Periodic orbits in the neighborhood of the triangular equilibrium points in the photogravitational restricted three body problem – Part 1

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Abstract

Objective: To find the periodic orbits using Fourier series expansions around the libration points L4 and L5 in the framework of planar restricted three-body problem by considering the more massive primary a source of radiation.

Methods/Statistical analysis: Period of the periodic orbits around the libration point L4 is found using the variational equations and function of the two finite masses. The period is independent on the size the orbit. When terms of higher order are retained in the analysis the period depends on the size of the orbit. Findings: The value of the critical mass is found in the photogravitational restricted three–body problem and is shown that the critical mass value corresponding to the small mass increases with the size of the orbit. It is shown that the classes of periodic orbits with infinitesimal limiting orbits L4 exist for values of the small mass greater than the critical mass value $m_0$. A comparison between these orbits with and without radiation pressure is made. Applications: Periodic orbits can be used to explore small solar system bodies, including asteroids and comets. Perturbation due to solar radiation pressure has to be understood and should be taken care of during human exploration mission.

Keywords: Photogravitational restricted problem; triangular libration point; periodic orbits; solar radiation pressure; Fourier series expansions

1 Introduction

Brown (1) considered the long period orbits around the triangular libration points by supposing finite amplitudes of libration and discussed in some detail the dependence of period and orbit shape on amplitude. Gylden (2) had begun the study of the infinitesimal periodic orbits around the libration points in the restricted body problem of three bodies. After that a number of investigations of the problem have appeared in the literature. E. Strömgren in 1928 has pointed out that the variational equations are suitable as a basis for numerical integrations. The period of the periodic orbits is found
using the variational equations around a libration point and a function of two finite masses, while it is independent of the size of the orbit. The study in\(^{(5)}\) showed that the period of the periodic orbits around the libration points \(L_4\) and \(L_5\) appears as a function of the size of the orbits, when terms of second and third order are retained in the series expression, however the investigation to the case that the masses have values in the neighbourhood of the critical values \(\mu_0\) and \(1 - \mu_0\). In this paper we have utilized the approach of\(^{(3)}\) in the planar RTBP when the more massive primary is a source of radiation.

Restricted three-body problem (RTBP) describes the motion of a massless body which moves under the gravitational effect of two finite masses called primaries. The primaries are supposed to move in circular orbits around their center of mass on account of their mutual attraction.\(^{(4)}\) is a fundamental book on the RTBP. During the last few decades many perturbing forces, such as oblateness and radiation forces of the primaries, variation in Coriolis and centrifugal forces, variation of the masses of the primaries have been included in the RTBP to study different aspects. In\(^{(5)}\) Radzievskii formulated the photo-gravitational restricted three-body problem (PRTBP) and studied the location of the equilibrium points. This arises from the classical problem when one of the masses is an intense emitter of radiation. Some of the papers in\(^{(6-16)}\) have studied the existence and stability of equilibrium points in the plane of motion of the primaries, where the primaries are sources of radiation. In\(^{(18,19)}\) the effect of radiation pressure is discussed in the four-body problem.

The effect of radiation pressure of a source can be expressed by a mass reduction factor \(q = 1 - \epsilon\), where the radiation coefficient \(\epsilon\) is the ratio of the force \(F_p\) which is caused by radiation to the force \(F_g\) which results from gravitation, i.e., \(\epsilon = \frac{F_p}{F_g}\). \(q\) is expressed in terms of particle radius \('a'\), density \('d'\) and radiation pressure efficiency \(\chi\) (in CGS system) as

\[
q = 1 - \frac{5.6 \times 10^{-5}}{a\delta} \chi
\]

Knowing the mass and the luminosity of the radiating body, \(\epsilon\) can be found for any given radius and density. Solar radiation pressure force \(F_p\) changes with distance by the same law of gravitational attraction force \(F_g\) and acts opposite to it. Thus, Sun's resulting force acting on the particle is\(^{(6,9)}\)

\[
F = F_g - F_p = \left(1 - \frac{F_p}{F_g}\right) F_g = (1 - \epsilon)F_g = (q)F_g
\]

If \(q = 1\), radiation has no effect. If \(q < 0\), then radiation surpasses gravity and if \(0 < q \leq 1\), gravitational force exceeds radiation.

### 2 Equations of Motion

In the dimensionless synodic coordinate system with origin of the system positioned at the center of mass of the primaries, considering the more massive primary at the location \((-\mu,0)\) and the smaller primary at \((1-\mu,0)\), the equations of motion for the circular photogravitational planar restricted three-body problem in the dimensionless barycentric synodic coordinates \((x, y)\) are:

\[
\begin{align*}
\ddot{x} - 2\dot{y} &= \Omega_x, \\
\ddot{y} + 2\dot{x} &= \Omega_y,
\end{align*}
\]

where

\[
\Omega = \frac{1}{2} \left[ (1 - \mu) \dot{r}_1^2 + \mu \dot{r}_2^2 \right] + \frac{\mu(1 - \mu)}{r_1} + \frac{\mu}{r_2}
\]

and \(r_1\) and \(r_2\) are determined by the equations

\[
\begin{align*}
r_1^2 &= (x + \mu)^2 + y^2, \\
r_2^2 &= (x - 1 + \mu)^2 + y^2
\end{align*}
\]

The position of the libration triangular points \(L_4\) and \(L_5\) in terms of \(\epsilon = 1 - q\) are:

\[
\left(\frac{1}{2}, \frac{-\epsilon}{3} - \mu, \frac{\sqrt{3}}{2} - \frac{\sqrt{3} \epsilon}{9}\right), \quad \left(\frac{1}{2}, \frac{-\epsilon}{3} - \mu, -\frac{\sqrt{3}}{2} + \frac{\sqrt{3} \epsilon}{9}\right)
\]
To investigate the motion of the infinitesimal mass in the neighborhood of the libration point $L_4$, we introduce a new coordinate system with its origin at $L_4$.

$$
\begin{align*}
    x &= \frac{1}{2} + \xi - \frac{\varepsilon}{3} - \mu, \\
    y &= \frac{\sqrt{3}}{2} + \eta - \frac{\sqrt{3}\varepsilon}{2}.
\end{align*}
$$

The differential equations of motion of the infinitesimal mass in the new co-ordinate system are

$$
\begin{align*}
    \xi - 2\eta &= \frac{\partial \Omega}{\partial \xi}, \\
    \eta + 2\xi &= \frac{\partial \Omega}{\partial \eta}
\end{align*}
$$

where

$$
\Omega = \frac{1}{2} \left[ \left( \frac{1}{2} + \xi - \frac{\varepsilon}{3} - \mu \right)^2 + \left( \frac{\sqrt{3}}{2} + \eta - \frac{\sqrt{3}\varepsilon}{9} \right)^2 \right] + \frac{q(1-\mu)}{\rho_1} + \frac{\mu}{\rho_2},
$$

and where $\rho_1$ and $\rho_2$ are determined by the equations

$$
\begin{align*}
    \rho_1^2 &= \left( \frac{\sqrt{3}}{2} - \frac{\varepsilon}{3\sqrt{3}} + \eta \right)^2 + \left( \frac{1}{2} - \frac{\varepsilon}{3} + \xi \right)^2, \\
    \rho_2^2 &= \left( \frac{\sqrt{3}}{2} - \frac{\varepsilon}{3\sqrt{3}} + \eta \right)^2 + \left( -\frac{1}{2} - \frac{\varepsilon}{3} + \xi \right)^2.
\end{align*}
$$

The Jacobi integral is

$$
\xi^2 + \eta^2 = 2 \Omega - C.
$$

With the help of series expansions, we get

$$
\frac{1}{\rho_1} = \left( \left( \frac{3\sqrt{3}}{\sqrt{27 - 18\varepsilon + 4\varepsilon^2}} + \frac{27(-9 + 2\varepsilon)\eta}{2(27 - 18\varepsilon + 4\varepsilon^2)^{3/2}} + \frac{81(\sqrt{3}(-135 + 36\varepsilon + 4\varepsilon^2)\eta^2}{2(27 - 18\varepsilon + 4\varepsilon^2)^{3/2}} - \frac{16(27 - 18\varepsilon + 4\varepsilon^2)^{7/2}}{27\sqrt{3}(-3 + 2\varepsilon)279(-9 + 2\varepsilon)(-3 + 2\varepsilon)\eta^2 + \frac{2187\sqrt{3}(-3 + 2\varepsilon)(297 - 108\varepsilon + 4\varepsilon^2)\eta^2}{16(27 - 18\varepsilon + 4\varepsilon^2)^{7/2}} - \frac{32805((-3 + 2\varepsilon)(2187 - 810\varepsilon - 108\varepsilon^2 + 40\varepsilon^4)\eta^3}{32(27 - 18\varepsilon + 4\varepsilon^2)^9/2} \right)^2 + \frac{\xi^2}{8(27 - 18\varepsilon + 4\varepsilon^2)^{5/2}} + \frac{2187(-243 + 1026\varepsilon - 612\varepsilon^2 + 88\varepsilon^3)\eta}{16(27 - 18\varepsilon + 4\varepsilon^2)^{7/2}} + \frac{6561\sqrt{3}(29889 - 73872\varepsilon + 48600\varepsilon^2 - 10944\varepsilon^3 + 6564\varepsilon^4)\eta^2}{64(27 - 18\varepsilon + 4\varepsilon^2)^{9/2}} + \frac{98415(452709 - 101039\varepsilon + 669400\varepsilon^2 - 171504\varepsilon^3 + 7632\varepsilon^4 + 1376\varepsilon^5)\eta^3}{128(27 - 18\varepsilon + 4\varepsilon^2)^{11/2}} + \frac{1295245(-3 + 2\varepsilon)(135 + 6\varepsilon - 44\varepsilon^2 + 8\varepsilon^3)\eta^2}{32(27 - 18\varepsilon + 4\varepsilon^2)^9/2} + \frac{295245(\sqrt{3}(-3 + 2\varepsilon)(-10449 + 15552\varepsilon + 18216\varepsilon^2 - 4992\varepsilon^3 + 368\varepsilon^4)\eta^2}{128(27 - 18\varepsilon + 4\varepsilon^2)^{11/2}} + \frac{30600435(-3 + 2\varepsilon)(-6561 + 106434\varepsilon - 103032\varepsilon^2 + 33264\varepsilon^3 - 3600\varepsilon^4 + 32\varepsilon^5)\eta^3}{256(27 - 18\varepsilon + 4\varepsilon^2)^{13/2}} \right)^2 + \frac{\xi^3}{16(27 - 18\varepsilon + 4\varepsilon^2)^{7/2}} + \frac{16(27 - 18\varepsilon + 4\varepsilon^2)^{7/2}}{16(27 - 18\varepsilon + 4\varepsilon^2)^{7/2}}.
\right)
$$
\[ \frac{1}{\rho_2} = \left( \frac{3\sqrt{3} + 27(-9 + 2e)\eta}{\sqrt{27 + 4e^2}} + \frac{81(\sqrt{3}(-135 + 108e + 4e^2))\eta^2}{8(27 + 4e^2)^{3/2}} - \frac{729(729 - 1782e + 108e^2 + 56e^2)\eta^2}{16(27 + 4e^2)^{3/2}} \right) + \frac{27\sqrt{3}(3 + 2e)\eta}{2(27 + 4e^2)^{3/2}} + \frac{729(-9 + 2e)(3 + 2e)\eta}{4(27 + 4e^2)^{3/2}} + \frac{2187\sqrt{3}(3 + 2e)(297 - 180e + 4e^2)\eta^2}{16(27 + 4e^2)^{3/2}} - \frac{32805(3 + 2e)(2187 - 2754e + 324e^2 + 40e^3)\eta^3}{32(27 + 4e^2)^{9/2}} \times \xi + \frac{81\sqrt{3}(-27 + 108e + 20e^2)}{8(27 + 4e^2)^{3/2}} + \frac{2187(-243 - 1566e - 36e^2 + 88e^3)\eta}{16(27 + 4e^2)^{3/2}} + \frac{6561\sqrt{3}(29889 + 68040e - 21384e^2 - 1008e^3 + 6564e^4)\eta^2}{64(27 + 4e^2)^{9/2}} + \frac{98415(452709 + 511758e - 600696e^2 - 132624e^3 + 46224e^4 + 1376e^5)\eta^3}{128(27 + 4e^2)^{11/2}} + \frac{295245(3 + 2e)(135 - 282e + 20e^2 + 8e^3)\eta}{32(27 + 4e^2)^{9/2}} + \frac{295245\sqrt{3}(3 + 2e)(-10449 + 58968e - 17208e^2 - 3360e^3 + 368e^4)\eta^2}{128(27 + 4e^2)^{11/2}} + \frac{18600435(3 + 2e)(-6561 - 190998e + 126360e^2 + 9936e^3 - 6480e^4 + 32e^5)\eta^3}{256(27 + 4e^2)^{13/2}} \right) \]

Inserting (9) in (6), we find the expression of \( \Omega \) after simplifications as:

\[ \Omega = \frac{1}{2} \left( \frac{\sqrt{3}}{2} - \frac{e}{3\sqrt{3} + \eta} \right)^2 + \frac{1}{2} \left( \frac{\sqrt{3}}{2} - \frac{e}{\sqrt{3}} - \mu + \xi \right)^2 + \mu \left( \frac{\sqrt{3}}{\sqrt{27 + 4e^2}} + \frac{27(-9 + 2e)\eta}{2(27 + 4e^2)^{3/2}} - \frac{81\sqrt{3}(-135 + 108e + 4e^2)\eta^2}{8(27 + 4e^2)^{3/2}} + \frac{729(729 - 1782e + 108e^2 + 56e^2)\eta^2}{16(27 + 4e^2)^{3/2}} \right) + \frac{27\sqrt{3}(3 + 2e)\eta}{2(27 + 4e^2)^{3/2}} + \frac{729(-9 + 2e)(3 + 2e)\eta}{4(27 + 4e^2)^{3/2}} + \frac{2187\sqrt{3}(3 + 2e)(297 - 180e + 4e^2)\eta^2}{16(27 + 4e^2)^{3/2}} - \frac{32805(3 + 2e)(2187 - 2754e + 324e^2 + 40e^3)\eta^3}{32(27 + 4e^2)^{9/2}} \times \xi + \frac{81\sqrt{3}(-27 + 108e + 20e^2)}{8(27 + 4e^2)^{3/2}} + \frac{2187(-243 - 1566e - 36e^2 + 88e^3)\eta}{16(27 + 4e^2)^{3/2}} + \frac{6561\sqrt{3}(29889 + 68040e - 21384e^2 - 1008e^3 + 6564e^4)\eta^2}{64(27 + 4e^2)^{9/2}} + \frac{98415(452709 + 511758e - 600696e^2 - 132624e^3 + 46224e^4 + 1376e^5)\eta^3}{128(27 + 4e^2)^{11/2}} + \frac{295245(3 + 2e)(135 - 282e + 20e^2 + 8e^3)\eta}{32(27 + 4e^2)^{9/2}} + \frac{295245\sqrt{3}(3 + 2e)(-10449 + 58968e - 17208e^2 - 3360e^3 + 368e^4)\eta^2}{128(27 + 4e^2)^{11/2}} + \frac{18600435(3 + 2e)(-6561 - 190998e + 126360e^2 + 9936e^3 - 6480e^4 + 32e^5)\eta^3}{256(27 + 4e^2)^{13/2}} \right) \]
From above equation, we obtain the expression of $\frac{\partial \Phi}{\partial \xi}$ and $\frac{\partial \Phi}{\partial \eta}$ and substitute in equation (5):

$$
\begin{align*}
\frac{\partial \Phi}{\partial \xi} &= \frac{1}{3} - \frac{e}{\eta} - \mu + \xi \left( \frac{27\sqrt{3}(3 + 2e)}{2(27 + 4e^2)^{3/2}} + \frac{729(-9 + 2e)(3 + 2e)\eta}{4(27 + 4e^2)^{5/2}} \right) - \frac{2187\sqrt{3}(3 + 2e)(297 - 180e + 4e^2)\eta^2}{16(27 + 4e^2)^{7/2}} + \\
\frac{\partial \Phi}{\partial \eta} &= \frac{1}{3} - \frac{e}{\eta} + \xi \left( \frac{27\sqrt{3}(3 + 2e)}{2(27 + 4e^2)^{3/2}} + \frac{729(-9 + 2e)(3 + 2e)\eta}{4(27 + 4e^2)^{5/2}} \right)
\end{align*}
$$

$$
\begin{align*}
&\frac{2187\sqrt{3}(3 + 2e)(297 - 180e + 4e^2)\eta^2}{16(27 + 4e^2)^{7/2}} + \\
&\frac{6561\sqrt{3}(29889 + 68040e - 21384e^2 - 10080e^3 + 6564e^4)\eta^2}{64(27 + 4e^2)^{7/2}}
\end{align*}
$$

$$
\begin{align*}
&\frac{98415(452709 + 511750e - 600696e^2 - 132624e^3 + 4624e^4 + 1376e^5)\eta^3}{128(27 + 4e^2)^{11/2}}
\end{align*}
$$

$$
\begin{align*}
&\frac{295245(3 + 2e)(135 - 282e + 20e^2 + 8e^3)\eta}{32(27 + 4e^2)^{9/2}} + \\
&\frac{295245\sqrt{3}(3 + 2e)(-10449 + 58968e - 17208e^2 - 3360e^3 + 368e^4)\eta^2}{18600435(3 + 2e)(-6561 - 190998e + 126360e^2 + 9936e^3 - 6480e^4 + 32e^5)\eta^3}
\end{align*}
$$

$$
\begin{align*}
&\frac{128(27 + 4e^2)^{11/2}}{64(27 + 4e^2)^{7/2}}
\end{align*}
$$

$$
\begin{align*}
&\frac{729(-9 + 2e)(-3 + 2e)\eta}{4(27 - 18e + 4e^2)^{5/2}} + \\
&\frac{2187\sqrt{3}(-3 + 2e)(297 - 108e + 4e^2)\eta^2}{16(27 - 18e + 4e^2)^{7/2}}
\end{align*}
$$

$$
\begin{align*}
&\frac{32805(-3 + 2e)(278 - 810e + 18216e^2 + 3360e^3 + 6564e^4)\eta^2}{64(27 - 18e + 4e^2)^{9/2}}
\end{align*}
$$

$$
\begin{align*}
&\frac{98415(452709 - 1010394e + 694008e^2 - 171504e^3 + 7632e^4 + 1376e^5)\eta^3}{2978989 - 73872e + 48600e^2 - 10944e^3 + 6564e^4)\eta^2}
\end{align*}
$$

$$
\begin{align*}
&\frac{128(27 - 18e + 4e^2)^{11/2}}{64(27 - 18e + 4e^2)^{7/2}}
\end{align*}
$$

$$
\begin{align*}
&\frac{295245(-3 + 2e)(135 + 6e - 44e^2 + 8e^3)\eta}{32(27 - 18e + 4e^2)^{9/2}} + \\
&\frac{295245\sqrt{3}(-3 + 2e)(-10449 + 15552e + 18216e^2 - 4992e^3 + 368e^4)\eta^2}{18600435(-3 + 2e)(-6561 + 106434e - 103032e^2 + 33264e^3 - 3600e^4 + 32e^5)\eta^3}
\end{align*}
$$

$$
\begin{align*}
&\frac{128(27 - 18e + 4e^2)^{11/2}}{256(27 - 18e + 4e^2)^{13/2}}
\end{align*}
$$
\[
\dot{\xi} + 2 \dot{\eta} = \frac{\sqrt{3}}{2} - \frac{\varepsilon}{3 \sqrt{3}} + \eta + \mu \left( \frac{27(-9 + 2\varepsilon)}{2(27 + 4\varepsilon)^{3/2}} - \frac{81 \sqrt{3}(-135 + 108\varepsilon + 4\varepsilon^2)\eta}{4(27 + 4\varepsilon)^{5/2}} - \frac{2187(729 - 1782\varepsilon + 108\varepsilon^2 + 56\varepsilon^3)\eta^2}{16(27 + 4\varepsilon)^{7/2}} + \frac{729(-9 + 2\varepsilon)(3 + 2\varepsilon)}{4(27 + 4\varepsilon)^{5/2}} + \frac{2187\sqrt{3}(3 + 2\varepsilon)(297 - 180\varepsilon + 4\varepsilon^2)\eta}{8(27 + 4\varepsilon)^{7/2}} - \frac{98415(3 + 2\varepsilon)(2187 - 2754\varepsilon + 324\varepsilon^2 + 40\varepsilon^3)\eta^2}{32(27 + 4\varepsilon)^{9/2}} + \frac{2187(-243 - 1566\varepsilon - 36\varepsilon^2 + 88\varepsilon^3)}{16(27 + 4\varepsilon)^{7/2}} + \frac{6561\sqrt{3}(29889 + 68040\varepsilon - 21384\varepsilon^2 - 10080\varepsilon^3 + 6564\varepsilon^4)\eta}{295245(452709 + 511758\varepsilon - 600696\varepsilon^2 - 132624\varepsilon^3 + 46224\varepsilon^4 + 1376\varepsilon^5)\eta^2} \right) \dot{\xi} + \frac{295245(3 + 2\varepsilon)(135 - 282\varepsilon + 20\varepsilon^2 + 8\varepsilon^3)}{32(27 + 4\varepsilon)^{9/2}} + \frac{295245\sqrt{3}(3 + 2\varepsilon)(-10449 + 5896\varepsilon - 17208\varepsilon^2 - 3360\varepsilon^3 + 368\varepsilon^4)\eta}{55801305(3 + 2\varepsilon)(-6561 - 190998\varepsilon + 126360\varepsilon^2 + 9936\varepsilon^3 - 6480\varepsilon^4 + 32\varepsilon^5)\eta^2} \dot{\xi} + (1 - \varepsilon)(1 - \mu) \right)
\]

(12)

If \( \varepsilon = 0 \), the equations of motion become the same as in Pedersen (3). The first equation of motion is:

\[
\ddot{\xi} - 2\dot{\eta} = \frac{3\varepsilon}{4} + \frac{3 \sqrt{3}}{4} (1 - 2\mu) \eta + \frac{21}{16} (1 - 2\mu) \dot{\xi}^2 - \frac{3 \sqrt{3}}{8} \eta \dot{\xi} - \frac{33}{16} (1 - 2\mu) \eta^2 - \frac{75 \sqrt{3}}{32} (1 - 2\mu) \eta \dot{\xi}^2 + \frac{123 \eta \dot{\xi}^2}{32} + \frac{45 \sqrt{3}}{32} (1 - 2\mu) \eta^3
\]

(13)

Finding the periodic orbits around L4 is equivalent to determining the coefficients a and b in the Fourier expansions of \( \xi \) and \( \eta \):

\[
\begin{align*}
\xi &= a_0 + a_1 \cos \omega t + a_2 \sin \omega t + a_3 \cos 2\omega t + a_4 \sin 2\omega t \\
\eta &= b_0 + b_1 \cos \omega t + b_2 \sin \omega t + b_3 \cos 2\omega t + b_4 \sin 2\omega t
\end{align*}
\]

(14)

\( a_1, a_2, b_1 \) and \( b_2 \) are of first order, while the other coefficients are of second order. Following the method of Pedersen (3), the first order terms are being retained.

\[
\begin{align*}
\dot{\xi} &= a_1 \cos \omega t + a_2 \sin \omega t, \\
\dot{\eta} &= b_1 \cos \omega t + b_2 \sin \omega t
\end{align*}
\]

(15)
Insert the values of $\xi$ and $\eta$ in equation (11) and (12), we get

\[
-a_1\omega^2\cos\omega t - a_2\omega^2\sin\omega t - 2(-b_1\cos\omega t + b_2\cos\omega t) = \frac{1}{2} + a_2\sin\omega t - \frac{\epsilon}{3} + \frac{4(27 - 18\epsilon + 4\epsilon^2)^{3/2}}{729b_2\sin\omega t(1 - \epsilon)(-9 + 2\epsilon)(-3 + 2\epsilon)(1 - \mu) + 2\sqrt{3}(1 - \epsilon)(-3 + 2\epsilon)(1 - \mu) + 2(27 - 18\epsilon + 4\epsilon^2)^{3/2}} + \frac{27\sqrt{3}(3 + 2\epsilon)\mu}{4(27 - 18\epsilon + 4\epsilon^2)^{3/2}} + \frac{81\sqrt{3}a_2\sin\omega t(-27 + 108\epsilon + 20\epsilon^2}\mu}{4(27 - 18\epsilon + 4\epsilon^2)^{3/2}} + \cos\omega t(a_1 + 2(27 + 4\epsilon^2)^{3/2}) + \frac{729b_1(1 - \epsilon)(-9 + 2\epsilon)(-3 + 2\epsilon)(1 - \mu)}{4(27 - 18\epsilon + 4\epsilon^2)^{3/2}} + \frac{81\sqrt{3}a_1(1 - \epsilon)(-27 - 36\epsilon + 20\epsilon^2)(1 - \mu)}{4(27 - 18\epsilon + 4\epsilon^2)^{3/2}} + \frac{729b_1(-9 + 2\epsilon)(3 + 2\epsilon)\mu}{4(27 + 4\epsilon^2)^{5/2}} + 
\]

Equating the coefficient of $\cos\omega t$ and $\sin\omega t$ in the equations (16) and (17) to zero, we get following system of four equation

\[
\begin{align*}
    a_2 \left( \frac{729(-9 + 2\epsilon)(-3 + 2\epsilon)(1 - \mu)}{4(27 - 18\epsilon + 4\epsilon^2)^{5/2}} + \frac{729(-9 + 2\epsilon)(3 + 2\epsilon)\mu}{4(27 + 4\epsilon^2)^{5/2}} \right) + 2a_1\omega + b_2 &= 0 \\
    b_1 \left( \frac{729(-9 + 2\epsilon)(-3 + 2\epsilon)(1 - \mu)}{4(27 - 18\epsilon + 4\epsilon^2)^{5/2}} + \frac{729(-9 + 2\epsilon)(3 + 2\epsilon)\mu}{4(27 + 4\epsilon^2)^{5/2}} \right) - 2a_2\omega &= 0 \\
    b_2 \left( \frac{729(-9 + 2\epsilon)(-3 + 2\epsilon)(1 - \mu)}{4(27 - 18\epsilon + 4\epsilon^2)^{5/2}} + \frac{729(-9 + 2\epsilon)(3 + 2\epsilon)\mu}{4(27 + 4\epsilon^2)^{5/2}} \right) - 2b_1\omega &= 0 \\
    a_2 \left( \frac{729(-9 + 2\epsilon)(-3 + 2\epsilon)(1 - \mu)}{4(27 - 18\epsilon + 4\epsilon^2)^{5/2}} + \frac{729(-9 + 2\epsilon)(3 + 2\epsilon)\mu}{4(27 + 4\epsilon^2)^{5/2}} \right) + \frac{81\sqrt{3}(-135 + 108\epsilon + 4\epsilon^2)\mu}{4(27 + 4\epsilon^2)^{5/2}} + \frac{81\sqrt{3}b_1(-9 + 2\epsilon)(3 + 2\epsilon)\mu}{4(27 + 4\epsilon^2)^{5/2}} &= 0
\end{align*}
\]
We get the following equations which determine $a_2$ and $b_2$ in terms of $a_1$ and $b_1$:

\[
\begin{align*}
(a_1 & \left( \frac{729(1-e)(-9+2e)(-3+2e)(1-\mu)}{4(27-18e+4e^2)^{5/2}} - \frac{729(-9+2e)(3+2e)\mu}{4(27+4e^2)^{5/2}} \right) + b_1(1 - \frac{81\sqrt{3}(1-e)(-135+36e+4e^2)(1-\mu)}{4(27-18e+4e^2)^{5/2}}) - \frac{81\sqrt{3}(1-e)(-9+2e)(3+2e)\mu}{4(27+4e^2)^{5/2}} + 4a_1\omega^2 + 4b_1\omega^2 = 0, \\
(-b_1 & \left( \frac{729(1-e)(-9+2e)(-3+2e)(1-\mu)}{4(27-18e+4e^2)^{5/2}} - \frac{729(-9+2e)(3+2e)\mu}{4(27+4e^2)^{5/2}} \right) - a_1(1 - \frac{81\sqrt{3}(1-e)(-135+36e+4e^2)(1-\mu)}{4(27-18e+4e^2)^{5/2}}) - \frac{81\sqrt{3}(1-e)(-9+2e)(3+2e)\mu}{4(27+4e^2)^{5/2}} + 4b_1\omega^2 + 4b_1\omega^2 = 0.
\end{align*}
\]

We get the following equations which determine $a_1$ and $b_1$.

\[
\begin{align*}
(1 & - \frac{81\sqrt{3}(1-e)(-135+36e+4e^2)(1-\mu)}{4(27-18e+4e^2)^{5/2}}) - \frac{81\sqrt{3}(1-e)(-9+2e)(3+2e)\mu}{4(27+4e^2)^{5/2}} + 4a_1\omega^2 = 0, \\
(1 & - \frac{81\sqrt{3}(1-e)(-27-36e+20e^2)(1-\mu)}{4(27-18e+4e^2)^{5/2}}) + \frac{81\sqrt{3}(1-e)(-27-36e+20e^2)\mu}{4(27+4e^2)^{5/2}} + 4\omega^2 = 0.
\end{align*}
\]

If now

\[
\begin{align*}
(1 & - \frac{81\sqrt{3}(1-e)(-135+36e+4e^2)(1-\mu)}{4(27-18e+4e^2)^{5/2}}) - \frac{81\sqrt{3}(1-e)(-9+2e)(3+2e)\mu}{4(27+4e^2)^{5/2}} + 4a_1\omega^2 = 0, \\
(1 & - \frac{81\sqrt{3}(1-e)(-27-36e+20e^2)(1-\mu)}{4(27-18e+4e^2)^{5/2}}) + \frac{81\sqrt{3}(1-e)(-27-36e+20e^2)\mu}{4(27+4e^2)^{5/2}} + 4\omega^2 + 4\omega^2 = 0
\end{align*}
\]
Then we have $a_1 = b_1 = a_2 = b_2 = 0$.

The mathematical condition for infinitesimal periodic orbit, therefore is

$$
\left( \frac{729(1 - \varepsilon)(-9 + 2\varepsilon)(-3 + 2\varepsilon)(1 - \mu)}{4(27 - 18\varepsilon + 4\varepsilon^2)^{5/2}} + \frac{729(-9 + 2\varepsilon)(3 + 2\varepsilon)\mu^2}{4(27 + 4\varepsilon^2)^{5/2}} \right)^2 - \frac{81\sqrt{3}(1 - \varepsilon)(-135 + 36\varepsilon + 4\varepsilon^2)(1 - \mu)}{4(27 - 18\varepsilon + 4\varepsilon^2)^{5/2}} - \frac{81\sqrt{3}(-135 + 108\varepsilon + 4\varepsilon^2)\mu}{4(27 + 4\varepsilon^2)^{5/2}} + \omega^2
$$

$$
\left( \frac{81\sqrt{3}(1 - \varepsilon)(-27 - 36\varepsilon + 20\varepsilon^2)(1 - \mu)}{4(27 - 18\varepsilon + 4\varepsilon^2)^{5/2}} + \frac{81\sqrt{3}(-27 + 108\varepsilon + 20\varepsilon^2)\mu}{4(27 + 4\varepsilon^2)^{5/2}} + \omega^2 \right) \left( \frac{4\omega^2}{324\sqrt{3}w^2} \right) + 4\omega^2 = 0
$$

(25)

If $\varepsilon = 0$, the equation (25) is same as in Pedersen (2).

$$
\left( \frac{3}{4}\sqrt{3}(1 - \mu) - \frac{3\sqrt{3}\mu}{4} \right) + 4\omega^2 - \left( 1 + \frac{1}{4}(1 - \mu) - \frac{\mu}{4} + \omega^2 \right) \left( 1 + \frac{5(1 - \mu)}{4} + \frac{5\mu}{4} + \omega^2 \right) = 0
$$

(26)

Now expanding equation (25)

$$
- \omega^2 + \frac{27}{16} + \frac{27(1 - 2\mu)}{2} \omega^2 + \frac{324\sqrt{3}w^2\omega^4}{18} = 0
$$

For $\varepsilon = 0.01$, equation (27) becomes

$$
\omega^4 - \omega^2 + 0.000017\mu \omega^2 + 6.76\mu - 6.76\mu^2 + 0.000117 = 0
$$

(28)

Further condition is that $\omega^2$ should be real and positive.

From (28), we get

$$
\omega^2 = \frac{1}{2} \left( 1 - 0.000017\mu + \sqrt{(1 - 0.000017\mu)^2 - 4(0.000017 + 6.76\mu - 6.76\mu^2)} \right)
$$

The condition for $\omega^2$ being real is

$$
\mu(1 - \mu) \leq 0.0367307
$$

(29)

or the product of the two finite masses should be less than or equal to 0.0367307.

Now we have

$$
\mu_0(1 - \mu_0) = 0.0367307
$$

(30)

The two equations:

$$
\xi = a_1\cos\omega(t - t_0) + a_2\sin\omega(t - t_0),
$$

$$
\eta = b_1\cos\omega(t - t_0) + b_2\sin\omega(t - t_0)
$$
correspond to the same motion as the two equations \((15)\), but rearranging the expressions on the right-hand side of the equations, we immediately see that other coefficients of \(\cos \omega t\) and \(\sin \omega t\) are obtained. The arbitrary constant \(t_0\), which may vary without changing the orbital motion, we shall call the phase constant. It is possible to choose the phase constant in such a way that the coefficients of \(\cos \omega t\) in the equation \(\eta = b_1 \cos \omega t - b_2 \sin \omega t\) vanishes; \(t_0\) then has to satisfy the equation \(b_1 \cos \omega (t - t_0) - b_2 \sin \omega (t - t_0) = 0\). Therefore, we can choose the phase constant in such a way that no terms with \(\cos \omega t\) appear in the expansion of \(\eta\).

Let us assume that the phase-constant has been fixed that \(b_1 = 0\).

Now we shall denote that \(\alpha = \lambda,\)

\[
2a_2 \omega = \lambda \left( \frac{729(1 - \epsilon)(-9 + 2\epsilon)(-3 + 2\epsilon)(1 - \mu) + 729(-9 + 2\epsilon)(3 + 2\epsilon)\mu}{4(27 - 18\epsilon + 4\epsilon^2)^{5/2}} \right)
\]

\[
2b_2 \omega = -\lambda \left( \frac{81\sqrt{3}(1 - \epsilon)(-27 - 36\epsilon + 20\epsilon^2)(1) + 81\sqrt{3}(-27 + 108\epsilon + 20\epsilon^2)\mu}{4(27 - 18\epsilon + 4\epsilon^2)^{5/2}} + \omega^2 \right) + \omega^2 \text{(31)}
\]

We summarize the above results as follows:

When \(\mu (1 - \mu) \leq 0.0367307\), infinitesimal period orbits around \(L_4\) exist, corresponding to the following equations:

\[
\xi = \lambda \cos \omega t + \frac{\lambda}{2w} \frac{729(1 - \epsilon)(-9 + 2\epsilon)(-3 + 2\epsilon)(1 - \mu) + 729(-9 + 2\epsilon)(3 + 2\epsilon)\mu}{4(27 - 18\epsilon + 4\epsilon^2)^{5/2}} \cos \omega t
\]

\[
\eta = \frac{\lambda}{2\omega} \left( \frac{81\sqrt{3}(1 - \epsilon)(-27 - 36\epsilon + 20\epsilon^2)(1 - \mu) + 81\sqrt{3}(-27 + 108\epsilon + 20\epsilon^2)\mu}{4(27 - 18\epsilon + 4\epsilon^2)^{5/2}} + \omega^2 \right) \sin \omega t \text{(32)}
\]

If \(\epsilon = 0\), equation \((32)\) becomes the same as in Pedersen \((3)\) i.e.

\[
\xi = \lambda \cos \omega t + \frac{3\lambda}{8w} (\sqrt{3}(1 - 2\mu) \sin \omega t
\]

\[
\eta = -\lambda \frac{3}{2\omega} \left( \frac{\sqrt{3}}{4} + \omega^2 \right) \sin \omega t \text{(33)}
\]

In order to find the Fourier coefficients in the Fourier expressions for \(\xi\) and \(\eta\), first we find the coefficients of the second order (coefficient scheme, Pedersen \((3)\) corresponding to \(\xi^2, \xi \eta\) and \(\eta^2\). Introducing the values found, together with \(a_0\) and \(b_0\), we can find the constant terms in the Fourier expressions \((11)\) and \((12)\). Putting these terms equal to zero two equations of the first order are found for the determination of \(a_0\) and \(b_0\). Similarly we compute the coefficient of the second order corresponding to \(\cos 2\omega t\) and \(\sin 2\omega t\) and \(\xi^2, \xi \eta\) and \(\eta^2\).

We get for more exact determination of \(a_1, a_2, b_1\) and \(b_2\) four equations of the form:

\[
-b_1 \left( \frac{729(1 - \epsilon)(-9 + 2\epsilon)(-3 + 2\epsilon)(1 - \mu) + 729(-9 + 2\epsilon)(3 + 2\epsilon)\mu}{4(27 - 18\epsilon + 4\epsilon^2)^{5/2}} \right) - 2b_2 \omega = 0
\]

\[
a_1 \left( \frac{81\sqrt{3}(1 - \epsilon)(-27 - 36\epsilon + 20\epsilon^2)(1 - \mu) + 81\sqrt{3}(-27 + 108\epsilon + 20\epsilon^2)\mu}{4(27 - 18\epsilon + 4\epsilon^2)^{5/2}} + \omega^2 \right) + \tau_1 \lambda = 0
\]

\[
-a_1 \left( \frac{729(1 - \epsilon)(-9 + 2\epsilon)(-3 + 2\epsilon)(1 - \mu) + 729(-9 + 2\epsilon)(3 + 2\epsilon)\mu}{4(27 - 18\epsilon + 4\epsilon^2)^{5/2}} \right) + 2a_2 \omega = 0
\]

\[
b_1 \left( \frac{81\sqrt{3}(1 - \epsilon)(-135 + 36\epsilon + 4\epsilon^2)(1 - \mu) - 81\sqrt{3}(-135 + 108\epsilon + 4\epsilon^2)\mu}{4(27 - 18\epsilon + 4\epsilon^2)^{5/2}} + \omega^2 \right) + \tau_2 \lambda = 0
\]
\[-b_2 \left( \frac{729(1 - \varepsilon)(-9 + 2\varepsilon)\varepsilon(1 - \mu)}{4(27 - 18\varepsilon + 4\varepsilon^2)^{3/2}} + \frac{729(-9 + 2\varepsilon)(3 + 2\varepsilon)\mu}{4(27 + 4\varepsilon^2)^{3/2}} \right) + 2b_1\omega -
\]
\[a_2 \left( 1 + \frac{81\sqrt{3}(1 - \varepsilon)(-27 - 36\varepsilon + 20\varepsilon^2)(1 - \mu)}{4(27 - 18\varepsilon + 4\varepsilon^2)^{3/2}} + \frac{81\sqrt{3}(-27 + 108\varepsilon + 20\varepsilon^2)\mu}{4(27 + 4\varepsilon^2)^{3/2}} + \omega^2 \right) + \tau_1\lambda = 0\]
\[-a_2 \left( \frac{729(1 - \varepsilon)(-9 + 2\varepsilon)(-3 + 2\varepsilon)\varepsilon(1 - \mu)}{4(27 - 18\varepsilon + 4\varepsilon^2)^{3/2}} + \frac{729(-9 + 2\varepsilon)(3 + 2\varepsilon)\mu}{4(27 + 4\varepsilon^2)^{3/2}} \right) - 2a_1\omega -
\]
\[b_2 \left( 1 - \frac{81\sqrt{3}(1 - \varepsilon)(-135 + 36\varepsilon + 4\varepsilon^2)(1 - \mu)}{4(27 - 18\varepsilon + 4\varepsilon^2)^{3/2}} - \frac{81\sqrt{3}(-135 + 108\varepsilon + 4\varepsilon^2)\mu}{4(27 + 4\varepsilon^2)^{3/2}} + \omega^2 \right) + \tau_1\lambda = 0\]

where \(\tau_1, \tau_2, \tau_3,\) and \(\tau_4\) are known functions of \(\lambda\), which are small quantities of the second order. 

From (34) and (35), we find \(a_2\) and \(b_2\) in terms of \(a_1\) and \(b_1\):
\[2a_2\omega + a_1 \left( \frac{729(1 - \varepsilon)(-9 + 2\varepsilon)(-3 + 2\varepsilon)(1 - \mu)}{4(27 - 18\varepsilon + 4\varepsilon^2)^{3/2}} + \frac{729(-9 + 2\varepsilon)(3 + 2\varepsilon)\mu}{4(27 + 4\varepsilon^2)^{3/2}} \right) + \frac{2a_1\omega}{b_1} \left( 1 - \frac{81\sqrt{3}(1 - \varepsilon)(-135 + 36\varepsilon + 4\varepsilon^2)(1 - \mu)}{4(27 - 18\varepsilon + 4\varepsilon^2)^{3/2}} - \frac{81\sqrt{3}(-135 + 108\varepsilon + 4\varepsilon^2)\mu}{4(27 + 4\varepsilon^2)^{3/2}} + \omega^2 \right) - \tau_2\lambda\]
\[2b_2\omega - b_1 \left( \frac{729(1 - \varepsilon)(-9 + 2\varepsilon)(-3 + 2\varepsilon)(1 - \mu)}{4(27 - 18\varepsilon + 4\varepsilon^2)^{3/2}} + \frac{729(-9 + 2\varepsilon)(3 + 2\varepsilon)\mu}{4(27 + 4\varepsilon^2)^{3/2}} \right) - \frac{2b_1\omega}{a_1} \left( 1 + \frac{81\sqrt{3}(1 - \varepsilon)(-27 - 36\varepsilon + 20\varepsilon^2)(1 - \mu)}{4(27 - 18\varepsilon + 4\varepsilon^2)^{3/2}} + \frac{81\sqrt{3}(-27 + 108\varepsilon + 20\varepsilon^2)\mu}{4(27 + 4\varepsilon^2)^{3/2}} + \omega^2 \right) + \tau_1\lambda\]

Inserting these values of \(a_2\) and \(b_2\) in the (36) and (37), we get the two equations:
\[-\left( \frac{729(1 - \varepsilon)(-9 + 2\varepsilon)(-3 + 2\varepsilon)(1 - \mu)}{4(27 - 18\varepsilon + 4\varepsilon^2)^{3/2}} + \frac{729(-9 + 2\varepsilon)(3 + 2\varepsilon)\mu}{4(27 + 4\varepsilon^2)^{3/2}} \right)\tau_1\lambda - \tau_2\lambda(-1) - \frac{81\sqrt{3}(1 - \varepsilon)(-27 - 36\varepsilon + 20\varepsilon^2)(1 - \mu)}{4(27 - 18\varepsilon + 4\varepsilon^2)^{3/2}} - \frac{81\sqrt{3}(-27 + 108\varepsilon + 20\varepsilon^2)\mu}{4(27 + 4\varepsilon^2)^{3/2}} + \omega^2 +
\]
\[b_1 \left( \frac{729(1 - \varepsilon)(-9 + 2\varepsilon)(-3 + 2\varepsilon)(1 - \mu)}{4(27 - 18\varepsilon + 4\varepsilon^2)^{3/2}} + \frac{729(-9 + 2\varepsilon)(3 + 2\varepsilon)\mu}{4(27 + 4\varepsilon^2)^{3/2}} \right)^2 + 4\omega^2 +
\]
\[1 - \frac{81\sqrt{3}(1 - \varepsilon)(-135 + 36\varepsilon + 4\varepsilon^2)(1 - \mu)}{4(27 - 18\varepsilon + 4\varepsilon^2)^{3/2}} - \frac{81\sqrt{3}(-135 + 108\varepsilon + 4\varepsilon^2)\mu}{4(27 + 4\varepsilon^2)^{3/2}} + \omega^2 \right) + 2\omega\tau_1\lambda = 0.
\]

If \(\varepsilon = 0\), equation (40) becomes same as in Pedersen (2) i.e.
\[-\left( \frac{3}{4}\sqrt{3}(1 - \mu) + \frac{3\sqrt{3}\mu}{4} \right)\tau_1\lambda - \tau_2\lambda(-1 + \frac{1 - \mu}{4} + \frac{\mu}{4} - \omega^2) +
\]
\[b_1 \left( \frac{3}{4}\sqrt{3}(1 - \mu) - \frac{3\sqrt{3}\mu}{4} \right)^2 + 4\omega^2 + (-1 + \frac{1 - \mu}{4} + \frac{\mu}{4} - \omega^2)
\]
\[1 + \frac{5(1 - \mu)}{4} + \frac{5\mu}{4} + \omega^2 \right) + 2\omega\tau_1\lambda = 0\]
\[ b_1(\frac{-27}{16}(1-2\mu)^2 - \omega^2 + \frac{27}{16} + \omega^4) = \tau_2 \lambda \left( \frac{3}{4} + \omega^2 \right) + 2\omega \tau_3 \lambda - \frac{3}{4} \sqrt{3}(1-2\mu) \tau_1 \lambda \]

\[
\begin{align*}
\tau_1 & = \left( \frac{729(1-\epsilon) - 9 + 2\epsilon)(-3 + 2\epsilon)(1 - \mu)}{4(27 - 18\epsilon + 4\epsilon^2)^{5/2}} \right) + \frac{729(-9 + 2\epsilon)(3 + 2\epsilon)\mu}{4(27 + 4\epsilon^2)^{5/2}} \tau_2 + (1 + \frac{81\sqrt{3}(1-\epsilon)(-135 + 36\epsilon + 4\epsilon^2)(1 - \mu)}{4(27 - 18\epsilon + 4\epsilon^2)^{5/2}})
\end{align*}
\]

One important characteristic of the solution of the nonlinear equation is that the orbit constant or the value of an orbital parameter is related to the orbital period

\[ T(\lambda) = T_0 + P(\lambda) \]

where \( P(\lambda) \) goes to zero as the orbital parameter, \( \lambda \), approaches zero. Here \( T(\lambda) \) is the period (considering the nonlinear system) and \( T_0 \) is the period of the orbit in the linear framework that is, when \( \lambda = 0 \). The quantity \( \lambda \) controls the size of the orbit and its related to the initial conditions.

Introducing now in equations (40) and (42), the values for the orbit constant \( a_1 \) and the phase constant \( b_1 \):

\[ a_1 = \lambda \text{ and } b_1 = 0 \]

we get the two equations:

\[
\begin{align*}
\tau_1 & = \left( \frac{729(1-\epsilon)(-9 + 2\epsilon)(-3 + 2\epsilon)(1 - \mu)}{4(27 - 18\epsilon + 4\epsilon^2)^{5/2}} \right) + \frac{729(-9 + 2\epsilon)(3 + 2\epsilon)\mu}{4(27 + 4\epsilon^2)^{5/2}} \tau_2 + \tau_1 \left( \frac{81\sqrt{3}(1-\epsilon)(-135 + 36\epsilon + 4\epsilon^2)(1 - \mu)}{4(27 - 18\epsilon + 4\epsilon^2)^{5/2}} \right) + \frac{81\sqrt{3}(-135 + 108\epsilon + 4\epsilon^2)\mu}{4(27 + 4\epsilon^2)^{5/2}} - 4\omega^2 + \left( \frac{-729(1-\epsilon)(-9 + 2\epsilon)(-3 + 2\epsilon)(1 - \mu)}{4(27 - 18\epsilon + 4\epsilon^2)^{5/2}} \right) + \frac{729(-9 + 2\epsilon)(3 + 2\epsilon)\mu}{4(27 + 4\epsilon^2)^{5/2}} \tau_2 + \tau_1 \left( \frac{81\sqrt{3}(1-\epsilon)(-27 - 36\epsilon + 20\epsilon^2)(1 - \mu)}{4(27 - 18\epsilon + 4\epsilon^2)^{5/2}} \right) + \frac{81\sqrt{3}(-27 + 108\epsilon + 20\epsilon^2)\mu}{4(27 + 4\epsilon^2)^{5/2}} + \omega^2 \right) \end{align*}
\]
From the above equation we find
\[
\frac{-729(1-\varepsilon)(-9+2\varepsilon)(-3+2\varepsilon)(1-\mu)}{4(27-18\varepsilon+4\varepsilon^2)^{5/2}} + \frac{729(-9+2\varepsilon)(3+2\varepsilon)\mu}{4(27+4\varepsilon^2)^{5/2}} \tau_1 - \frac{\tau_2(-1 - \frac{81\sqrt{3}(1-\varepsilon)(-27-36\varepsilon+20\varepsilon^2)(1-\mu)}{4(27-18\varepsilon+4\varepsilon^2)^{5/2}})}{4(27+4\varepsilon^2)^{5/2}} - \frac{81\sqrt{3}(-27+108\varepsilon+20\varepsilon^2)\mu}{4(27+4\varepsilon^2)^{5/2}} - \omega^2 + 2\omega \tau_3 = 0
\]
(44)

If \( \varepsilon = 0 \) equations (43) and (44) becomes
\[
\frac{-27}{16} (1 - 2\mu)^2 - \omega^2 + \frac{27}{16} + \omega^4 - \tau_1 \left( \frac{9}{4} + \omega^2 \right) + 2\omega \tau_4 + \frac{3}{4} \sqrt{3} (1 - 2\mu) \tau_2 = 0
\]
(45)

\[
(-\frac{3}{4} \sqrt{3} (1 - 2\mu)) \tau_1 + \tau_2 (\frac{3}{4} + \omega^2) + 2\omega \tau_3 = 0
\]

For \( \varepsilon = 0.01 \), equation (43) becomes
\[
0.0001176 + 6.764\mu - 6.764\mu^2 + (1.29 - 2.60\mu) \tau_2 \lambda - \omega^2 + 0.000117 \omega^2 + \omega^4 + \tau_1 \lambda (2.25 + 0.0150\mu - \omega^2) + 2\omega \tau_4 \lambda = 0
\]
(46)

Now considering the values of \( \mu \) in the neighborhood of the critical value of \( \mu_0 \), the smaller root of the equation:
\[
\mu_0 (1 - \mu_0) = 0.0367307
\]

From the above equation we find
\[
\mu_0 = 0.0381891.
\]
(47)

The corresponding values of \( \omega \), determined with the aid of the variational equations, is found from the following equation
\[
\omega^2 = \frac{1}{2} (1 - 0.000017\mu + \sqrt{(1 - 0.000017\mu)^2 - 4(0.000117 + 6.764\mu - 6.7645\mu^2)})
\]
\[
\omega_0^2 = \frac{1}{2}
\]

Now we have assumed that \( \mu \) lies in the neighborhood of \( \mu_0 \), we can put
\[
\mu (1 - \mu) = 0.0367307 + \delta
\]
(48)

where \( \delta \) is infinitely small.

Introducing (48) in (43), we get
\[
(-\frac{1}{2} + \omega^2)^2 + \frac{27\delta}{4} - \left( \frac{9}{4} + \omega^2 - 0.015\mu \right) \tau_1 + \frac{3}{4} \sqrt{3} (1 - 2\mu) \tau_2 + 2\omega \tau_3 = 0.
\]
(49)

The last three terms in equation (49) are small quantities of the second order. As only terms of the second order are retained in this equation, we can substitute \( \omega_0 \) and \( \mu_0 \) for \( \omega \) and \( \mu \) in these terms. The condition for real values of \( \omega \) and therefore the condition for infinitesimal periodic orbits around \( L_4 \) is
\[
27\delta \leq 9\tau_1 - 3\sqrt{3}(k)\tau_2 - 4\sqrt{2}\tau_4
\]
(50)

and we have found the upper limit of the mass product
\[
\frac{1}{27} (1 + 9\tau_1 - 3\sqrt{3}(k)\tau_2 - 4\sqrt{2}\tau_4)
\]
(51)

Now we turn to the computation of the quantities \( \tau_1, \tau_2, \tau_3 \) and \( \tau_4 \), which are functions of \( \mu \) and \( \lambda \) of the following form
\[
\tau_n = f_n(\mu)\lambda^2 \quad (n = 1, 2, 3, 4)
\]
We compute the coefficients of the second order in the coefficient scheme corresponding to

\[ \mu_0 \text{ and } \omega_0 \] for \( \mu \) and \( \omega \). The computation is based on following equation

\[ a_1 = \lambda_2 \quad b_1 = 0 \]

\[
2a_2\omega = \lambda \left( \frac{729(1-\varepsilon)(-9+2\varepsilon)(-3+2\varepsilon)(1-\mu)}{4(27-18\varepsilon+4\varepsilon^2)^{5/2}} + \frac{729(-9+2\varepsilon)(3+2\varepsilon)\mu}{4(27+4\varepsilon^2)^{5/2}} \right),
\]

\[
2b_2\omega = -\lambda \left(1 + \frac{81\sqrt{3}(1-\varepsilon)(-27-36\varepsilon+20\varepsilon^2)(1-\mu)}{4(27-18\varepsilon+4\varepsilon^2)^{5/2}} + \frac{81\sqrt{3}(-27+108\varepsilon+20\varepsilon^2)\mu}{4(27+4\varepsilon^2)^{5/2}} + \omega^2 \right)
\]

Substitute the values of \( \mu = \mu_0 \) and \( \omega = \omega_0 \) inequations (31) become

\[ a_1 = \lambda, \quad b_1 = 0, \]

\[
\sqrt{2}a_2 = \left( \frac{6.959(-9+2\varepsilon)(3+2\varepsilon)}{(27+4\varepsilon^2)^{5/2}} + \frac{175.29(1-\varepsilon)(-9+2\varepsilon)(-3+2\varepsilon)}{(27-18\varepsilon+4\varepsilon^2)^{5/2}} \right)\lambda
\]

\[
\sqrt{2}b_2 = -\lambda \left(1 + \frac{81\sqrt{3}(1-\varepsilon)(-27-36\varepsilon+20\varepsilon^2)(1-\mu)}{4(27-18\varepsilon+4\varepsilon^2)^{5/2}} + \frac{81\sqrt{3}(-27+108\varepsilon+20\varepsilon^2)\mu}{4(27+4\varepsilon^2)^{5/2}} + \omega^2 \right)
\]

For \( \varepsilon = 0.01 \), we get

\[ a_2 = 0.848\lambda, \quad b_2 = -0.880\lambda \]

With the values given by (54), we find the coefficients of the second order of the coefficient scheme (Pedersen \(^2\)) corresponding to \( \xi^2 \), \( \eta^2 \) and \( \xi\eta \),

\[
\begin{array}{c|c}
\xi^2 & 0.859\lambda^2 \\
\xi\eta & -0.374\lambda^2 \\
\eta^2 & 0.390\lambda^2 \\
\end{array}
\]

Introducing these values in the differential equation (11) and (12), we get

\[
0.0000184672 + 0.7454a_0 + 1.196b_0 + 0.537\lambda^2 = 0
\]

\[
0.000033331 + 1.196a_0 + 2.254b_0 + 0.773\lambda^2 = 0
\]

Which have the following roots

\[ a_0 = -0.000007026 - 1.148\lambda^2 \]

\[ b_0 = -0.0000110557 + 0.266\lambda^2 \]

We compute the coefficients of the second order in the coefficient scheme corresponding to \( \cos 2\omega t \) and \( \sin 2\omega t \) and \( \xi^2 \), \( \eta^2 \) and \( \xi\eta \)

\[
\begin{array}{c|cc|cc}
\xi^2 & 0.140\lambda^2 & 0.848\lambda^2 \\
\xi\eta & 0.373\lambda^2 & -0.440\lambda^2 \\
\eta^2 & -0.387\lambda^2 & 0 \\
\end{array}
\]
Introducing these values in the differential equations (11) and (12), we get for the determination of \(a_3, a_4, b_3\) and \(b_4\) the four equations:

\[
\begin{align*}
-2.82b_4 - 2.74a_3 - 1.19b_3 &= 0.67\lambda^2 \\
2.82b_3 - 2.74a_4 - 1.19b_4 &= 1.31\lambda^2 \\
2.82a_4 - 4.25b_3 - 1.19a_3 &= -1.85\lambda^2 \\
-2.82a_3 - 4.25b_4 - 1.19a_4 &= 1.41\lambda^2
\end{align*}
\] (57)

which have the roots:

\[
\begin{align*}
a_3 &= -0.43\lambda^2, & a_4 &= 0.57\lambda^2 \\
b_3 &= 0.94\lambda^2, & b_4 &= -0.21\lambda^2
\end{align*}
\] (58)

With these values of \(a_0, b_0, a_3, a_4, b_3\) and \(b_4\) we can compute the coefficients in the coefficients scheme corresponding to \(\cos \omega t\) and \(\sin \omega t\).

\[
\begin{array}{|c|c|}
\hline
\xi \eta & -2.24\lambda^3 & -0.99\lambda^3 \\
\eta^2 & 0.40\lambda^3 & 0.54\lambda^3 \\
\xi^2 & 0.18\lambda^3 & 0.36\lambda^3 \\
\hline
\end{array}
\]

Introducing these values in the differential equations (11) and (12), we get for the more exact determination of \(a_1, a_2, b_1\) and \(b_2\) of four equations (35),(36),(37) and (38). Comparing two systems of equations we can determine the values of \(t_1, t_2, t_3\) and \(t_4\).

\[
\begin{align*}
t_1 &= 3.18\lambda^2, & t_2 &= 0.96\lambda^2 \\
t_3 &= 1.84\lambda^2, & t_4 &= 5.37\lambda^2
\end{align*}
\] (59)

With the values of \(t_1, t_2, t_3\) and \(t_4\) found we can show that equation (44) is satisfied (We have obtained accuracy in the range of \(10^{-2}\)).

Now we insert the values of \(a_1 = \lambda, b_1 = 0, t_1 = 3.18 \lambda^2\) and \(t_2 = 0.96\lambda^2\) in equations (38) and (39) and get the following values:

\[
\begin{align*}
a_1 &= \lambda, b_1 = 0 \\
2a_2 \omega &= 0.96\lambda^3 + 1.3\lambda (1 - 2\mu) \\
2b_2 \omega &= 3.18\lambda^3 - \lambda (0.75 + \omega^2)
\end{align*}
\] (60)

Inserting the values of \(t_1, t_2\) and \(t_4\) in (49), we get the equation

\[
(-0.5 + \omega^2)^2 + 6.75\delta - 0.032\lambda^2 = 0.
\] (61)

The condition for real values of \(\omega\) and hence for infinitesimal periodic orbits around \(L_4\) is

\[
27\delta \leq 0.128\lambda^2
\] (62)

and the upper limit of the mass product is

\[
0.03703 \left[1 + 0.128\lambda^2\right]
\] (63)

We can determine the orbit constant \(\lambda_0\) by the equation

\[
27\delta = 0.128\lambda_0^2.
\] (64)
Brown [1] found that for mass ratio $\mu > \mu_0$, two families of periodic orbits exist at $L_4$ and both families depend analytically on a real parameter $\lambda$. This orbital parameter admits a strictly positive lower bound $\delta$ the same for both families. $\delta$ is an analytical function of mass ratio $\mu$, when $\lambda$ goes to $\delta$, both families close to a common periodic orbit, which is called the limiting orbit.

We did not carry out the computation of the terms of the third order due to the computation of the expression. Therefore we carry out the determination of the Fourier coefficients of $\xi$ and $\eta$ for limiting orbits. For the limiting orbits the equation of the condition (61) yields only one value of $\omega^2 (\omega^2 = \frac{1}{2})$. The orbit constant of the limiting orbit is given by equation (64)

$$27\delta = 0.128\lambda_0^2$$  \hspace{1cm} (65)

and $\delta$ is given by equation (48):

$$\mu (1 - \mu) = 0.0367307 + \delta$$ \hspace{1cm} (66)

From (64) and (66) we get the following equation

$$27\mu (1 - \mu) = 0.991 + 0.128\lambda_0^2$$ \hspace{1cm} (67)

and from (67) we find,

$$1 - 2\mu = 0.92 \left(1 - 0.0103\lambda_0^2\right)$$ \hspace{1cm} (68)

Introducing $\omega = 0.707$ and values of $1-2\mu$ given by (68) in equations (60), we get the following values of the coefficients of the first order for the limiting orbits:

$$a_1 = \lambda, \quad b_1 = 0$$
$$a_2 = 0.636\lambda^3 + 0.85\lambda$$
$$b_2 = 2.24\lambda^3 - 0.88\lambda$$ \hspace{1cm} (69)

Now we can find the coefficients of the second order by substituting $\lambda_0$ for $\lambda$

$$a_0 = -0.0000070 - 1.1484976\lambda_0^2$$
$$b_0 = -0.000011 + 0.266356\lambda_0^2$$
\[ a_3 = -0.43\lambda_0^2, \quad a_4 = 0.57\lambda_0^2 \]
\[ b_3 = 0.94\lambda_0^2, \quad b_4 = -0.21\lambda_0^2 \]  
(70)

Introducing (69) and (70) in the following equations
\[
\xi = a_0 + a_1\cos\omega t + a_2\sin\omega t + a_3\cos2\omega t + a_4\sin2\omega t, \quad \xi = a_0 + a_1\cos\omega t + a_2\sin\omega t + a_3\cos2\omega t + a_4\sin2\omega t, \\
\eta = b_0 + b_1\cos\omega t + b_2\sin\omega t + b_3\cos2\omega t + b_4\sin2\omega t \quad \eta = b_0 + b_1\cos\omega t + b_2\sin\omega t + b_3\cos2\omega t + b_4\sin2\omega t
\]  
(71)

we get the Fourier expressions for the limiting orbits
\[
\xi = -1.15\lambda_0^2 + \lambda_0\cos\omega_{0}t + (0.85\lambda_0 + 0.636\lambda_0^3)\sin\omega_{0}t - 0.43\lambda_0^2\cos2\omega_{0}t + 0.57\lambda_0^2\sin2\omega_{0}t \\
\eta = 0.266\lambda_0^2 + (-0.88\lambda_0 + 2.24\lambda_0^3)\sin\omega_{0}t + 0.94\lambda_0^2\cos2\omega_{0}t - 0.21\lambda_0^2\sin2\omega_{0}t.
\]  
(72)

where \(\omega_{0} = 0.707\)

Now we introduce the values of 0.1 and 0.2 for \(\lambda_0\). With \(\lambda_0 = 0.1\), we get from (72) the Fourier expressions:
\[
10^{5}\xi = -1150 - 430\cos2\omega_{0}t + 10000\cos\omega_{0}t + 570\sin2\omega_{0}t + 8563\sin\omega_{0}t, \\
10^{5}\eta = 260.35 + 940\cos2\omega_{0}t - 210\sin2\omega_{0}t - 8576\sin\omega_{0}t
\]  
(73)

and with \(\lambda_0 = 0.2\), we get
\[
10^{5}\xi = -4600 - 1720\cos2\omega_{0}t + 20000\cos\omega_{0}t + 2280\sin2\omega_{0}t + 17508\sin\omega_{0}t, \\
10^{5}\eta = 1065.4 + 3760\cos2\omega_{0}t - 840\sin2\omega_{0}t - 15808\sin\omega_{0}t
\]  
(74)

The corresponding values of \(\mu\) are found with the aid of equation (68). Denoting the value of \(\mu\) corresponding to \(\lambda_0 = 0.1\) and \(\lambda_0 = 0.2\) by \(\mu_1\) and \(\mu_2\), respectively, we get
\[
\mu_1 = 0.040047 = \mu_0 + 0.00185 \\
\mu_2 = 0.040108 = \mu_0 + 0.00200
\]  
(75)

Figure 2 correspond to the equations (73) and (74).

3 Results

When \(\mu > \mu_0\) and orbital parameter reach a critical value \(\lambda_0\), at which the value of the two periodic orbits coincide and their period becomes equal. Figure 2 shows two limiting orbits belonging to various values of the mass parameter for \(\varepsilon = 0.01\). First corresponding to the short period limiting orbit (inner orbit) with \(\mu_1 = 0.04007\), the second belonging to the long period limiting orbit (outer orbit) with \(\mu_2 = 0.040108\). At \(\mu = \mu_0\), \(\lambda_0 = 0\). Figure 3 shows the effect of radiation pressure (\(\varepsilon = 0.01\)) of more massive primary on the limiting orbit (inner orbit) around \(L_4\), when the small mass has the value \(\mu_1 = \mu_0 + 0.00185\). A comparison between these orbits with and without radiation pressure is made and presented in Figure 4. The semi major axis a and eccentricity e of the periodic orbits are calculated and presented in Table 1. It can be noticed that with the inclusion of radiation pressure, the semi-major axis and the eccentricity of the limiting orbit increases.

<table>
<thead>
<tr>
<th>Table 1. Semi major axis and eccentricity of limiting orbits</th>
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<tbody>
<tr>
<td><strong>Inner Orbit</strong></td>
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<tr>
<td>Semi Major Axis</td>
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<tr>
<td>Pedersen</td>
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<tr>
<td>With perturbation</td>
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https://www.indjst.org/
Fig 2. The limiting orbits around L₄ for ε = 0.01, when the small mass has the value μ₁ = μ₀ + 0.00185 (Inner orbit) and μ₂ = μ₀ + 0.00200 (outer orbit).

Fig 3. Comparison of the limiting orbit (inner orbit) with and without perturbation around L₄, when the small mass has the value μ₁ = μ₀ + 0.00185.
4 Conclusion

Fourier expansion of the periodic orbits in the immediate vicinity of the triangular libration points in the photogravitational restricted three-body problem are obtained. Periodic orbits are generated around the libration points $L_4$ and $L_5$ by considering the more massive primary as a source of radiation. We observe that the shape of periodic orbits changed with the inclusion of radiation pressure. A comparison between these orbits with and without radiation pressure is made. The value of the critical mass is found and is shown that the critical mass value corresponding to the small mass increases with the size of the orbit. Calculations were carried out using Wolfram Mathematica 11 and MATLAB 2015B software.

References


