alpha_{\beta\gamma}$ in the Finsler space F^n satisfies the relation

$$\nabla_\varepsilon Q^\alpha_{\beta\gamma} = R_\varepsilon\ Q^\alpha_{\beta\gamma} \tag{2.12}$$

then F^n is termed as Q-recurrent with recurrence vector field R_ε.

Consequently, we have a theorem:

Theorem 2.2:

If F^n admits the projectively flat Q-recurrent space then the relation $\nabla_\varepsilon Q^\alpha_{\beta\gamma} + \nabla_\beta Q^\alpha_{\gamma\varepsilon} + \nabla_\gamma Q^\alpha_{\varepsilon\beta} = 0$ holds good.

Proof:

If the space is projectively flat then from equation (1.12), we have

$$H^\alpha_{\beta\gamma} + Q^\alpha_{\beta\gamma} = 0 \tag{2.13}$$

Differentiating (2.13) covariantly with respect to x^ε, we get

$$\nabla_\varepsilon H^\alpha_{\beta\gamma} + \nabla_\varepsilon Q^\alpha_{\beta\gamma} = 0 \tag{2.14}$$

Taking the cyclic permutation in β, γ, ε and adding, we have

$$\left(\nabla_\varepsilon H^\alpha_{\beta\gamma} + \nabla_\beta H^\alpha_{\gamma\varepsilon} + \nabla_\gamma H^\alpha_{\varepsilon\beta}\right) + (\nabla_\varepsilon Q^\alpha_{\beta\gamma} + \nabla_\beta Q^\alpha_{\gamma\varepsilon} + \nabla_\gamma Q^\alpha_{\varepsilon\beta}) = 0 \tag{2.15}$$

The first part of equation (2.15) vanishes due to commutation formula ([8], equation (6.13), p.128), hence we obtain

$$\nabla_\varepsilon Q^\alpha_{\beta\gamma} + \nabla_\beta Q^\alpha_{\gamma\varepsilon} + \nabla_\gamma Q^\alpha_{\varepsilon\beta} = 0 \tag{2.16}$$

3 Neo-Pseudo Projective Recurrent Space:

Definition 3.1:

Neo-pseudo projective curvature tensor $Q^\alpha_{\beta\gamma\delta}$ of a Finsler space satisfies the relation

$$\nabla_\rho \nabla_\varepsilon Q^\alpha_{\beta\gamma\delta} = K_{\varepsilon\rho} Q^\alpha_{\beta\gamma\delta} \tag{3.1}$$

wherein $K_{\varepsilon\rho}$ is non-zero recurrent tensor, then it is called neo-pseudo projective recurrent space of second order or briefly a Q-birecurrent space[3,4].

Definition 3.2:

If the covariant derivative of neo-pseudo projective curvature tensor $Q^{\alpha}_{\beta\gamma\delta}$ vanishes identically then the space is termed as Q-bisymmetric.

As a consequence of above definition follows the result

$$\nabla_{\rho}\nabla_{\varepsilon}Q^{\alpha}_{\beta\gamma\delta}=0 \tag{3.2}$$

In this regard we shall now establish the following theorem:

Theorem 3.1:

The necessary and sufficient condition for Finsler space to be Q-bisymmetric space is that the neo-pseudo projective tensor vanishes identically.

Proof:

Since neo-pseudo projective tensor vanishes i.e. $Q^{\alpha}_{\beta\gamma\delta}=0$. Consequently from equation (3.1) it follows that $\nabla_{\rho}\nabla_{\varepsilon}Q^{\alpha}_{\beta\gamma\delta}=0$. This manifests that the space to be Q-bisymmetric.

Conversely, if the space to be Q-bisymmetric then the converse of theorem is immediately proof.

Remark 3.1:

It is noteworthy that every Q-recurrent is necessarily Q-birecurrent.

Theorem 3.2:

In a Finsler space F^n, the recurrent tensor field $K_{\varepsilon\rho}$ is not symmetric in general.

Proof:

Contracting α and δ in equation (3.1) yields

$$\nabla_{\rho}\nabla_{\varepsilon}Q_{\beta\gamma}=K_{\varepsilon\rho}Q_{\beta\gamma} \tag{3.3}$$

Interchanging ε and ρ in equation (3.3) and subtracting the new equation from equation (3.3), we obtain

$$\nabla_{\rho}\nabla_{\varepsilon}Q_{\beta\gamma}-\nabla_{\varepsilon}\nabla_{\rho}Q_{\beta\gamma}=(K_{\varepsilon\rho}-K_{\rho\varepsilon})Q_{\beta\gamma} \tag{3.4}$$

Contracting (3.4) with $\dot{x}^{\beta}\ \dot{x}^{\gamma}$ and use of equation (1.7), we get

$$\nabla_{\rho}\nabla_{\varepsilon}Q-\nabla_{\varepsilon}\nabla_{\rho}Q=(K_{\varepsilon\rho}-K_{\rho\varepsilon})Q \tag{3.5}$$

Using commutation formula ([5], equation (6.10), p.126) and equation (2.2), we obtain

$$\left(K_{\varepsilon\rho}-K_{\rho\varepsilon}\right)Q=(\dot{\partial}_{\sigma}Q)H^{\sigma}_{\rho\varepsilon} \tag{3.6}$$

From equations (1.9) and (3.6), consequently, follows

$$K_{\varepsilon\rho}-K_{\rho\varepsilon}=\dot{\partial}_{\sigma}\left(logT\right)H^{\sigma}_{\rho\varepsilon} \tag{3.7}$$

Yields the result

$$K_{\varepsilon\rho}\neq K_{\rho\varepsilon} \tag{3.8}$$

Theorem 3.3:

If a Finsler space F^n admits projectively flat Q-birecurrent space then the relation $K_{\varepsilon\rho}Q^{\alpha}_{\beta\gamma}+K_{\beta\rho}Q^{\alpha}_{\gamma\varepsilon}+K_{\gamma\rho}Q^{\alpha}_{\varepsilon\beta}=0$ holds good.

Proof:

Differentiating equation (2.16) covariantly with respect to x^{ρ}, we get

$$\nabla_{\rho}\nabla_{\varepsilon}Q^{\alpha}_{\beta\gamma}+\nabla_{\rho}\nabla_{\beta}Q^{\alpha}_{\gamma\varepsilon}+\nabla_{\rho}\nabla_{\gamma}Q^{\alpha}_{\varepsilon\beta}=0 \tag{3.9}$$

Since the space is Q-birecurrent then equation (3.2) assumes the form

$$K_{\varepsilon\rho}Q^{\alpha}_{\beta\gamma}+K_{\beta\rho}Q^{\alpha}_{\gamma\varepsilon}+K_{\gamma\rho}Q^{\alpha}_{\varepsilon\beta}=0 \tag{3.10}$$

4 Generalised Neo-Pseudo Projective Recurrent Space:

Let us consider the relation

$$\nabla_\rho \nabla_\varepsilon Q^\alpha_{\beta\gamma\delta} = R_\rho \; \nabla_\varepsilon Q^\alpha_{\beta\gamma\delta} + K_{\varepsilon\rho} \; Q^\alpha_{\beta\gamma\delta} \tag{4.1}$$

wherein R_ρ and $K_{\varepsilon\rho}$ are recurrence vector and recurrence tensor fields respectively.

Definition 4.1:

The neo-pseudo projective curvature tensor $Q^\alpha_{\beta\gamma\delta}$ of Finsler space F^n satisfying the condition (4.1) is called generalised neo-pseudo projective recurrent curvature tensor[(3,4)].

Definition 4.2:

Finsler space F^n equipped with the generalised neo-pseudo projective recurrent curvature tensor $Q^\alpha_{\beta\gamma\delta}$ is called generalised neo-pseudo projective recurrent Finsler space[(3,9)].

In this regard, we have the following theorems:

Theorem 4.1:

The necessary and sufficient condition for Finsler space F^n to be Q-symmetric is that the space has to be Q-birecurrent.

Proof:

If the space is to be Q-symmetric i.e. $\nabla_\varepsilon Q^\alpha_{\beta\gamma\delta} = 0$,

Consequently, from equation (4.1) follows

$$\nabla_\rho \nabla_\varepsilon Q^\alpha_{\beta\gamma\delta} = K_{\varepsilon\rho} \; Q^\alpha_{\beta\gamma\delta} \tag{4.2}$$

which is the condition of Q-birecurrent.

Conversely, let us assume that the space be Q-birecurrent, follows the condition (4.2). Inserting equation (4.2) into equation (4.1), we obtain $\nabla_\varepsilon Q^\alpha_{\beta\gamma\delta} = 0$. Hence the space is Q-symmetric.

Theorem 4.2:

If the space F^n is Q-symmetric and Q-flat then its generalised neo-pseudo projective recurrent space vanishes identically.

Proof:

If the space be Q-symmetric i.e. $\nabla_\varepsilon Q^\alpha_{\beta\gamma\delta} = 0$ and Q-flat i.e. $Q^\alpha_{\beta\gamma\delta} = 0$. Consequently, from equation (4.1) follows $\nabla_\rho \nabla_\varepsilon Q^\alpha_{\beta\gamma\delta} = 0$. This establishes the validity of the theorem.

Remark 4.1:

It is noteworthy that if F^n to be Q-symmetric and Q-flat follows that the generalised neo-pseudo projective recurrent space necessarily vanishes. Consequently, the space is simply generalised Q-symmetric one.

Theorem 4.3:

If space F^n admits Q-symmetric and Q-flat then the space is a generalised Q-symmetric one.

Proof:

It follows immediately from theorem 4.2.

Theorem 4.4:

In Finsler space F^n, if the space is Q-birecurrent then the generalised Q-recurrent space is Q-symmetric.

Proof:

It is obvious from equations (2.4), (3.1) and (4.1).

Theorem 4.5:

For the recurrence vector R_ρ the relation

$$R_\rho \nabla_\varepsilon T - R_\varepsilon \nabla_\rho T = 0 \tag{4.3}$$

holds good.

Proof:

Contracting (4.1) by α and δ, we obtain

$$\nabla_\rho \nabla_\varepsilon Q_{\beta\gamma} = R_\rho \nabla_\varepsilon Q_{\beta\gamma} + K_{\varepsilon\rho} Q_{\beta\gamma} \tag{4.4}$$

Interchanging ε and ρ in equation (4.4) and subtracting the new equation from equation (4.4), we get

$$\nabla_\rho \nabla_\varepsilon Q_{\beta\gamma} - \nabla_\varepsilon \nabla_\rho Q_{\beta\gamma} = R_\rho \nabla_\varepsilon Q_{\beta\gamma} - R_\varepsilon \nabla_\rho Q_{\beta\gamma} + (K_{\varepsilon\rho} - K_{\rho\varepsilon}) Q_{\beta\gamma} \tag{4.5}$$

Contracting (4.5) with $\dot{x}^{\beta}\ \dot{x}^{\gamma}$ and use of equation (1.7), we get

$$\nabla_{\rho}\nabla_{\varepsilon}Q-\nabla_{\varepsilon}\nabla_{\rho}Q=R_{\rho}\nabla_{\varepsilon}Q-R_{\varepsilon}\nabla_{\rho}Q+(K_{\varepsilon\rho}-K_{\rho\varepsilon})Q \tag{4.6}$$

By virtue of equations (3.5) and (4.6), we get

$$R_{\rho}\nabla_{\varepsilon}Q-R_{\varepsilon}\nabla_{\rho}Q=0 \tag{4.7}$$

Inserting equation (1.9) in equation (4.7), we get the desired result.

Theorem 4.6:

For the projective flat generalised Q-recurrent space the relation $\nabla_{\rho}\nabla_{\varepsilon}Q^{\alpha}_{\gamma\delta}+\nabla_{\rho}\nabla_{\gamma}Q^{\alpha}_{\delta\varepsilon}+\nabla_{\rho}\nabla_{\delta}Q^{\alpha}_{\varepsilon\gamma}=0$ holds good.

Proof:

Contracting (4.1) with $\dot{x}^{\beta}$, we obtain

$$\nabla_{\rho}\nabla_{\varepsilon}Q^{\alpha}_{\gamma\delta}=R_{\rho}\ \nabla_{\varepsilon}Q^{\alpha}_{\gamma\delta}+K_{\varepsilon\rho}\ Q^{\alpha}_{\gamma\delta} \tag{4.8}$$

Taking the cyclic permutation in ε, γ, δ and on making use of equations (2.16) and (3.10), we observe that

$$\nabla_{\rho}\nabla_{\varepsilon}Q^{\alpha}_{\gamma\delta}+\nabla_{\rho}\nabla_{\gamma}Q^{\alpha}_{\delta\varepsilon}+\nabla_{\rho}\nabla_{\delta}Q^{\alpha}_{\varepsilon\gamma}=0 \tag{4.9}$$

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