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Alpha Power Transformed Aradhana Distributions, Its Properties and Applications

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Abstract

Objectives: To introduce a new two-parameter lifetime distribution that will be more flexible in modeling real lifetime data over the existing common lifetime distributions. **Methods:** The new two-parameter lifetime distribution is generated by using the Alpha Power Transformed model developed by Mahdavi and Kundu. In this method, the probability density function and cumulative distribution function of Aradhana distribution are used as a base distribution for generating Alpha Power Transformed Aradhana Distribution. The probability density function and cumulative distribution function of the Aradhana distribution are substituted in the Alpha Power Transformed model to get the new and more flexible lifetime distribution for modeling real-life data. **Findings:** The authors reveal that the hazard rate of the Alpha Power Transformed Aradhana is increasing. They also found that the Alpha Power Transformed Aradhana distribution gives a much close fit than the two-parameter Aradhana distribution, Power Aradhana, Length Biased Garima, Exponential and Garima distribution. **Novelty :** In this study, a novel probability distribution is introduced. The Alpha Power Transformed Aradhana distribution is capable of modeling upside-down bathtub shaped hazard rates. The model is appropriate to fit the asymmetrical data that are not correctly fitted by other distributions. The said distribution can be applied to different fields like insurance, earthquake data for analysis, reliability etc.

Keywords: Reliability Analysis; Moments; Parameter Estimation

1 Introduction

The statistical analysis and fitting of lifetime data are important in all applied fields. The number of distributions of one and two parameters for modeling lifetime data is proposed by many researchers using different methods. The new lifetime model and regression model were discussed by Yousof et al. [1] and studied the characterization of the model with the application. A new Two-Parameter lifetime model is discussed by Alizadeh et al. [2] and studied the properties of the distribution with the application. A new generalization of the lifetime model based on the generalization of the half-normal

distribution are introduced by Altun, Yousof et al. [3] along with the properties of the model and regression models. Another generalization of the generalized exponential distribution along with properties and applications are developed by Zelibe et al. [4]. Para and Jan [5] discussed the new lifetime distribution a generalization of Log-Logistic distribution for modeling medical data and obtained various properties of the model. Para and Jan [6] discussed the three-parameter weighted Pareto Type II distribution and studied the properties with application in medical sciences.

Since the 1980s, researchers have diverted their attention from existing approaches to adding a parameter to existing distributions. The approach/method used by the researchers for generating new distributions were the beta-generated method, Transformed-Transformer method, Exponentiated generalized method etc. These methods were applied to the existing distributions for generating the new distributions which can overcome the lacking of the existing distributions.

Shanker [7] introduced a new one-parameter distribution viz Aradhana distribution for modeling lifetime data. Let Y be the random variable having Aradhana distribution with parameter θ, then the probability density function and cumulative distribution function of the distribution is given by

$$f(y,\theta) = \frac{\theta^3}{\theta^2 + 2\theta + 2}(1+y)^2 e^{-\theta y}, \quad y > 0, \theta > 0 \tag{1}$$

$$F(y,\theta) = 1 - \left(1 + \frac{\theta y(\theta y + 2\theta + 2)}{\theta^2 + 2\theta + 2}\right) e^{-\theta y}, \quad y > 0, \theta > 0 \tag{2}$$

The Aradhana distribution is a combination of two components, exponential distribution (θ) and gamma distribution (3, θ). The author has discussed various mathematical properties of the distribution. The parameters of the Aradhana distribution are estimated by using the method of Maximum Likelihood estimation. In this article, the real lifetime data set is used to check the validity of the Aradhana distribution over one and two-parameter distributions. The authors found that the Aradhana distribution gives a much closer fit in modeling life-time data than Akash, Exponential, Shanker, Lindley distributions. Some of the modifications of the Aradhana distribution and their properties are introduced by Shanker.

Shanker and Welday [8] introduced a generalized Aradhana distribution by using the concept of mixture models for modeling the data sets. The probability density function and cumulative distribution function of the two-parameter Aradhana distribution with parameters θ and α, is given by

$$f(y,\theta,\alpha) = \left(\frac{\theta^3}{\theta^2 + 2\alpha\theta + 2\alpha^2}\right)(1+\alpha y)^2 e^{-\theta y}, \, y > 0, \theta > 0, \alpha \geq 0$$

$$F(y,\theta,\alpha) = 1 - \left(1 + \frac{\theta\alpha y(2\theta + \theta\alpha y + 2\alpha)}{\theta^2 + 2\theta\alpha x + 2\alpha^2}\right) e^{-\theta y}$$

The said distribution is a combination of one-parameter exponential (θ) distribution, gamma (3,θ) distribution and gamma (2,θ) distribution. The Aradhana distribution is a special case of two-parameter Aradhana distribution. The properties of the distribution are discussed by the authors. The parameters of the two-parameter Aradhana distribution are estimated by using the method of Maximum Likelihood estimation. The authors used real lifetime data to verify the validity and flexibility of the Aradhana distribution on one and two-parameter distributions.

Quasi Aradhana distribution was proposed by Shanker and Shukla [9] for modeling engineering data set. The probability density function and cumulative distribution function of the distribution having parameters θ and α is given by

$$f(y,\theta,\alpha) = \left(\frac{\theta}{\alpha^2 + 2\alpha + 2}\right)(\alpha + \theta y)^2 e^{-\theta y}, \, y > 0, \theta > 0, \alpha^2 + 2\alpha + 2 > 0$$

$$F(y,\theta,\alpha) = 1 - \left(1 + \frac{\theta y(\theta y + 2 + 2\alpha)}{\alpha^2 + 2\alpha + 2}\right) e^{-\theta y}$$

The Quasi Aradhana distribution is the mixture of exponential (θ), a gamma (2, θ), and gamma (3, θ) distribution. Various mathematical properties are discussed by the authors. The validation of the distribution is verified by using datasets and compared with gamma, Weibull, Lognormal, Aradhana, Lindley and exponential distributions.

A two-parameter power Aradhana distribution was discussed by Shanker and Shukla [10]. They have obtained the said distribution by the method of power transformation $Y = X^{\frac{1}{\alpha}}$.the probability density function and cumulative distribution

function of the distribution is given by

$$f(y, \theta, \alpha) = \left(\frac{\alpha\theta^2}{\theta^2 + 2\theta + 2}\right)(y)^{\alpha-1}\left(y^{2\alpha} + 2y^{\alpha} + 1\right)e^{-\theta y^{\alpha}} , \ y > 0, \theta > 0, \alpha > 0$$

The two-parameter power Aradhana distribution is a mixture of Weibull and generalized gamma distribution. The pdf of generalised gamma distribution introduced by Stacy (1962)[11] is given by

$$f(y, \beta, \theta, \alpha) = \frac{\beta}{\theta\Gamma\alpha}\left(\frac{y}{\theta}\right)^{\alpha\beta-1}e^{-(y/\theta)^{\beta}} , \ y \geq 0, \ \beta, \theta > 0, \alpha > 0$$

The purpose of this study is to develop another generalization of the Aradhana distribution called the Alpha Power Transformed Aradhana (APTAT) distribution using the APT model proposed by Mahdavi and Kundu[12]. The cdf of the APT family of distributions as

$$F_{APT}(y) = \left\{\begin{array}{ll} \left(\dfrac{\alpha^{F(y)} - 1}{\alpha - 1}\right), & \text{if } \alpha > 0, \alpha \neq 1, y \in \mathbb{R} \\ F(y) & , \text{if } \alpha > 0, \alpha = 1, y \in \mathbb{R} \end{array}\right\} \tag{3}$$

The corresponding probability density function as

$$f_{APT}(y) = \left\{\begin{array}{ll} \left(\dfrac{\log\alpha}{\alpha - 1}\right)f(y)\alpha^{F(y)}, & \text{if } \alpha > 0, \alpha \neq 1, y \in \mathbb{R} \\ f(y) & \text{if } \alpha > 0, \alpha = 1, y \in \mathbb{R} \end{array}\right\} \tag{4}$$

Recently, some extensive work has been done on the Alpha Power family of distributions. Alpha power Transformed Frechet Distribution was introduced by Suleman et al.[13] for modeling real-life data sets. Some of the statistical properties of the distribution such as quantile function, moments, mean residual life, generating function, entropy, stochastic ordering, etc. are discussed. The method of maximum likelihood estimation is used for estimating the parameters of the distribution.

The Alpha Power Transformed family, properties and Applications was developed by Mead, Cordeiro, Afify and Al-Mofleh[14]. The author(s) reveal that the probability density function of the Alpha Power Transformation is the weighted function of the probability density function of the base distribution, here $\alpha^{G(x)}$ is the weight function for the model $g(x)$. The Statistical properties of the APT model are studied such as Linear representation of the model, Moments, Incomplete moments, moment generating function and order statistics. Further, the work was extended to introduce the Alpha Power Exponentiated Weibull distribution. The Alpha Power Exponentiated Weibull distribution uses the cdf of the Exponentiated Weibull distribution in the APT model. The Statistical properties of the APEW model is studied. This includes linear representation of the model, Moments, Incomplete moments, moment generating function and order statistics. A Simulation Study is also carried out to investigate the behaviour of the MLEs for different sample size.

Alpha Power Transformed Power Lindley distribution was suggested by Hassan[15]. Some statistical properties of the model are discussed such as quantile function, moments, probability weighted moments and stochastic ordering. The parameters of the model are estimates by maximum likelihood estimator and Maximum Product of Spacing Estimators.

Moreover, Ghosh[16] introduced and studied properties of the Alpha Power Transformed Lindley distribution with application to Earthquake data. Alpha Power Transformed Inverse Power Lindley distribution was proposed by Nassar and Kumar[17] and discussed properties of the model with the application. Alpha Power Transformed Quasi Lindley distribution was introduced by Patrick and Harrison[18] and studied the properties of the model. Alpha Power Transformed Weibull-G was proposed by Golam Kibria[19] with application to failure data. The distribution is generated by combining the two families of distributions APT-G family and Weibull-G family. The statistical properties of the APTW-G are derived and discussed. The APTW-G reduces to Alpha power transformed Weibull exponential distribution, Alpha power transformed Weibull Rayleigh distribution and Alpha power transformed Weibull Lindley distribution. Alpha Power Transformed Pareto distribution was introduced by Sakthivel[20] and studied various properties of the distribution with the application. Alpha Power Transformed extended exponential distribution was proposed by Hassan[21] and studied the properties of the model with the application. The alpha power extended exponential distribution reduces to alpha power exponential, alpha power Lindley, exponential, Lindley and gamma distribution. Alpha Power Transformed Inverse Lomax distribution was suggested by Hassan[1] and studied the properties and different methods for estimating the parameters.

2 Alpha Power Transformed Aradhana (APTAD) Distribution

Let Y be a random variable which is distributed as Alpha Power Transformed Aradhana distribution (APTAD) with scale parameter θ and shape parameter α, and denoted by APTAD(θ, α). The pdf of the APTAD is obtained by substituting Equation (1) and Equation (2) in Equation (4) is given as

$$f_{APT}(y, \theta, \alpha) = \left(\frac{\log \alpha}{\alpha - 1} \right) \frac{\theta^3}{\theta^2 + 2\theta + 2} (1+y)^2 e^{-\theta y} \alpha^{1 - \left(1 + \frac{\theta y(\theta y + 2\theta + 2)}{\theta^2 + 2\theta + 2}\right) e^{-\theta y}} \quad y > 0, \theta, \alpha > 0, \alpha \neq 1 \tag{5}$$

The corresponding cumulative distribution of APTAD is obtained by substituting Equation (2) in Equation (3) as

$$F_{APT}(y, \theta, \alpha) = \frac{\alpha^{1 - \left(1 + \frac{\theta y(\theta y + 2\theta + 2)}{\theta^2 + 2\theta + 2}\right) e^{-\theta y}} - 1}{\alpha - 1} \quad \alpha > 0, \alpha \neq 1 \tag{6}$$
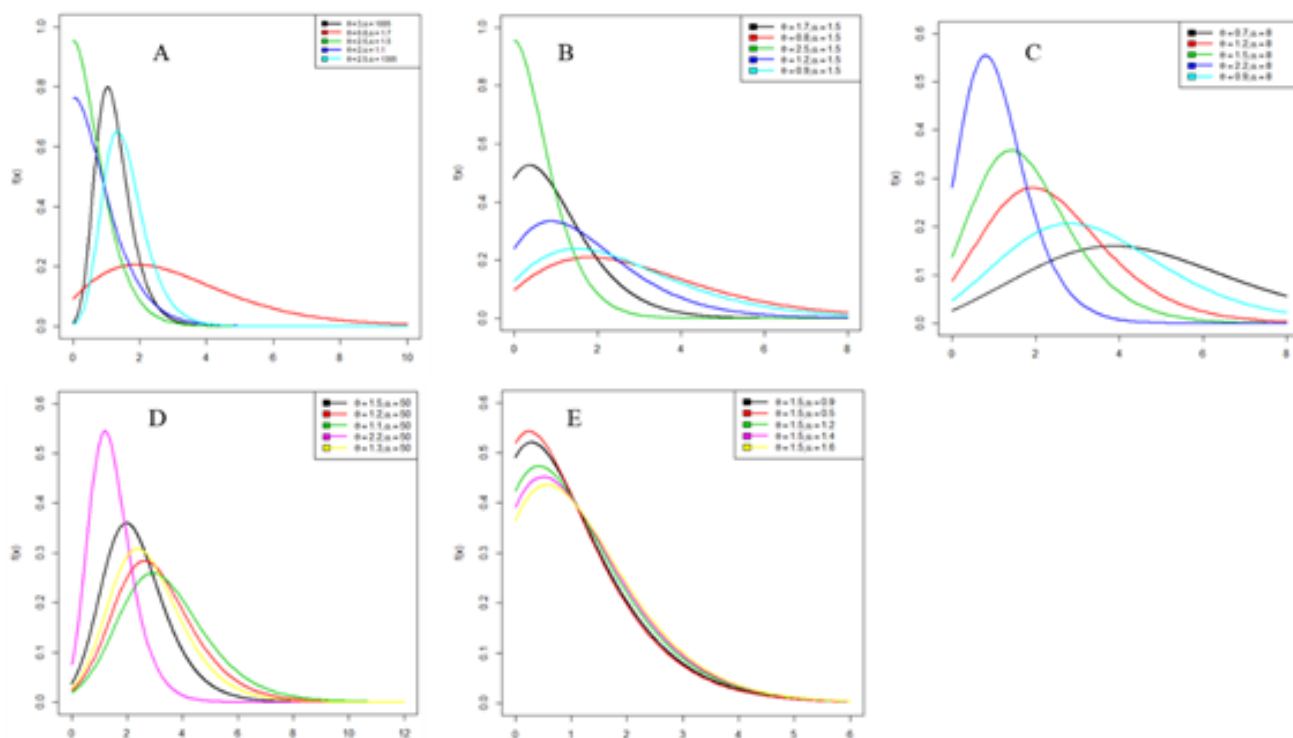

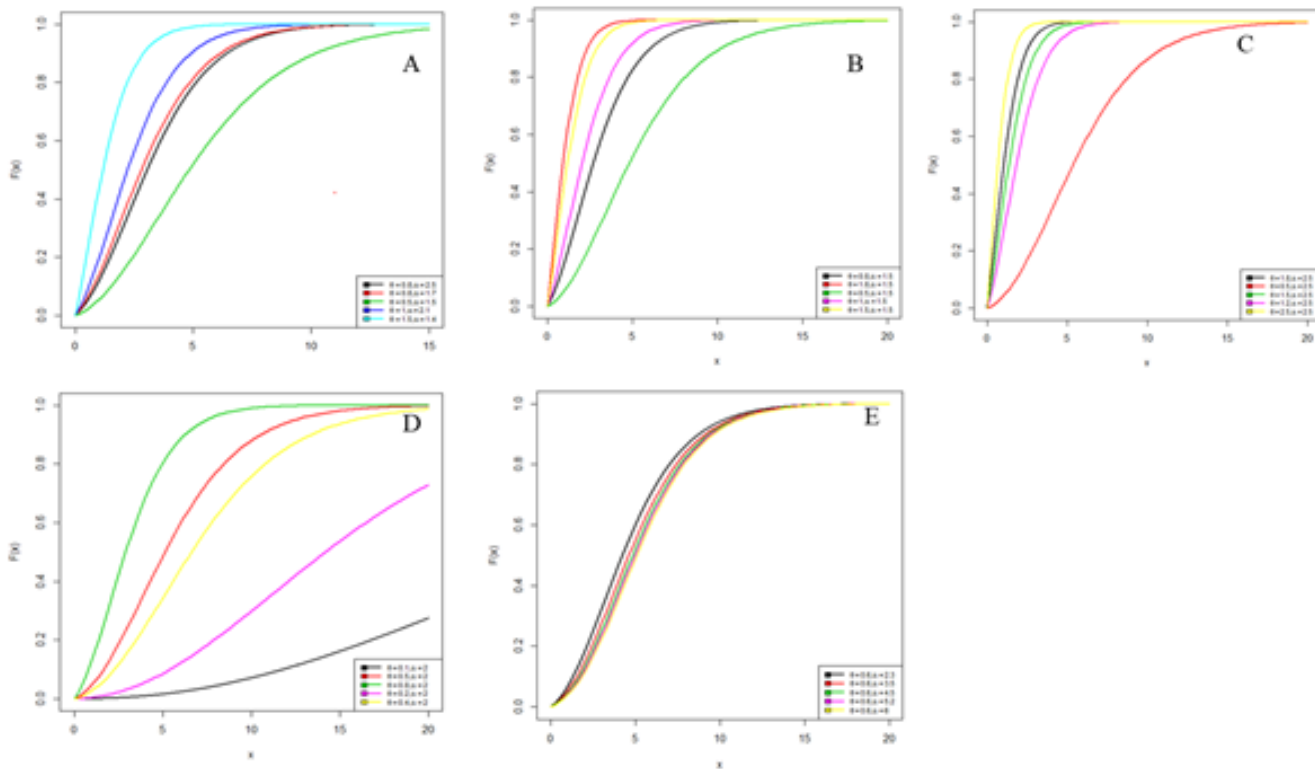**Fig 1.** (A-E) pdf plot of Alpha Power Transformed Aradhana distribution

Fig 2. (A - E) cdf plot of Alpha Power Transformed Aradhana distribution

The reliability or survival function of the Alpha Power Transformed Aradhana distribution is given as

$$S(y, \theta, \alpha) = \frac{\alpha - \alpha^{1 - \left(1 + \frac{\theta y (\theta y + 2\theta + 2)}{\theta^2 + 2\theta + 2}\right) e^{-\theta y}}}{\alpha - 1} \tag{7}$$

The hazard function of the Alpha Power Transformed Aradhana distribution can be obtained as

$$h(y, \theta, \alpha) = \frac{f(y, \theta, \alpha)}{S(y, \theta, \alpha)}$$

Therefore,

$$h(y, \theta, \alpha) = \frac{\log \alpha \, \theta^3 (1 + y)^2 e^{-\theta y} \, \alpha^{1 - \left(1 + \frac{\theta y (\theta y + 2\theta + 2)}{\theta^2 + 2\theta + 2}\right) e^{-\theta y}}}{(\alpha - \alpha^{1 - \left(1 + \frac{\theta y (\theta y + 2\theta + 2)}{\theta^2 + 2\theta + 2}\right) e^{-\theta y}})(\theta^2 + 2\theta + 2)} \tag{8}$$
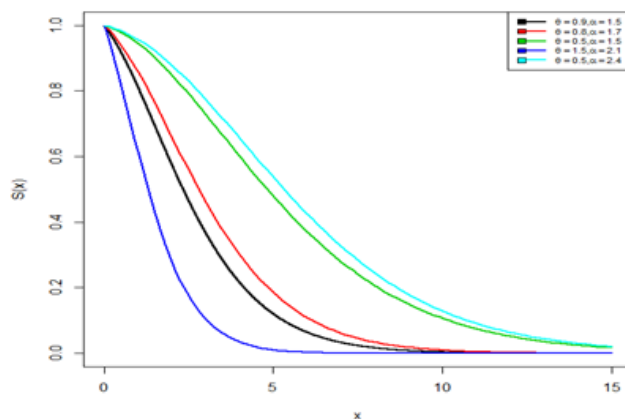
Fig 3. Survival plot of Alpha Power Transformed Aradhana distribution
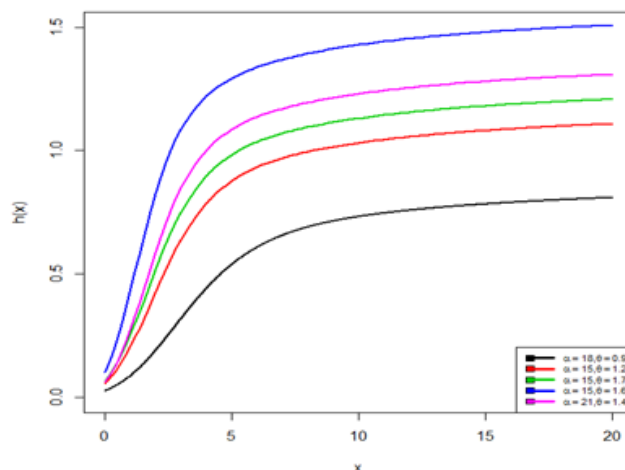
Fig 4. Hazard plot of Alpha Power transformed Aradhana distribution

3 Statistical Properties

In this section, some of the properties of the Alpha Power Transformed Aradhana distribution are discussed.

Moments

Let Y denote the random variable follows Alpha Power Transformed Aradhana distribution then r^{th} order moment about origin of μ_r' APTAD is

$$E\left(Y^r\right) = \int_0^\infty y^r \, f_{APT}\left(y\right) \, dy$$

$$E\left(Y^r\right) = \int_0^\infty y^r \left(\frac{log\alpha}{\alpha-1}\right)\left(\frac{\theta^3}{\theta^2+2\theta+2}\right)(1+y)^2 e^{-\theta y}\alpha^{1-\left(1+\frac{\theta y\left(\theta y+2\theta+2\right)}{\theta^2+2\theta+2}\right)e^{-\theta y}} \, dy \qquad (9)$$

Using power series expansion to Equation (9),

$$\alpha^z = \sum_{i=0}^{\infty} \frac{(\log \alpha)^i}{i!} z^i$$

the Equation (9) becomes,

$$E(Y^r) = \int_0^{\infty} y^r \sum_{i=0}^{\infty} \frac{(-\log \alpha)^i}{i!} \left(\left(1 + \frac{\theta y (\theta y + 2\theta + 2)}{\theta^2 + 2\theta + 2} \right) e^{-\theta y} \right)^i (1+y)^2 e^{-\theta y} \left(\frac{\theta^3}{2 + 2\theta + \theta^2} \right) dy \tag{10}$$

We apply the binomial expansion $(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$ to Equation (10), after simplification of the above Equation (10), we obtain the rth moment of the Alpha Power Transformed Aradhana distribution as

$$E(Y^r) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} \sum_{l=0}^{k} \sum_{m=0}^{l} (-1)^j \binom{l}{m} \binom{i}{j} \binom{j}{k} \binom{k}{l} \left(\frac{(\log \alpha)^k}{k!} \right) \left(\frac{1}{\theta^2 + 2\theta + 2} \right)^{k+1} \theta^{k-m+4}$$

$$\left\{ \frac{(r+2k-l)!}{(\theta(j+1))^{r+2k-l+1}} + \frac{(r+2k-l+1)!}{(\theta(j+1))^{r+2k-l+2}} + \frac{(r+2k-l+2)!}{(\theta(j+1))^{r+2k-l+3}} \right\} \tag{11}$$

Now we obtain the first four moments of the alpha power transformed Aradhana distribution by putting $r = 1, 2, 3, 4 \dots$ in Equation (11) as

$$E(Y) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} \sum_{l=0}^{k} \sum_{m=0}^{l} (-1)^j \binom{l}{m} \binom{i}{j} \binom{j}{k} \binom{k}{l} \left(\frac{(\log \alpha)^k}{k!} \right) \left(\frac{1}{\theta^2 + 2\theta + 2} \right)^{k+1} \theta^{k-m+4}$$

$$\left(\frac{(1+2k-l)!}{\theta^{2k-l+2}(j+1)^{2k-l+2}} + \frac{(2k-l+3)!}{\theta^{2k-l+3}(j+1)^{2k-l+3}} + \frac{(2k-l+4)!}{\theta^{2k-l+4}(j+1)^{2k-l+4}} \right)$$

$$E(Y^2) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} \sum_{l=0}^{k} \sum_{m=0}^{l} (-1)^j \binom{l}{m} \binom{i}{j} \binom{j}{k} \binom{k}{l} \left(\frac{(\log \alpha)^k}{k!} \right) \left(\frac{1}{\theta^2 + 2\theta + 2} \right)^{k+1} \theta^{k-m+4}$$

$$\left(\frac{(2+2k-l)!}{\theta^{2k-l+3}(j+1)^{2k-l+3}} + \frac{(2k-l+4)!}{\theta^{2k-l+4}(j+1)^{2k-l+4}} + \frac{(2k-l+5)!}{\theta^{2k-l+5}(j+1)^{2k-l+5}} \right)$$

$$E(Y^3) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} \sum_{l=0}^{k} \sum_{m=0}^{l} (-1)^j \binom{l}{m} \binom{i}{j} \binom{j}{k} \binom{k}{l} \left(\frac{(\log \alpha)^k}{k!} \right) \left(\frac{1}{\theta^2 + 2\theta + 2} \right)^{k+1} \theta^{k-m+4}$$

$$\left(\frac{(3+2k-l)!}{\theta^{2k-l+4}(j+1)^{2k-l+4}} + \frac{(2k-l+5)!}{\theta^{2k-l+5}(j+1)^{2k-l+5}} + \frac{(2k-l+6)!}{\theta^{2k-l+6}(j+1)^{2k-l+6}} \right)$$

$$E(Y^4) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} \sum_{l=0}^{k} \sum_{m=0}^{l} (-1)^j \binom{l}{m} \binom{i}{j} \binom{j}{k} \binom{k}{l} \left(\frac{(\log \alpha)^k}{k!} \right) \left(\frac{1}{\theta^2 + 2\theta + 2} \right)^{k+1} \theta^{k-m+4}$$

$$\left(\frac{(5+2k-l) \cdot}{\theta^{2k-l+5}(j+1)^{2k-l+5}} + \frac{(2k-l+6)!}{\theta^{2k-l+6}(j+1)^{2k-l+6}} + \frac{(2k-l+7)!}{\theta^{2k-l+7}(j+1)^{2k-l+7}} \right)$$

Moment Generating Function

Let Y be random variable follows Alpha Power Transformed Aradhana distribution, then the moment generating function (mgf) of Y is obtained as

$$M_Y(t) = E(e^{ty}) = \int_0^{\infty} e^{ty} f_{APT}(y) dy$$

$$M_Y(t) = E(e^{ty}) = \int_0^{\infty} \left(1 + ty + \frac{(ty)^2}{2!} + \dots \right) f(y) dy$$

$$= \int_0^{\infty} \sum_{r=0}^{\infty} \frac{t^r}{r!} y^r f(y) dy$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$$

$$M_T(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} \sum_{l=0}^{k} \sum_{m=0}^{l} (-1)^j \sum_{r=0}^{\infty} \frac{t^r}{r!} \begin{pmatrix} l \\ m \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} \begin{pmatrix} j \\ k \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix} \left(\frac{(\log \alpha)^k}{k!} \right) \left(\frac{1}{\theta^2+2\theta+2} \right)^{k+1} \theta^{k-m+4}$$

$$\left(\frac{(r+2k-l)!}{\theta^{r+2k-l+1}(j+1)^{r+2k-l+1}} + \frac{(2k-l+r+2)!}{\theta^{r+2k-l+2}(j+1)^{r+2k-l+2}} + \frac{(2k-l+r+3)!}{\theta^{r+2k-l+3}(j+1)^{r+2k-l+3}} \right)$$

Similarly, the characteristic function of Alpha Power Transformed Aradhana distribution can be obtained as

$$\phi_Y(t) = M_Y(it)$$

$$M_Y(it) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \sum_{k=0}^{j} \sum_{l=0}^{k} \sum_{r=0}^{\infty} \sum_{m=0}^{l} (-1)^j \frac{(it)^r}{r!} \begin{pmatrix} l \\ m \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix} \begin{pmatrix} j \\ k \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix} \left(\frac{(\log \alpha)^k}{k!} \right) \left(\frac{1}{\theta^2+2\theta+2} \right)^{k+1} \theta^{k-m+4}$$

$$\left(\frac{(r+2k-l)!}{\theta^{r+2k-l+1}(j+1)^{r+2k-l+1}} + \frac{(2k-l+r+2)!}{\theta^{r+2k-l+2}(j+1)^{r+2k-l+2}} + \frac{(2k-l+r+3)!}{\theta^{r+2k-l+3}(j+1)^{r+2k-l+3}} \right)$$

4 Order Statistics

Let $Y_{(1)}, Y_{(2)}, Y_{(3)} \ldots Y_{(n)}$ be the order statistics of a random variable $Y_1, Y_2, Y_3 .. Y_n$ drawn from the continuous population. $f(y)$ is the pdf while $F(y)$ is the cdf of the distribution, then the pdf of rth order statistic $Y_{(r)}$ is given by

$$f_{Y(y)}(y) = \frac{n!}{(r-1)!(n-r)!} f(y)[F(y)]^{r-1}[1-F(y)]^{n-r} \tag{12}$$

Inserting Equation (5) and Equation (6) in Equation (12), the pdf of rth order statistic $Y_{(r)}$ of the Alpha Power Aradhana distribution is given by

$$f_{Y(r)}(x) = \frac{n!}{(r-1)!(n-r)!} \left(\frac{\log \alpha}{\alpha-1} \right) \left(\frac{\theta^3}{\theta^2+2\theta+2} \right) (1+y)^2 e^{-\theta y} \alpha^{1-\left(1+\frac{\theta y(\theta y+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta y}}$$

$$\left[\frac{\alpha^{1-\left(1+\frac{\theta y(\theta y+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta y}}-1}{\alpha-1} \right]^{r-1} \left[1 - \frac{\alpha^{1-\left(1+\frac{\theta y(\theta y+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta y}}-1}{\alpha-1} \right]^{n-r}$$

Therefore the pdf of the higher-order statistic $Y_{(n)}$ can be obtained as

$$f_{Y(y,)}(y) = n \left(\left(\frac{\log \alpha}{\alpha-1} \right) \left(\frac{\theta^3}{\theta^2+2\theta+2} \right) (1+y)^2 e^{-\theta y} \alpha^{1-\left(1+\frac{\theta y(\theta y+2\theta+2)}{\theta^2+2\theta+2}\right)^{-\theta y}} \right) \left(\frac{\alpha^{1-\left(1+\frac{\theta y(\theta y+2\theta+2)}{\theta^2+2\theta+2}\right)^{-\theta y}}-1}{\alpha-1} \right)^{n-1}$$

and the pdf of the first order statistic $Y_{(1)}$ can be obtained as

$$f_{Y(y,)}(y) = n \left(\left(\frac{\log \alpha}{\alpha-1} \right) \left(\left(\frac{\log \alpha}{\alpha-1} \right) \left(\frac{\theta^3}{\theta^2+2\theta+2} \right) (1+y)^2 e^{-\theta y} \alpha^{1-\left(1+\frac{\theta y(\theta y+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta y}} \right) \right)$$

$$\left(1 - \frac{\alpha^{1-\left(1+\frac{\theta y(\theta y+2\theta+2)}{\theta^2+2\theta+2}\right)e^{-\theta y}}-1}{\alpha-1} \right)$$

5 Parameter estimation

Let $y_1, y, y_3, y_4, \ldots \ldots y_n$ be a random sample of size n from Alpha Power Transformed Aradhana Distribution. The likelihood function, L of Alpha Power Transformed Aradhana Distribution is given by

$$L = \prod_{i=1}^{n} \left(\frac{\log \alpha}{\alpha - 1} \right) \left(\frac{\theta^3}{\theta^2 + 2\theta + 2} \right) (1 + y_i)^2 e^{-\theta y_i} \alpha^{1 - 1 + \left(\frac{\theta y_i (\theta y_1 + 2\theta + 2)}{\theta^2 + 2\theta + 2} \right)^{-\theta y_i}}$$

$$L = \left(\frac{\log \alpha}{\alpha - 1} \right)^n \left(\frac{\theta^3}{\theta^2 + 2\theta + 2} \right)^n \prod_{i=1}^{n} (1 + y_i)^2 e^{-\theta y_i} \alpha^{1 - \left(1 + \frac{\theta y_i (\theta y_i + 2\theta + 2)}{\theta^2 + 2\theta + 2} \right)^{-\theta y_i}}$$

(12)

Taking log on both sides of Equation (12), we get the log-likelihood function of the APTAD as

$$\log L = n \log(\log \alpha) - n \log(\alpha - 1) + 3n \log \theta - n \log \left(\theta^2 + 2\theta + 2 \right) - \theta y_i + \sum_{i=1}^{n} \log (1 + y_i)^2$$

$$+ \log \alpha \sum_{i=1}^{n} \left(1 - \left(1 + \frac{\theta y_i (\theta y_i + 2\theta + 2)}{\theta^2 + 2\theta + 2} \right) e^{-\theta y_i} \right)$$

(13)

By taking the first partial derivatives of the log-likelihood function with respect to the two parameters (α, θ)

$$\frac{\partial \log L}{\partial \alpha} = \left(\frac{n}{\alpha \log \alpha} \right) - \left(\frac{n}{\alpha - 1} \right) + \frac{1}{\alpha} \sum_{i=1}^{n} \left(1 - \left(1 + \frac{\theta y_i (\theta y_i + 2\theta + 2)}{\theta^2 + 2\theta + 2} \right) e^{-\theta y_i} \right)$$

(14)

$$\frac{\partial \log L}{\partial \alpha} = \left(\frac{3n}{\theta} \right) - \left(\frac{n(2\theta + 2)}{\theta^2 + 2\theta + 2} \right) + \log \alpha \sum_{i=1}^{n} \left(1 - \left(1 + \frac{\theta y_i (\theta y_i + 2\theta + 2)}{\theta^2 + 2\theta + 2} \right) e^{-\theta y_i} \right)$$

(15)

The maximum likelihood estimates $(\hat{\alpha}, \hat{\theta})$ equations $\frac{\partial L}{\partial \alpha} = 0, \frac{\partial L}{\partial \theta} = 0$ The Equation (14) and Equation (15) cannot be solved as they both are in closed forms. So we compute the parameters of the Alpha Power Transformed Aradhana distribution using R software [22]. The optim function Nelder-Mead in R Statistical parameters.

6 Application of Alpha Power Transformed Aradhana distribution

The flexibility and performance of the Alpha Power Transformed Aradhana distribution are evaluated on competing models viz Aradhana distribution, Power Aradhana, Length Biased Garima, Exponential and Garima. Here, the distribution is fitted to two data sets, the performance of the distribution was compared with Two Parameter Aradhana distribution, Power Aradhana, Length Biased Garima, Exponential and Garima distributions for the data sets using Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Akaike Information Criterion Corrected (AICC) and -2lnL. Distribution with the lowest AIC, BIC, AICC and -2InL is considered the most flexible and superior distribution for a given dataset. The results are presented in the tables below.

Data Set 1

The first data set taken by Nwikpe [23] is the length of time (in years) that 81 randomly selected Nigerian graduates stayed without job before been employed by the universal basic education commission.

2, 5, 7, 5, 6, 7, 7, 6, 6, 9, 9, 6, 6, 7, 5, 4, 5, 2, 9, 8, 5, 9, 6, 6, 7, 2, 8, 3, 6, 6,2, 8, 5, 7, 4, 5, 6, 8, 8, 9, 3, 7, 6, 2, 6, 8, 9, 7, 6, 6, 9, 5, 9, 5, 5, 5, 3, 9, 8, 6, 6, 6, 7, 9, 4, 4, 6, 9, 7, 8, 8, 9, 4, 6, 3, 5, 4, 7, 6, 6, 5.

Data Set 2

The second data set is the strength data of glass of the aircraft window given by Fuller [24]. The data set is given below

18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.50, 25.52, 25.80, 26.69, 26.77, 26.78, 27.05, 27.67, 9.90, 31.11, 33.20, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98,37.08, 37.09, 39.58, 44.045, 45.29, 45.381.

Table 1. Goodness of fit the Alpha Power Transformed Aradhana distribution for data set 1

Model	ML Estimates	Standard Deviation	AIC	BIC	AICC	-2ln L
Two Parameter Aradhana	$\hat{\theta}$=0.3821 $\hat{\alpha}$=0.5528	0.0396 0.2254	368.6978	373.4867	369.0095	364.6978
Alpha power transformed Aradhana Distribution	$\hat{\theta}$=0.7793 $\hat{\alpha}$=1.0856	0.0481 1.4801	351.5837	356.3726	351.8954	347.5838
LB Garima	$\hat{\theta}$=0.4287	0.0299	394.6677	397.0622	394.7183	392.6677
Exponential	$\hat{\theta}$=0.1639	0.0182	456.91	459.3044	456.9606	454.9100
Garima	$\hat{\theta}$=0.2484	0.0234	442.3109	444.7053	442.3615	440.3109
Power Aradhana Distribution	$\hat{\theta}$=0.2202 $\hat{\alpha}$=0.1907	0.0227 0.0189	1427.962	1432.751	1428.273	1423.962

Table 2. Goodness of fit the Alpha Power Transformed Aradhana distribution for data set 2

Model	ML Estimates	Standard Deviation	AIC	BIC	AICC	-2ln L
Two Parameter Aradhana	$\hat{\theta}$=0.0794 $\hat{\alpha}$=0.0996	0.0111 0.0433	231.6854	234.5534	232.5743	227.6855
Alpha Power Transformed Aradhana	$\hat{\theta}$= 0.1975 $\hat{\alpha}$=1.9082	4.9346 1.6787	213.9172	216.7852	214.8061	209.9173
LB Garima	$\hat{\theta}$=0.0877	0.0099	248.5801	250.0141	248.7180	246.5801
Exponential	$\hat{\theta}$=0.0324	0.0058	276.5289	277.9629	276.6668	274.5289
Garima	$\hat{\theta}$=0.0515	0.0079	269.2780	270.7120	269.4159	267.2780
Power Aradhana Distribution	$\hat{\theta}$=0.2170 $\hat{\alpha}$=0.1015	0.0365 0.0165	691.8346	694.7025	692.7234	687.8346

7 Conclusion

In this paper, a novel probability distribution is introduced. The new distribution is a two- parameter distribution, Alpha Power Transformed Aradhana distribution. Some of the properties are derived and discussed like moments, order statistics, reliability analysis, and hazard rate. The method of maximum likelihood estimation is used for determining the parameters. The performance of the new model is determined by fitting it to two real-life data using the goodness of fit criteria such as AIC, BIC, AICC and 2 log L. It is found that Alpha Power Transformed Aradhana distribution gives a better fit to the data set as compared with exponential, Power Aradhana, Length Biased Garima, Exponential and Garima distributions. Further, The Alpha Power Transformed can be applied to various areas. The Alpha Power Transformed Aradhana distribution may suitable for most of the lifetime data and provides better outcomes than other well-known distributions.

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