Direct Product of GK Algebra

J Kavitha1,2 *, R Gowri3

1 Assistant Professor, Department of Mathematics, Dwaraka Doss Goverdhan Doss Vaishnav College (Autonomous), Chennai, India
2 Research Scholar, Department of Mathematics, Government College for women (Autonomous), Affiliated to Bharathidasan University, Kumbakonam, India
3 Assistant Professor, Department of Mathematics, Government College for women (Autonomous), Kumbakonam, India

Abstract

Objectives: To find the direct product of an algebraic structure namely as GK algebra. Methods/Findings: We derive some important results in which direct product of two GK algebra is again GK algebra as a particular case and also, derive the general case of the same then after investigate the direct product of kernel of GK algebra.

Keywords: Direct Product; Kernel; isomorphism; Homomorphism; GK algebra

1 Introduction

BCK-algebras and BCI-algebras are abridged to two B-algebras. The BCK algebra was coined in 1966 by the Japanese mathematicians, Y. Imai and K. Iseki1. Two B-algebras are created from two different provenances. In 2007, the new algebraic structure which is said to be BF algebra, was explored by Andrzej J Walendziak2 which is a generalization of BCI/BCK/B-algebras. In 2008, the generalization of B algebra called as BG algebra initiated by Kim & Kim3. In 2009, another algebra which is generalization of BE algebra and dual BCK/BCI/BCH algebras, namely CI algebra was initiated by Biao long Meng4.

Direct product plays an important role in algebraic structures. In 2019, Slamet Widianto, Sri Gemawati, Kartini5–7 were discussed about the Direct product of BG algebra. Likewise, many authors have discussed this topic in their work. Motivated by these, in this paper we discuss about direct product of GK algebra and obtain its some interesting results. In 2018, we introduced the new algebraic structure namely GK algebra8 and discussed about its characteristics and investigated some results. In this paper we discuss about the direct product of GK algebra and investigate its properties.

2 Direct product of GK algebra

2.1 Definition

Let \((M, \oplus, I_M)\) and \((N, \circ, I_N)\) be GK algebras. Direct product \(M \times N\) is defined as a structure \(M \times N = (M \times N; \oplus, \circ, \{(1_M; I_N)\})\), where \(M \times N\) is the set \(\{(m, n) | m \in M, n \in N\}\)
and ⊗ is given by

\[(m_1, n_1) \otimes (m_2, n_2) = (m_1 \otimes m_2, n_1 \otimes n_2)\]

This shows that the direct product of two sets of GK algebra \(M\) and \(N\) is denoted by \(M \times N\), which each \((m, n)\) is an ordered pair.

### 2.2 Theorem

Direct product of any two GK algebras is again a GK algebra.

**Proof:**

Let \(M\) and \(N\) be GK algebras, let \(m_1, m_2 \in M\) and \(n_1, n_2 \in N\). We know that \(M \times N = (M \times N; \otimes; (1_M; 1_N))\). Since \(1_M \in M, 1_N \in N\), this implies that \((1_M, 1_N) \in M \times N\).

\[\therefore M \times N\text{ is non-empty.}\]

Now let us prove it is GK algebra.

Let \(m_1, m_2 \in M\) and \(n_1, n_2 \in N\)

1. \((m_1, n_1) \otimes (m_1, n_1) = (m_1 \otimes m_1, n_1 \otimes n_1) = (1_M, 1_N)\) by definition of GK algebra
2. \((m_1, n_1) \otimes (1_M, 1_N) = (m_1 \otimes 1_M, n_1 \otimes 1_N) = (m_1, n_1)\) by definition GK algebra
3. If \((m_1, n_1) \otimes (m_2, n_2) = (1_M, 1_N)\) and \((m_2, n_2) \otimes (m_1, n_1) = (1_M, 1_N)\)
   then \((m_1 \otimes m_2, n_1 \otimes n_2) = (1_M, 1_N)\)
   \(\implies m_1 \otimes m_2 = 1_M\ and\ n_1 \otimes n_2 = 1_N\)
   \(\implies m_1 = m_2\ and\ n_1 = n_2\) by definition GK algebra.
4. \([m_2, n_2] \otimes (m_3, n_3) \otimes ([m_1, n_1] \otimes [m_2, n_2])\)
   \(\implies (m_2 \otimes m_3, n_2 \otimes n_3) \otimes (m_1 \otimes m_3, n_1 \otimes n_3)\)
   \(\implies \{([m_2 \otimes m_3] \otimes (m_1 \otimes m_3)) \otimes ([n_2 \otimes n_3] \otimes (n_1 \otimes n_3))\}\)
   \(\implies (m_2 \otimes m_1, n_2 \otimes n_1)\)
   \(\implies (m_2, n_2) \otimes (m_1, n_1)\).
5. \([m_1, n_1] \otimes (m_2, n_2) \otimes [1_M, 1_N] \otimes (m_2, n_2)\]
   \(\implies [(m_1 \otimes m_2), (n_1 \otimes n_2)] \otimes [(1_M \otimes m_2), (1_N \otimes n_2)]\)
   \(\implies (m_1 \otimes m_2) \otimes (1_M \otimes m_2), [n_1 \otimes n_2] \otimes (1_N \otimes n_2)]\)
   \(\implies (m_1 \otimes 1_M, n_1 \otimes 1_N)\)
   \(\implies (m_1, n_1)\)

Hence \(M \times N\) is a GK algebra.

### 2.3 Theorem

Let \(\{M_i, (M_i \otimes; 1) : i = 1, 2, 3, \ldots n\}\) and \(\{N_i, (N_i \otimes; 1) : i = 1, 2, 3, \ldots n\}\) be the family of GK algebras and let \(\zeta_i : M_i \rightarrow N_i, i = 1, 2, 3, \ldots n\) be the set of isomorphism.

If \(\zeta\) from \(\prod \{M_i, (M_i \otimes; 1) : i = 1, 2, 3, \ldots n\}\) given by \(\zeta_i (m_i), (i = 1, 2, 3, \ldots n) = \zeta_i (m_i), i = 1, 2, 3, \ldots n\), then \(\zeta\) is also an isomorphism.

**Proof**:

Let \(\{M_i, (M_i \otimes; 1) : i = 1, 2, 3, \ldots n\}\) and \(\{N_i, (N_i \otimes; 1) : i = 1, 2, 3, \ldots n\}\) be the family of GK algebras and let \(\zeta_i : M_i \rightarrow N_i, i = 1, 2, 3, \ldots n\) be the set of isomorphism.

To prove \(\zeta\) is an isomorphism.

If \((m_1, n_1) \in \prod M_i\), then \(\zeta\ [(m_1, m_2, \ldots m_n) \otimes (n_1, n_2, \ldots n_n)]\)

\(= \zeta [m_1 \otimes m_2 \otimes \ldots m_n \otimes n_n]\)

\(= (\xi_1 (m_1 \otimes n_1), \xi_2 (m_2 \otimes n_2), \ldots, \xi_n (m_n \otimes n_n))\)

\(= ((\xi_1 (m_1), \xi_2 (m_2), \ldots, \xi_n (m_n))) \otimes ((\xi_1 (n_1), \xi_2 (n_2), \ldots, \xi_n (n_n)))\)

\(= \zeta (m_1, m_2, \ldots, m_n) \otimes \xi (n_1, n_2, \ldots, n_n)\)
Therefore \( \zeta \) is a homomorphism.

We have to prove \( \zeta \) is onto, we have \( \zeta \) is onto, where \( i=1,2,3\ldots n \).

Let \( (n_1, n_2, \ldots, n_m) \in N_1 \times N_2 \times \ldots \times N_n \)

\[ \implies \text{Since } \zeta \text{ is onto, } n_i \in N_i, \text{ there exists } m_i \in M_i \text{ such that } \zeta(m_i) = n_i \text{ for } i = 1,2,3\ldots n \]

\[ \implies (n_1, n_2, \ldots, n_n) = [(\zeta_1(m_1), \zeta_2(m_2), \ldots, \zeta_n(m_n))] = (\zeta(m_1, m_2, \ldots, m_n)) \]

\[ \implies \zeta \text{ is onto.} \]

Now, to prove \( \zeta \) is 1-1.

\[ \zeta(m_1, m_2, \ldots, m_n) = \zeta(n_1, n_2, \ldots, n_n) \]

\[ [(\zeta_1(m_1), \zeta_2(m_2), \ldots, \zeta_n(m_n))] = [(\zeta_1(n_1), \zeta_2(n_2), \ldots, \zeta_n(n_n))] \]

\[ \implies \zeta_i(m_i) = \zeta_i(n_i) \]

\[ \implies m_i = n_i \text{, where } i=1,2,3\ldots n, \text{ since } \zeta_i \text{ is 1-1.} \]

\[ \implies (m_1, m_2, \ldots, m_n) = (n_1, n_2, \ldots, n_n) \]

\[ \implies \zeta \text{ is 1-1.} \]

Hence \( \zeta \) is an isomorphism.

### 2.4 Theorem

Let \( M_i, N_i \), \( i = 1,2 \) be GK algebras. Consider the mapping \( \zeta_1 : M_1 \rightarrow N_1 \) and

\[ \zeta_2 : M_2 \rightarrow N_2 \text{ where } \zeta_1, \zeta_2 \text{ are homomorphisms. If the map } \zeta : M_1 \times M_2 \rightarrow N_1 \times N_2 \text{ given by} \]

\[ \zeta(m_1, m_2) = (\zeta_1(m_1), \zeta_2(m_2)), \text{ then} \]

1. \( \zeta \) is a homomorphism.
2. \( \text{Ker } \zeta = \text{kern } \zeta_1 \times \text{kern } \zeta_2. \)

**Proof:**

Let us consider the mapping \( \zeta_1 : M_1 \rightarrow N_1 \) and \( \zeta_2 : M_2 \rightarrow N_2 \) where \( \zeta_1, \zeta_2 \) are homomorphisms.

If the map \( \zeta : M_1 \times M_2 \rightarrow N_1 \times N_2 \) given by \( \zeta(m_1, n_1) = (\zeta_1(m_1), \zeta_2(n_2)) \),

for \( m_1, m_2 \in M_1 \text{ and } n_1, n_2 \in M_2 \) then

- \( \zeta \) \((m_1, n_1) \odot (m_2, n_2)) = \zeta(m_1 \odot m_2, n_1 \odot n_2) = (\zeta_1(m_1 \odot m_2), \zeta_2(n_1 \odot n_2)) = (\zeta_1(m_1), \zeta_2(n_1)) \odot (\zeta_1(m_2), \zeta_2(n_2)) = \zeta_1(m_1, n_1) \odot \zeta_2(m_2, n_2)

Therefore \( \zeta \) is a homomorphism.

- Let \((m, n) \in \text{ker } \zeta \iff \zeta(m, n) = (1_{M_1}, 1_{M_2}) \]

\[ \iff (\zeta_1(m), \zeta_2(n)) = (1_{M_1}, 1_{M_2}) \]

\[ \iff \zeta_1(m) = 1_{M_1}, \zeta_2(n) = 1_{M_2} \]

\[ \iff m \in \text{ker } \zeta_1, n \in \text{ker } \zeta_2 \]

\[ \iff (m, n) \in \text{ ker } \zeta_1 \times \text{ ker } \zeta_2. \]

Hence \( \text{Ker } \zeta = \text{ker } \zeta_1 \times \text{ ker } \zeta_2. \)

### 3 Conclusion

In this article we discussed about the concept of the direct product of GK algebra. We derived the finite form of direct product of GK algebra is isomorphism and also, we investigated and applied the concept of direct product of GK algebra in GK homomorphism and GK kernel, then obtained interesting results.
References


