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A view on characterizations of the J shaped statistical distribution

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Abstract

Objectives: In recent years, characterization of any distribution has become important in the field of probability distribution.The objective of the study is to characterize the power function distribution to see its usefulness under different real life situations such as Engineering and medical sciences.
Methods: The study proposed the characterization of Power function distribution based on mean inactivity times (MIT), mean residual function (MRF), conditional moments, conditional variance (CV), doubly truncated mean (DTM), incomplete moments and reverse hazard function. **Findings**:We have characterized the power function distribution using different method, and conclude that thesufficient and necessary conditions of different methods mentioned above meet the results of Power function distribution. **Application**: Power function distribution has wide applicability in the field of Engineering. The findings of the paper may help the Engineers to know more about the Power function distribution.

Keywords: Characterization; mean inactivity time; mean residual function; power function distribution

1 Introduction

The characterization of probability distribution is important before its application to any real world phenomena to confirm whether the given probability distribution is suitable for the specific data set or not. There are several characterizing functions associated with a probability distribution that uniquely define it.

We assume that the distribution of random variable called "Z" follows Power function distribution (PFD) with following cumulative density function

$$\Phi(z) = \left(\frac{z}{\beta}\right)^{\omega}; \qquad 0 < z < \beta \tag{1}$$

Then $W = Z^{-1}$has the Pareto distribution with parameters $\delta = \beta^{-1}$and $\gamma = \omega$.

$$G(w) = 1 - \left(\frac{\delta}{w}\right)^{\gamma}; \qquad 0 < \delta \leq w$$

The application of this distribution was discoursed by Meniconi and Barry[(1)] to test the reliability of any module. The characterization of PFD has been introduced by Fisz[(2)], Basu[3], Govindarajulu[4] and Dallas[5]. Many authors have studied the characterization of different distributions; see Dallas[6], Deheuvels[7], Gupta[8], Nagaraja[9], Rao et al.[10] and Huang et al.[11].

Su and Huang[12] presented the following function for the cdf "Φ" of "T"

$$v(t) = E(g(T) \mid T > t), \quad a < t < b$$

Where g may be observed as continuous function and (a,b) is the support of Φ. They have also showed that if T has a pdf, then Φ (t), a<t<b, $\delta(t)$, a<t<b, or E(T|T>t), a<t<b, are equivalent, in the sense that given on them , the other two can be determined, where:

$$\delta(t) = \frac{f(t)}{1-\Phi(t)} \quad ; \qquad a < t < b$$

Gupta and Kirmani[13] consider the case a=0 and b=∞ and used the ratio of $\delta(t)$and E(T|T>t) to characterize the distribution of T. Nair and Shudheesh[14] showed that, for a continuous function g and a differential function h, the pdf "f" satisfies the differential equation. Unnikrishnan and Sudheesh[15] studied the characterization of continuous distributions by properties of conditional variance. Also, characterizations based on the properties of the failure rate function have been considered by many authors. Elbatal et al.[16], Huang et al.[17], Bhatt[18], Ahsanullah et al.[19] and references therein. Imen et al.[20] discussed the characterization of Exponential q-distribution.[21],[22] and[23] discussed the properties and characterizations of the J shaped distributions. Zaka et al.[24] introduced the exponentiated generalized class of Power function distribution.

This study discusses a novel characterization of PFD established on mean inactivity times (MIT), mean residual function (MRF), conditional moments, conditional variance, doubly truncated mean (DTM), incomplete moments and reverse hazard function (RHF). The paper is organized as follows. We characterized the PFD by MIT, MRF and doubly truncated mean in section-2. Under Section -3, we introduce characterization based on different types of moments. In Section 4, we obtain the PFD from RHF. The concluding remarks are provided in Section 5.

2 Materials and Methods

2.1. Characterization of PFD under mean inactivity times

We may write the pdf and cdf for the PFD

$$f(z) = \omega z^{\omega-1} \quad ; \qquad 0 < z < 1$$

$$\Phi(z) = z^{\omega} \quad ; \qquad 0 < z < 1$$

Also let $\bar{\Phi}(z)$ as the survival function for the PFD. Then "Z" has PFD with parameter "ω" if and only if

$$M_z(t) = t - \frac{\omega t}{\omega+1}$$

Where M_z(t): Mean Inactivity Times for PFD.

Proof:

Necessary part:

From (1), by taking β=1.

$$\Phi(z) = z^{\omega}$$

Since $M_z(t) = t - E(z|z \leq t)$

$$M_z(t) = t - \frac{\int_0^t z\left(\omega z^{\omega-1}\right) dz}{\Phi(t)}$$

$$M_z(t) = t - \frac{\omega \int_0^t (z^{\omega})\, dz}{\Phi(t)}$$

Since Φ (t) = t^{ω}

$$M_z(t) = t - \frac{\omega t}{\omega+1}$$

Also

$$\mathrm{M_z(t) - t} = \frac{\omega \mathrm{t}}{\omega+1} = \mathrm{E(z \mid z \leq t)}$$

$$E(z \mid z \leq t) = \frac{\omega t}{\omega+1} \tag{2}$$

Sufficient Part:

Now $E(z \mid z \leq t)$ can be written as:

$$E(z \mid z \leq t) = \frac{\int_0^t z f(z) dz}{\Phi(t)}$$

Using integration by parts

$$E(z \mid z \leq t) = t - \frac{\int_0^t \Phi(z) dz}{\Phi(t)} \tag{3}$$

Equating (2) and (3), we get

$$t - \frac{\int_0^t \Phi(z) dz}{\Phi(t)} = \frac{\omega t}{\omega+1}$$

$$\int_0^t \Phi(z) dz = \left(\frac{t}{\omega+1}\right) \Phi(t)$$

Differentiating "t" on both sides

$$\Phi(\mathrm{t}) \left\{1 - \frac{1}{\omega+1}\right\} = \frac{\mathrm{t}}{\omega+1} \mathrm{f(t)}$$
$$\frac{\mathrm{f(t)}}{\Phi(\mathrm{t})} = \frac{\omega}{\mathrm{t}}$$

Apply Integral on both sides

ln (Φ (t)) = $\omega\, ln(t)$

Therefore Φ (t) = t^{ω}

and f(t)= $\omega t^{\omega-1}$

so the random variable "z" has PFD, "f(z) = $\omega z^{\omega-1}$" with parameter "ω" if and only if

$$M_z(t) = t - \frac{\omega t}{\omega+1} \; ; \qquad where\ M_z(t) : Mean\ Inactivity\ Times$$

2.2 Characterization of PFD under mean residual function

We may write the pdf and cdf for the PFD

$$f(z) = \omega z^{\omega-1}; \quad 0 < z < 1$$
$$\Phi(z) = z^{\omega}$$

Also let $\bar{\Phi}(z)$ as the survival function for the PFD. Then the random variable "Z" has PFD with parameter "ω" if and only if

$$M(t) = \frac{\omega}{\omega+1} \frac{\left(1 - t^{(\omega+1)}\right)}{\bar{\Phi}(t)} - t \; ;$$

Where M (t): Mean Residual Function.
Proof:
Necessary part:

$$\Phi(z) = z^{\omega}$$

Since

$$\mathrm{M(t) = E(z \mid z \geq t) - t} \tag{4}$$

$$E(z \mid z \geq t) = \frac{\int_t^1 z\left(\omega z^{\omega-1}\right) dz}{\bar{\Phi}(t)}$$

$$E(z \mid z \geq t) = \frac{\omega}{\bar{\Phi}(t)} \left\{ \frac{1 - t^{(\omega+1)}}{\omega+1} \right\}$$

Therefore, (4) becomes

$$M(t) = \frac{\omega}{\omega+1} \frac{(1 - t^{(\omega+1)})}{\bar{\Phi}(t)} - t$$

$$M(t) + t = \frac{\omega}{\omega+1} \frac{(1 - t^{(\omega+1)})}{\bar{\Phi}(t)}$$

$$E(z \mid z \geq t) = \frac{\omega}{\omega+1} \frac{\left(1 - t^{(\omega+1)}\right)}{\bar{\Phi}(t))} \tag{5}$$

Sufficient Part:
Now $E(z \mid z \geq t)$ can be written as:

$$\mathrm{E(z \mid z \geq t) = \frac{\int_t^1 zf(z)dz}{\bar{\Phi}(z)}}$$

Using integration by parts

$$\mathrm{E(z \mid z \geq t) = -t - \frac{\int_t^1 \bar{\Phi}(z)dz}{\bar{\Phi}(z)}} \tag{6}$$

By equating (5) and (6)

$$-t\,\bar{\phi}(t) - \int_t^1 \bar{\phi}(z)\,dz = \frac{\omega}{\omega+1} \left(1 - t^{(\omega+1)}\right)$$

$$\int_t^1 \bar{\phi}(z)\,dz = -t\,\bar{\phi}(t) - \frac{\omega}{\omega+1} \left(1 - t^{(\omega+1)}\right)$$

Since ϕ (t) = t^{ω}

$$(1-t)-\int_{t}^{1}\phi(z)\,dz = -t+t^{\omega+1}-\frac{\omega}{\omega+1}\left(1-t^{(\omega+1)}\right)$$

$$\int_{t}^{1}\phi(z)\,dz=\left(1+\frac{\omega}{\omega+1}\right)\left(1-t^{(\omega+1)}\right)$$

Differentiating "t" on both sides

$$-\phi\,(t)=\left(\frac{2\omega+1}{\omega+1}\right)\left(-\left(\omega+1\right)\,t^{\omega}\right)$$

$$\phi\,(t)=(2\omega+1)\,t^{\omega}$$

By Differentiating "t" on both sides

$$f\,(t)=(2\omega+1)\,\omega\,t^{\omega-1}$$

$$\frac{1}{(2\omega+1)}\int_{0}^{1}f\,(t)\;dt=1$$

Therefore, $f\,(t)=\omega\,t^{\omega-1}$

Therefore, it is proved that the random variable "z" has PFD, f(z) = $\omega\, z^{\omega-1}$" with parameter "ω" if and only if

$$M\,(t)=\frac{\omega}{\omega+1}\,\frac{(1-t^{(\omega+1)})}{\bar{\Phi}(t)}-t\;;\qquad where\;M\,(t):Mean\;Residual\;Function$$

2.3 Characterization Of PFD under doubly truncated mean

We may write the pdf and cdf for the PFD

$$\begin{aligned}&\mathrm{f(z)}=\omega \mathrm{z}^{\omega-1};\quad 0<\mathrm{z}<1\\&\phi(\mathrm{z})=\mathrm{z}^{\omega};\quad 0<\mathrm{z}<1\end{aligned}$$

And let $\bar{\Phi}\,(z)$ be the survival function .Then the random variable "Z" has PFD with parameter "ω" if and only if

$$E\,(Z)\,z<Z<w)=\frac{\omega}{\omega+1}\,\frac{(w\,\phi\,(w)-z\,\phi\,(z)\}}{(\phi\,(w)-\phi\,(z)\}}\;\;;$$

$$where\;E\,(Z)\,z<Z<w):Doubly\;Truncated\;Mean\;.$$

Proof:

Necessary Condition:

$$E\,(Z)\,z<Z<w)=\frac{\int_{z}^{w}z\,f\,(z)\;dz}{\phi\,(w)-\phi\,(z)}$$

$$E\left(Z\right)z<Z<w)=\frac{\omega}{\omega+1}\frac{\left(w^{\omega+1}-z^{\omega+1}\right\}}{\left(\phi\left(w\right)-\phi(z)\right\}}$$

Since ϕ (z) = z^{ω}

$$E\left(Z\right)z<Z<w)=\frac{\omega}{\omega+1}\frac{\left(w\,\phi\left(w\right)-z\,\phi(z)\right\}}{\left(\phi\left(w\right)-\phi(z)\right\}} \tag{7}$$

Sufficient Condition:

Also $E\left(Z\right)z<Z<w)$ can be written as

$$E\left(Z\right)z<Z<w)=\frac{\int_{z}^{w}z\,f\left(z\right)\,dz}{\phi\left(w\right)-\phi(z)}$$

$$E\left(Z\right)z<Z<w)=\frac{w\,\phi\left(w\right)-z\,\phi(z)-\int\limits_{z}^{w}\phi\left(z\right)\,dz}{\phi\left(w\right)-\phi(z)}$$

By equating (7) and (8), we get

$$\frac{w\,\phi\left(w\right)-z\,\phi(z)-\int\limits_{z}^{w}\phi\left(z\right)\,dz}{\phi\left(w\right)-\phi(z)}=\frac{\omega}{\omega+1}\frac{\left(w\,\phi\left(w\right)-z\,\phi(z)\right\}}{\left(\phi\left(w\right)-\phi(z)\right\}}$$

Differentiating with respect to ‘w’ on both sides

$$w\,f\left(w\right)+\phi\left(w\right)-\phi(w)=\frac{\omega}{\omega+1}\left\{w\,\phi\left(w\right)+\phi(w)\right\}$$

$$\left(\frac{1}{\omega+1}\right)w\,f\left(w\right)=\frac{\omega}{\omega+1}\,\phi\left(w\right)$$

$$\frac{f\left(w\right)}{\phi(w)}=\frac{\omega}{w}$$

By integrating on both sides

ln (ϕ (w)) = ln (w^{ω})

Therefore

f(w)= $\omega\ w^{\omega-1}$

3 Characterization Of PFD by conditional variances, conditional moments and incomplete moments

3.1. Characterization of PFD by conditional variances

We may write the pdf and cdf for the PFD

$$f(z)=\omega z^{\omega-1};\quad 0<z<1$$
$$\phi(z)=z^{\omega}$$

And let $\bar{\phi}\left(z\right)$ be the survival function respectively. Then the random variable “z” has PFD with parameter “ω” if and only if

$$V(z\mid z\ \le t)=t^{2}\left\{\frac{\omega}{\omega+2}-\frac{\omega^{2}}{(\omega+1)^{2}}\right\}$$

$$where\ E\,(Z)\,z < Z \leq t) : Conditional\ Variance(with\ respect\ to\ F\,(z))\ .$$

Proof:

Necessary part:

ϕ (z) = z^{ω}

Since

$$V\,(z\mid z\ \leq t) = E\left(z^2\mid z\ \leq t\right) - \{E(z\mid z\ \leq t)\}^2 \tag{9}$$

$$E(z^2|z\leq t) = \frac{\omega\int_0^t \left(z^{\omega+1}\right)\,dz}{\phi(t)}$$

Since ϕ (t) = t^{ω}

$$E(z^2|z\leq t) = \frac{\omega\,t^2}{\omega+2}$$

Also

$$\mathrm{M_z(t) - t} = \frac{\omega \mathrm{t}}{\omega+1} = \mathrm{E(z\mid z\leq t)}$$

Therefore (9) becomes

$$\mathrm{V(z\mid z\leq t)} = \frac{\omega \mathrm{t}^2}{\omega+2} - \left(\frac{\omega \mathrm{t}}{\omega+1}\right)^2$$

$$V\,(z\mid z\ \leq t) = t^2\left\{\frac{\omega}{\omega+2} - \frac{\omega^2}{(\omega+1)^2}\right\} \tag{10}$$

Sufficient Part:

Now $V\,(z\mid z\ \leq t)$ can be written as:

$$\mathrm{V(z\mid z\leq t)} = \frac{\int_0^{\mathrm{t}}\mathrm{z^2f(z)dz}}{\phi(\mathrm{t})} - \left(\frac{\int_0^{\mathrm{t}}\mathrm{zf(z)dz}}{\phi(\mathrm{t})}\right)^2$$

Using integration by parts

$$V\,(z\mid z\ \leq t) = \frac{t^2F(t) - \int_0^t 2z\,\phi(z)dz}{\phi(t)} - \left\{\frac{t\phi(t) - \int_0^t \phi(z)dz}{\phi(t)}\right\}^2$$

$$\mathrm{V(z\mid z\leq t)} = \left\{\mathrm{t}^2 - \frac{2\int_0^{\mathrm{t}}\mathrm{z}\phi(\mathrm{z})\mathrm{dz}}{\phi(\mathrm{t})}\right\} - \left\{\mathrm{t} - \frac{\int_0^{\mathrm{t}}\phi(\mathrm{z})\mathrm{dz}}{\phi(\mathrm{t})}\right\}^2 \tag{11}$$

Equating (10) and (11), we get

$$\left\{t^2 - \frac{2\int_0^t z\,\phi(z)dz}{\phi(t)}\right\} - \left\{t - \frac{\int_0^t\phi(z)dz}{\phi(t)}\right\}^2 = t^2\left\{\frac{\omega}{\omega+2} - \frac{\omega^2}{(\omega+1)^2}\right\}$$

Since $t - \frac{\int_0^t \phi(z)\,dz}{\phi(t)} = \frac{\omega t}{\omega+1}$

$$\left\{\mathrm{t}^2 - \frac{2\int_0^{\mathrm{t}} \mathrm{z}\phi(\mathrm{z})\mathrm{dz}}{\phi(\mathrm{t})}\right\} - \left\{\frac{\omega \mathrm{t}}{\omega+1}\right\}^2 = \mathrm{t}^2\left\{\frac{\omega}{\omega+2} - \frac{\omega^2}{(\omega+1)^2}\right\}$$

$$t^2 - \frac{2\int_0^t z\,\phi(z)dz}{\phi(t)} = \frac{t^2\omega}{\omega+2}$$

$$\int_0^t z\,\phi(z)\;dz = \frac{t^2}{\omega+2}\phi(t)$$

Differentiating "t" on both sides

$$\mathrm{tF(t)} = \frac{1}{\omega+2}\left[\mathrm{t}^2\mathrm{f(t)} + \phi(\mathrm{t})2\mathrm{t}\right]$$
$$\omega\phi(\mathrm{t}) = \mathrm{tf(t)}$$
$$\frac{\mathrm{f(t)}}{\phi(\mathrm{t})} = \frac{\omega}{\mathrm{t}}$$

ln (ϕ (t)) = $\omega\, ln(t)$

Therefore ϕ (t) = t^{ω}

and f(t)= $\omega\, t^{\omega-1}$

Therefore, it is proved that the "z" has PFD f(z) =$\omega\, z^{\omega-1}$ with parameter "ω" if and only if

$$\mathrm{V(z \mid z \leq t)} = \mathrm{t}^2\left\{\frac{\omega}{\omega+2} - \frac{\omega^2}{(\omega+1)^2}\right\}$$

where $\mathrm{E(z^r \mid z \leq t)}$: Conditional rth Moment (with respect to $\Phi(\mathrm{z})$)

3.2 Characterization of PFD By Conditional Moments

We may write the pdf and cdf for the PFD

$$\mathrm{f(z)} = \omega \mathrm{z}^{\omega-1}, \quad 0 < \mathrm{z} < 1$$
$$\phi(\mathrm{z}) = \mathrm{z}^{\omega}$$

And let $\bar{\phi}\,(z)$ be the survival function respectively. Then the random variable "z" has PFD with parameter "ω" if and only if

$$\mathrm{E}\left(\mathrm{z^r \mid z \leq t}\right) = \left\{\frac{\omega \mathrm{t^r}}{\omega+\mathrm{r}}\right\}, \text{ where } \mathrm{E}\left(\mathrm{z^r \mid z \leq t}\right): \text{ Conditional rth Moment (with respect to } \Phi(\mathrm{z}))$$

Proof:

Necessary part:

ϕ (z) = z^{ω}

$$E(z^r|z \leq t) = \frac{\int_0^t z^r\,\left(\omega\, z^{\omega-1}\right)dz}{\phi(t)}$$

Since ϕ (t) = t^{ω}

$$E(z^r|z \leq t) = \frac{\omega t^r}{\omega+r}$$

Sufficient Part:

Now $E(z^r|z \leq t)$ can be written as:

$$\mathrm{E}(\mathrm{z}^{\mathrm{r}} \mid \mathrm{z} \leq \mathrm{t}) = \frac{\int_0^t \mathrm{z}^{\mathrm{r}} \mathrm{f}(\mathrm{z}) \mathrm{dz}}{\phi(\mathrm{t})}$$

Using integration by parts

$$\mathrm{E}(\mathrm{z}^{\mathrm{r}} \mid \mathrm{z} \leq \mathrm{t}) = \frac{\mathrm{t}^{\mathrm{r}} \phi(\mathrm{t}) - \mathrm{r} \int_0^t \mathrm{z}^{\mathrm{r}-1} \phi(\mathrm{z}) \mathrm{dz}}{\phi(\mathrm{t})}$$

$$\mathrm{E}(\mathrm{z}^{\mathrm{r}} \mid \mathrm{z} \leq \mathrm{t}) = \mathrm{t}^{\mathrm{r}} - \frac{\mathrm{r} \int_0^t \mathrm{z}^{\mathrm{r}-1} \phi(\mathrm{z}) \mathrm{dz}}{\phi(\mathrm{t})} \tag{13}$$

Equating (12) and (13), we get

$$\begin{aligned} &\mathrm{t}^{\mathrm{r}} - \frac{\mathrm{r} \int_0^t \mathrm{z}^{\mathrm{r}-1} \phi(\mathrm{z}) \mathrm{dz}}{\phi(\mathrm{t})} = \frac{\omega \mathrm{t}^{\mathrm{r}}}{\omega + \mathrm{r}} \\ &\int_0^t \mathrm{z}^{\mathrm{r}-1} \phi(\mathrm{z}) \mathrm{dz} = \frac{\phi(\mathrm{t}) \mathrm{t}^{\mathrm{r}}}{\omega + \mathrm{r}} \end{aligned}$$

Differentiating “t” on both sides

$$\begin{aligned} &\mathrm{t}^{\mathrm{r}-1} \phi(\mathrm{t}) = \frac{1}{\omega + \mathrm{r}} \left[\mathrm{t}^{\mathrm{r}} \mathrm{f}(\mathrm{t}) + \phi(\mathrm{t}) \mathrm{r} \mathrm{t}^{\mathrm{r}-1}\right] \\ &\omega \phi(\mathrm{t}) = \mathrm{t} \mathrm{f}(\mathrm{t}) \\ &\frac{\mathrm{f}(\mathrm{t})}{\phi(\mathrm{t})} = \frac{\omega}{\mathrm{t}} \end{aligned}$$

Apply Integral on both sides

ln (ϕ (t)) = $\omega \, ln(t)$

Therefore ϕ (t) = t^{ω}

and f(t)= $\omega \, t^{\omega-1}$

Therefore it is proved that “z” has PFD, “f(z) = $\omega z^{\omega-1}$” with parameter “ω” if and only if

$$E(z^r \mid z \leq t) = \left\{ \frac{\omega t^r}{\omega + r} \right\}$$

$$where\ V(z^r|z \leq t) : Conditional\ Variance(with\ respect\ to\ \varnothing(z)) \ .$$

3.3 Characterization of PFD by incomplete moments

We may write the pdf and cdf for the PFD

$$\begin{aligned} &\mathrm{f}(\mathrm{z}) = \omega \mathrm{z}^{\omega-1}; \quad 0 < \mathrm{z} < 1 \\ &\phi(\mathrm{z}) = \mathrm{z}^{\omega} \end{aligned}$$

And let $\bar{\phi}(z)$ be the survival function respectively. Then the random variable “z” has PFD with parameter “ω” if and only if

$$\mu_{z|\,(\omega,\beta=1);r}(t) = \left(\frac{\omega t^{r+\omega}}{\omega + r} \right\};$$

$$where\ \mu_{z|\,(\omega,\beta=1);r}(t) : Incomplete\ rth\ Moment$$

Proof:

Necessary part:

$\phi(z) = z^{\omega}$

$$\mu_{z|(\omega,\beta=1);r}(t) = \int_0^t z^r \left(\omega z^{\omega-1}\right) dz$$

$$\mu_{z|(\omega,\beta=1);r}(t) = \frac{\omega\, t^{r+\omega}}{\omega + r}$$

Sufficient Part:

Now $\mu_{z|(\omega,\beta=1);r}(t)$ can be written as:

$$\mu_{z|(\omega,\beta=1);r}(t) = \int_0^t z^r f(z) dz$$

Using integration by parts

$$\mu_{z|(\omega,\beta=1);r}(t) = t^r \phi(t) - r\int_0^t z^{r-1}\phi(z)\, dz$$

Equating (14) and (15), we get

$$\mathrm{t^r F(t) - r\int_0^t z^{r-1}\phi(z)dz = \frac{\omega t^{r+\omega}}{\omega + r}}$$

Since $\phi(t) = t^{\omega}$

$$r\int_0^t z^{r-1}\phi(z)dz = t^r t^{\omega} - \frac{\omega t^{r+\omega}}{\omega + r}$$

Differentiating "t" on both sides

$$t^{r-1}\phi(t) = \frac{(\omega + r)t^{r+\omega-1}}{\omega + r}$$

Therefore ϕ (t) = t^{ω}

and f(t)= $\omega\, t^{\omega-1}$

Therefore it is proved that the "z" has PFD, f(z) = $\omega z^{\omega-1}$ with parameter "ω" if and only if

$$\mu_{z|(\omega,\beta=1);r}(t) = \left(\frac{\omega\, t^{r+\omega}}{\omega + r}\right\};$$

$$where\ \mu_{z|(\omega,\beta=1);r}(t) : Incomplete\ rth\ Moment$$

4 Characterization Of PFD, using reverse hazard function

We may assume any real number "t" such that Φ_z(t)>0 , r_w(t)=a r_z(t) with a>0 iff

$$a_W^{(n)}(t) = (-lnt^{\gamma})^n + \frac{n}{a\,\phi_W(t)}\int_0^t (-lnw^{\gamma})^{n-1} f_W(w)\, dw$$

Where U(t)= $-ln\phi_Z(t)$, $U'(t) = \frac{d}{dt}U(t) = -\, r_Z(t)$, $a_W^{(n)}(t) = E(U^n(W)\,|W < t)$,

"r_z(t)" and "r_w(t)" are the reverse hazard rates. Also follows the PFD.

Proof:

Necessary Part:

$$a_{\mathrm{W}}^{(\mathrm{n})}(\mathrm{t}) = \mathrm{E}\left(\mathrm{U}^{\mathrm{n}}(\mathrm{W}) \mid \mathrm{W} < \mathrm{t}\right)$$

$$a_W^{(n)}(t) = \frac{1}{\phi_W(t)} \int_0^t (-ln\phi_W(w))^n f_W(w)\, dw$$

$$a_W^{(n)}(t) = \frac{(-ln\phi_W(t))^n \phi_W(t)}{\phi_W(t)} - \frac{n}{\phi_W(t)} \int_0^t (-ln\phi_W(w))^{n-1}(-r_z(w))\, \phi_W(w)\, dw$$

Since $r_w(t)$=a $r_z(t)$

$$a_W^{(n)}(t) = (-ln\phi_W(t))^n + \frac{n}{\phi_W(t)} \int_0^t (-ln\phi_W(w))^{n-1}\left(\frac{r_W(w)}{a}\right) \phi_W(w)\, dw$$

$$a_W^{(n)}(t) = (-ln\phi_W(t))^n + \frac{n}{a\,\phi_W(t)} \int_0^t (-ln\phi_W(w))^{n-1} f_W(w)\, dw$$

Since $\phi_W(t) = t^\gamma$

$$a_W^{(n)}(t) = (-lnt^\gamma)^n + \frac{n}{a\,\phi_W(t)} \int_0^t (-lnw^\gamma)^{n-1} f_W(w)\, dw$$

Sufficient Part:

$$\frac{1}{\phi_W(t)} \int_0^t (-lnw^\gamma)^n f_W(w)\, dw = (-lnt^\gamma)^n + \frac{n}{a\,\phi_W(t)} \int_0^t (-lnw^\gamma)^{n-1} f_W(w)\, dw$$

$$\int_0^t (-lnw^\gamma)^n f_W(w)\, dw = \phi_W(t)(-lnt^\gamma)^n + \frac{n}{a} \int_0^t (-lnw^\gamma)^{n-1} f_W(w)\, dw$$

Taking derivative on both sides

$$(-\ln t^\gamma)^{\mathrm{n}} \mathrm{f_W}(\mathrm{t}) = (-\ln t^\gamma)^{\mathrm{n}} \mathrm{f_W}(\mathrm{t}) + \phi_{\mathrm{W}}(\mathrm{t})\mathrm{n}(-\ln t^\gamma)^{\mathrm{n}-1}\left(-\frac{1}{\mathrm{t}\gamma}\gamma t^{\gamma-1}\right) + \frac{\mathrm{n}}{\mathrm{a}}(-\ln t^\gamma)^{\mathrm{n}-1} \mathrm{f_W}(\mathrm{t}) \tag{16}$$

$$\phi_W(t)(-lnt^\gamma)^{n-1}\left(\frac{\gamma}{t}\right) = \frac{1}{a}(-lnt^\gamma)^{n-1} f_W(t)$$

$$a\left(\frac{\gamma}{t}\right) = \frac{f_W(t)}{\phi_W(t)}$$

$$a\gamma \ln(t) = \ln(\phi_w(t))$$
$$\phi_W(t) = t^{a\gamma}$$
$$f_W(t) = a\gamma t^{a\gamma-1}$$

Also (16) becomes

$$\phi_W(t)\; n\,(-lnt^\gamma)^{n-1}(-r_Z(t)) + \frac{n}{a}(-lnt^\gamma)^{n-1} f_W(t) = 0$$

$$\phi_W(t)\;\; (-lnt^\gamma)^{n-1} r_Z(t) = \frac{1}{a}(-lnt^\gamma)^{n-1} f_W(t)$$

$$r_Z(t) = \tfrac{1}{a} r_W(t)$$
$$r_W(t) = a\, r_Z(t)$$

5 Conclusion

The objective of the study is to acquire the characterization of PFD based on mean inactivity times (MIT), mean residual function (MRF), conditional moments, conditional variance, doubly truncated mean (DTM), incomplete moments and reverse hazard function (RHF). We hope that this study will be useful for the statisticians in various fields of studies.

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