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Symmetries of Icosahedral group and classification of G-circuits of length six

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Abstract

Objectives: To determine the exact number of equivalence classes of G-circuits of length $q \geq 2$. **Methods/Statistical Analysis:** To classify G-orbits of $Q(\sqrt{m})/Q$ containing G-circuits of length 6. **Findings:** The equivalence classes of G-circuits of length 6 is ten in number and determine the exact number of G-orbits and structure of G-orbits corresponding to each of ten equivalence classes of G-circuits. Furthermore, we describe some generalized G-circuits of length 2t corresponding to each of these ten equivalence classes and the structure of these G-circuits with conditions on t. **Applications/Improvements:** We employ Symmetries of Icosahedral group to explore cyclically equivalence classes of G-circuits and similar G-circuits of length 6 corresponding to each of these ten equivalence classes. This study helps us in classifying reduced numbers lying in $PSL(2, Z)$ -orbits. These results are verified by some suitable example.

Keywords: Rotational symmetries of icosahedral group; partition function; equivalence classes of G-circuits; reduced quadratic irrational numbers

1 Introduction

If $n = d^2m$, here $d \in \mathbb{N}$ and $m \neq e^2$ for some integer e . Then $Q^*(\sqrt{n}) = \{\alpha = \frac{a+\sqrt{n}}{c} : a, b = \frac{a^2-n}{c}, c \in \mathbb{Z}, c \neq 0 \text{ and } (a, b, c) = 1\}$ and $Q_{red}^*(\sqrt{n}) = \{\gamma \in Q^*(\sqrt{n}) : \gamma > 1 \text{ and } -1 < \bar{\gamma} < 0\}$. Then $(Q(\sqrt{m})/Q) = U_{k \in \mathbb{N}} Q^*(\sqrt{k^2m})$ contain $Q^*(\sqrt{n})$ and $Q_{red}^*(\sqrt{n})$ as G-subset and subset respectively.

For $\alpha = \frac{a+\sqrt{n}}{c} \in Q^*(\sqrt{n})$, its algebraic conjugate $\bar{\alpha} = \frac{a-\sqrt{n}}{c}$ may or may not have the same sign as that of α . If α and $\bar{\alpha}$ have opposite signs, then α is an ambiguous number, that is, α is ambiguous number if and only if $a^2 < n$ ⁽¹⁾. An ambiguous number γ is reduced if $\gamma > 1$ and $-1 < \bar{\gamma} < 0$. Note that if γ is a reduced number, then $\bar{\gamma}$, $-\bar{\gamma}$ and $-\gamma$ are the ambiguous numbers which are not reduced.⁽¹⁾ The modular group is actually the group of 2×2 matrices whose entries are integers and determinant is one.

Throughout this paper $p(q)$ denote the number of partitions of q whereas D_n stands for Dihedral group of order $2n$ and S_n stands for Symmetric group of order $n!$. We know that $|S_n| = \frac{(n-1)!}{2} \times 2n = (A_{n-1}) \times (D_n)$. Particularly, $|S_6| = \frac{(6-1)!}{2} \times 12 = (A_5) \times (D_6)$ and it is well-known that Symmetries of Icosahedral group is isomorphic to alternating group A_5 .

In ⁽²⁾ numbers of distinct circuits of length 4 are obtained by using fixed number of T triangles. In ⁽³⁾ numbers of distinct homomorphic images are obtained by contracting all the pairs of vertices. In ⁽⁴⁾ authors discussed action of two Hecke groups. One of them is the modular group $H(\lambda_3)$ and other is $H(\lambda_4)$. In ⁽⁵⁾ authors determine the transitive H -set of $Q(\sqrt{m}) \setminus Q$ by using structure of circuits.

In ⁽⁶⁾, it was proved that the ambiguous numbers in the orbit $\gamma^G, \gamma \in Q^*(\sqrt{n})$ makes a G-circuit or simply circuit. Thus it becomes interesting to classify circuit.

In ⁽⁷⁾ authors determine that exact number of equivalence classes of G-circuits of length 4 is 4 in number and classify G-orbits of $Q(\sqrt{m})/Q$ containing G-circuits of length 4. Thus it becomes interesting to find to determine exact number of equivalence classes of G-circuits of length $q \geq 2$.

If $l'_1, l'_3, l'_5, \dots, l'_{2t-1}$ and $l'_2, l'_4, l'_6, \dots, l'_{2t}$ are positive integers, then we say $[l'_1, l'_2, l'_3, l'_4, \dots, l'_{2t-1}, l'_{2t}]$ is a circuit if it has l'_{2i-1} triangles has two edges outside the circuit, l'_{2i} triangles has two edges inside the circuit where $1 \leq i \leq t$.

Throughout this paper G-circuit (resp.G-orbit) will be simply denoted by circuit (resp.orbit). The concept of circuit grew out of the study of Group action on $Q(\sqrt{m}) \cup \{\infty\}$ and the study of G-subsets $Q^*(\sqrt{n})$.

In this paper, we define what a circuit of specific length is and we classify non equivalent circuits of length 6 so as to classify orbits containing these circuits. We also consider some of the elementary concepts associated with equivalent circuits, cyclically equivalent circuits and similar circuits are introduced to explore transitive G-subsets (called orbits) of $Q^*(\sqrt{n})$.

The definition of an equivalent circuit that is now standard was a long time in being formulated.

Two circuits $[l'_1, l'_2, l'_3, l'_4, \dots, l'_{2t-1}, l'_{2t}]$ and $[l_1, l_2, l_3, l_4, l_5, l_6, \dots, l_{2t-1}, l_{2t}]$ are said to be equivalent if and only if $[l'_{(1)\theta}, l'_{(2)\theta}, l'_{(3)\theta}, l'_{(4)\theta}, \dots, l'_{(2t-1)\theta}, l'_{(2t)\theta}] = [l_1, l_2, l_3, l_4, l_5, l_6, \dots, l_{2t-1}, l_{2t}]$,

where $\theta \in S_{2t}$. That is the circuits are equivalent to $[l'_1, l'_2, l'_3, l'_4, \dots, l'_{2t-1}, l'_{2t}]$ if and only if they are obtained just by permuting the entries $l'_1, l'_2, l'_3, l'_4, \dots, l'_{2t-1}, l'_{2t}$. Notation for equivalent is " \sim ". It is easy to see that being equivalent of circuits is an equivalence relation. Thus, a property which is possessed by one circuit that is also possessed by all equivalent circuits. Such properties which are preserved under equivalent are called equivalent properties or circuit invariant.

Two circuits $[l'_1, l'_2, l'_3, l'_4, \dots, l'_{2t-1}, l'_{2t}]$ and $[l_1, l_2, l_3, l_4, l_5, l_6, \dots, l_{2t-1}, l_{2t}]$ are said to be cyclically equivalent if and only if the circuit $[l'_{(1)\theta}, l'_{(2)\theta}, l'_{(3)\theta}, l'_{(4)\theta}, \dots, l'_{(2t-1)\theta}, l'_{(2t)\theta}] = [l_1, l_2, l_3, l_4, l_5, l_6, \dots, l_{2t-1}, l_{2t}]$,

where $\theta \in D_t$. Notation for cyclically equivalent is " \sim_c ". It is easy to see that being cyclically equivalent of circuits is an equivalence relation.

Two circuits are said to be similar if they represent the same orbit. That is two circuits $[l'_1, l'_2, l'_3, l'_4, \dots, l'_{2t-1}, l'_{2t}]$ and $[l_1, l_2, l_3, l_4, l_5, l_6, \dots, l_{2t-1}, l_{2t}]$ are said to be similar if and only if $[l'_{(1)\theta}, l'_{(2)\theta}, l'_{(3)\theta}, l'_{(4)\theta}, \dots, l'_{(2t-1)\theta}, l'_{(2t)\theta}] = [l_1, l_2, l_3, l_4, l_5, l_6, \dots, l_{2t-1}, l_{2t}]$,

where $\theta \in C_t = \langle (1\ 3\ 5 \dots 2t-1)(2\ 4\ 6 \dots 2t) \rangle$. Notation for cyclically equivalent is " \sim_s ".

It is interesting to note that the orbit containing circuit $[l'_1, l'_2, l'_3, l'_4, \dots, l'_{2t-1}, l'_{2t}]$ has exactly

$2(l'_1 + l'_2 + l'_3 + \dots + l'_{2t-1} + l'_{2t})$ ambiguous numbers while this circuit consists of only t number of reduced numbers. Thus studying orbits with the help of reduced numbers is fruitful and economical.

Throughout this paper $l'_1, l'_2, l'_3, l'_4, l'_5, l'_6$ denote distinct positive integers and the expression $\frac{d + \sqrt{d^2 + 4ef}}{2f}$ is replaced by $\frac{d}{h} + \frac{\sqrt{d^2 + 4ef}}{h^2}$ when $(d, (2e, 2f)) = h > 1$. Classification of non-equivalent circuits and cyclically equivalent circuits play a significant role to obtain the orbits of $Q(\sqrt{m}) \setminus Q$ because with this task of finding orbits is simplified.

In ⁽⁸⁾, Authors have found G-subsets of $Q^*(\sqrt{k^2m}) \subseteq (Q(\sqrt{m})|Q) = U_{k \in N} Q^*(\sqrt{k^2m})$. So it becomes interesting to explore transitive G-subsets called orbits.

2 Materials and Methods

The following results of ⁽⁸⁻¹¹⁾ are used in the sequel.

Lemma 2.1 ⁽⁹⁾ If γ^G has a circuit $[l'_1, l'_2, l'_3, l'_4, l'_5, l'_6]$, then $[l'_2, l'_3, l'_4, l'_5, l'_6, l'_1]$, $[l'_1, l'_6, l'_5, l'_4, l'_3, l'_2]$, $[l'_6, l'_5, l'_4, l'_3, l'_2, l'_1]$ are the circuits of $(-\gamma)^G$, $(-\bar{\gamma})^G$, $(\bar{\gamma})^G$ respectively.

Lemma 2.2 ⁽¹⁰⁾ An ambiguous number $\alpha = \frac{a+\sqrt{a^2-4bc}}{c}$ with $c > 0$ is reduced if and only if $|b+c| < 2a$.

Lemma 2.3 ⁽⁸⁾ If $l'_1, l'_2, l'_3, l'_4, \dots, l'_{2t-1}, l'_{2t}$ are positive integers then the circuit whose period length is $2t'$ with t' divides t is impossible to exist.

Lemma 2.4 ⁽¹¹⁾ The number of different arrangements of q objects of which q_1 are alike, q_2 are alike, q_3 are alike, \dots , q_r are alike is $\frac{q!}{q_1! q_2! q_3! \dots q_r!}$ where $q_1 + q_2 + q_3 + \dots + q_r = q$.

3 Classification of Circuits of Length Six

Circuits are very important in the study of modular group acting on quadratic field.

We start this section with a consideration of finding the non equivalent relation of circuits in a specific length q . Given the positive integer q , we say that the sequence of positive integers $q_1, q_2, q_3, q_4, \dots, q_{r-1}, q_r$ with $q_1 \geq q_2 \geq q_3 \geq q_4 \geq \dots \geq q_{r-1} \geq q_r$ constitute a partition of q if $q = q_1 + q_2 + q_3 + \dots + q_r$.

Theorem 3.1: Number of equivalence classes of length $q > 2$ are exactly $p(q) - 1$, where $p(q)$ denotes the partition of q .

Proof: Let $p(q)$ denote the partition of q and we are looking in determining all equivalence classes of equivalent circuits of length q . For a given circuit of length q , to find all equivalence classes of equivalent circuits of length q . We adopt partitions of $q = q_1 + q_2 + q_3 + \dots + q_r$, in the sense that if q_1 entries are alike, q_2 entries are alike, q_3 entries are alike q_r entries are alike, where $r \geq 2$. We get non-equivalent circuits $[l'_1, l'_1, l'_1, \dots, l'_1, l'_2, l'_2, \dots, l'_2, \dots, l'_r, l'_r, \dots, l'_r]$ of length q correspond to each partitions of q .

where l'_i repeats q_i times, $1 \leq i \leq r$.

Here $r \neq 1$ because if $q = q_1$ then circuits corresponding to this partition is $[l'_1, l'_1, l'_1, \dots, l'_1]$, where l'_1 repeats q times. This circuit of length $q > 2$ is not possible by Lemma 2.3. So distinct classes of equivalent circuit of length $q > 2$ are exactly $p(q) - 1$.

Remark 3.1.1: For $q = 2$, there are precisely two partitions namely 2 and 1+1. Circuits of length 2 corresponding to these partitions are $[l'_1, l'_1]$ and $[l'_1, l'_2]$. Furthermore, structure of these circuits are $\gamma^G = (\bar{\gamma})^G = (-\bar{\gamma})^G = (-\gamma)^G$ and $\gamma^G = (-\bar{\gamma})^G$ as these circuits are special cases of Note 3.7.2 and Note 3.6.1.

In the following theorems, we discuss Equivalence classes of circuits, Cyclically Equivalence classes of circuits, Equivalent circuits, Cyclically Equivalent circuits, Similar circuits of lengths 6.

Corollary 3.2: There are precisely ten non-equivalent circuits of length six.

Proof: Let $l'_1, l'_2, l'_3, l'_4, l'_5, l'_6$ be different positive integers. Then by Theorem 3.1 We have ten non-equivalent circuits namely

$$[l'_1, l'_2, l'_3, l'_4, l'_5, l'_6], [l'_1, l'_1, l'_2, l'_3, l'_4, l'_5], [l'_1, l'_1, l'_1, l'_2, l'_3, l'_4], [l'_1, l'_1, l'_2, l'_2, l'_3, l'_4], [l'_1, l'_1, l'_2, l'_2, l'_3, l'_3]$$

$$[l'_1, l'_1, l'_1, l'_2, l'_2, l'_3], [l'_1, l'_1, l'_1, l'_1, l'_2, l'_3], [l'_1, l'_1, l'_1, l'_1, l'_2, l'_2], [l'_1, l'_1, l'_1, l'_1, l'_1, l'_2], [l'_1 l'_1, l'_1, l'_2, l'_2, l'_2]$$

corresponding to the number $p(6) - 1$ namely 1+1+1+1+1+1, 2+1+1+1+1, 3+1+1+1, 2+2+1+1, 2+2+2, 3+2+1, 4+1+1, 4+2, 5+1, 3+3 respectively.

Circuit corresponding to summand 6 is $[l'_1, l'_1, l'_1, l'_1, l'_1, l'_1]$. This circuit is not possible by Lemma 2.3. So these are the only ten non-equivalent circuits of length 6. The notation used in this paper for equivalence classes of circuits of length 6 is E_{T_i} and number of circuits equivalent to T_i is denoted by $|E_{T_i}|$. Similarly, the notation for cyclically equivalent classes is $E_{T_i}^c$ and $|E_{T_i}^c|$ denotes the number of circuits cyclically equivalent to T_i .

The number of distinct orbits corresponding to C_{T_i} is denoted by $|O_{T_i}|$. Furthermore, each cyclically equivalent class $E_{T_i}^c$ is discussed in each corresponding relevant corollaries.

Corollary 3.3: There are 720 equivalent circuits of length 6 in which all numbers are different.

Proof: It is well-known that $S_n = \{(D_n) \varphi : \varphi \in A_{n-1}\}$. In our case $S_6 = \{(D_6) \varphi : \varphi \in A_5\}$. We know that there are $6! = 720$ arrangements of six different numbers $l'_1, l'_2, l'_3, l'_4, l'_5, l'_6$ taken all at time and so circuits corresponding to these arrangements are $[l'_{(1)\theta}, l'_{(2)\theta}, l'_{(3)\theta}, l'_{(4)\theta}, l'_{(5)\theta}, l'_{(6)\theta}]$ for each $\theta \in S_6$. Hence, the proof.

Corollary 3.4: If $l'_1, l'_2, l'_3, l'_4, l'_5, l'_6$ are distinct positive integers, then there exists 60 cyclically equivalent classes $E_{T_1}^C$.

Proof: Let $l'_1, l'_2, l'_3, l'_4, l'_5, l'_6$ be different positive integers. It is well known that Icosahedral group is isomorphic to Alternating group A_5 . Now the cyclically equivalent classes $E_{T_i}^C$ are obtained by $E_{[l'_{(1)\varphi}, l'_{(2)\varphi}, l'_{(3)\varphi}, l'_{(4)\varphi}, l'_{(5)\varphi}, l'_{(6)\varphi}]}$ for each $\varphi \in A_5$.

These are exactly 60 cyclically equivalent classes namely

$$\begin{aligned}
 & E_{[l'_1, l'_2, l'_3, l'_4, l'_5, l'_6]}^C, E_{[l'_2, l'_1, l'_4, l'_2, l'_5, l'_6]}^C, E_{[l'_2, l'_1, l'_5, l'_4, l'_3, l'_6]}^C, E_{[l'_4, l'_3, l'_2, l'_1, l'_5, l'_6]}^C, E_{[l'_5, l'_3, l'_2, l'_4, l'_1, l'_6]}^C, \\
 & E_{[l'_3, l'_4, l'_1, l'_2, l'_5, l'_6]}^C, E_{[l'_5, l'_4, l'_2, l'_2, l'_1, l'_6]}^C, E_{[l'_5, l'_2, l'_4, l'_2, l'_1, l'_6]}^C, E_{[l'_4, l'_2, l'_5, l'_1, l'_3, l'_6]}^C, E_{[l'_2, l'_1, l'_3, l'_5, l'_4, l'_6]}^C, \\
 & E_{[l'_3, l'_2, l'_1, l'_5, l'_4, l'_6]}^C, E_{[l'_3, l'_5, l'_1, l'_4, l'_2, l'_6]}^C, E_{[l'_4, l'_5, l'_3, l'_1, l'_2, l'_6]}^C, E_{[l'_1, l'_4, l'_5, l'_2, l'_3, l'_6]}^C, \\
 & E_{[l'_1, l'_5, l'_4, l'_3, l'_2, l'_6]}^C, E_{[l'_2, l'_4, l'_2, l'_1, l'_5, l'_6]}^C, E_{[l'_2, l'_5, l'_3, l'_4, l'_1, l'_6]}^C, E_{[l'_3, l'_2, l'_4, l'_1, l'_5, l'_6]}^C, \\
 & E_{[l'_3, l'_2, l'_5, l'_4, l'_1, l'_6]}^C, E_{[l'_4, l'_2, l'_3, l'_5, l'_1, l'_6]}^C, E_{[l'_2, l'_3, l'_1, l'_4, l'_5, l'_6]}^C, E_{[l'_3, l'_1, l'_2, l'_4, l'_5, l'_6]}^C, \\
 & E_{[l'_4, l'_1, l'_3, l'_2, l'_5, l'_6]}^C, E_{[l'_5, l'_1, l'_3, l'_4, l'_2, l'_6]}^C, \\
 & E_{[l'_4, l'_2, l'_1, l'_3, l'_5, l'_6]}^C, E_{[l'_1, l'_4, l'_2, l'_3, l'_5, l'_6]}^C, E_{[l'_1, l'_2, l'_4, l'_5, l'_3, l'_6]}^C, E_{[l'_1, l'_3, l'_5, l'_4, l'_2, l'_6]}^C, \\
 & E_{[l'_1, l'_2, l'_5, l'_3, l'_4, l'_6]}^C, E_{[l'_5, l'_2, l'_2, l'_1, l'_4, l'_6]}^C, E_{[l'_1, l'_3, l'_4, l'_2, l'_5, l'_6]}^C, E_{[l'_1, l'_5, l'_3, l'_2, l'_4, l'_6]}^C, E_{[l'_1, l'_4, l'_2, l'_5, l'_2, l'_6]}^C, E_{[l'_5, l'_2, l'_1, l'_4, l'_2, l'_6]}^C \\
 \\
 & E_{[l'_1, l'_5, l'_2, l'_4, l'_3, l'_6]}^C, E_{[l'_2, l'_3, l'_4, l'_5, l'_1, l'_6]}^C, E_{[l'_2, l'_4, l'_5, l'_3, l'_1, l'_6]}^C, E_{[l'_3, l'_4, l'_2, l'_5, l'_1, l'_6]}^C, E_{[l'_3, l'_5, l'_4, l'_2, l'_1, l'_6]}^C, E_{[l'_4, l'_3, l'_5, l'_2, l'_1, l'_6]}^C, \\
 & E_{[l'_4, l'_5, l'_2, l'_3, l'_1, l'_6]}^C, E_{[l'_3, l'_1, l'_4, l'_5, l'_2, l'_6]}^C, E_{[l'_4, l'_1, l'_5, l'_3, l'_2, l'_6]}^C, E_{[l'_4, l'_3, l'_1, l'_5, l'_2, l'_6]}^C, E_{[l'_5, l'_3, l'_4, l'_1, l'_2, l'_6]}^C, \\
 & E_{[l'_3, l'_4, l'_5, l'_1, l'_2, l'_6]}^C, E_{[l'_4, l'_1, l'_2, l'_5, l'_3, l'_6]}^C, E_{[l'_5, l'_4, l'_2, l'_1, l'_3, l'_6]}^C, E_{[l'_5, l'_1, l'_4, l'_2, l'_3, l'_6]}^C, E_{[l'_2, l'_3, l'_5, l'_1, l'_4, l'_6]}^C, \\
 & E_{[l'_3, l'_5, l'_2, l'_1, l'_4, l'_6]}^C, E_{[l'_3, l'_1, l'_5, l'_2, l'_4, l'_6]}^C, E_{[l'_5, l'_3, l'_1, l'_2, l'_4, l'_6]}^C, E_{[l'_2, l'_5, l'_1, l'_3, l'_4, l'_6]}^C, E_{[l'_2, l'_5, l'_4, l'_1, l'_3, l'_6]}^C, E_{[l'_5, l'_1, l'_2, l'_3, l'_4, l'_6]}^C, \\
 & E_{[l'_1, l'_3, l'_2, l'_5, l'_4, l'_6]}^C, E_{[l'_5, l'_4, l'_1, l'_3, l'_2, l'_6]}^C, E_{[l'_2, l'_4, l'_1, l'_5, l'_3, l'_6]}^C, E_{[l'_4, l'_5, l'_1, l'_2, l'_3, l'_6]}^C.
 \end{aligned}$$

Following is the generalization of the above circuit T_1 whose length is “2t” and its proof can be obtained by applying induction on “t”.

Note 3.4.1: If $[l'_1, l'_2, l'_3, l'_4, \dots, l'_{2t-1}, l'_{2t}]$ is circuit contained in the orbit γ^G then $\gamma^G, (-\gamma)^G, (\bar{\gamma})^G$ and $(-\bar{\gamma})^G$ are all distinct.

Corollary 3.5: If $l'_1, l'_2, l'_3, l'_4, l'_5, l'_6$ are distinct positive integers, then each cyclically equivalent class contain 12 cyclically equivalent circuits.

Proof: Circuits cyclically equivalent to circuit $[l'_1, l'_2, l'_3, l'_4, l'_5, l'_6]$ are $[l'_{(1)\theta}, l'_{(2)\theta}, l'_{(3)\theta}, l'_{(4)\theta}, l'_{(5)\theta}, l'_{(6)\theta}]$ for each $\theta \in D_6$ which is shown in Table 1. Similarly we can find cyclically equivalent classes corresponding to remaining circuits.

Circuits of length 6 which are given in Table 1 are cyclically equivalent, Moreover by Lemma 2.1, $[l'_1, l'_2, l'_3, l'_4, l'_5, l'_6] \sim_s [l'_3, l'_4, l'_5, l'_6, l'_1, l'_2] \sim_s [l'_5, l'_6, l'_1, l'_2, l'_3, l'_4]$ is the circuit contained in γ^G , $[l'_2, l'_3, l'_4, l'_5, l'_6, l'_1] \sim_s [l'_4, l'_5, l'_6, l'_1, l'_2, l'_3] \sim_s [l'_6, l'_1, l'_2, l'_3, l'_4, l'_5]$ is the circuit contained in $(-\gamma)^G$, $[l'_1, l'_6, l'_5, l'_4, l'_3, l'_2] \sim_s [l'_5, l'_4, l'_3, l'_2, l'_1, l'_6] \sim_s [l'_3, l'_2, l'_1, l'_6, l'_5, l'_4]$ is the circuit contained in $(-\bar{\gamma})^G$, $[l'_6, l'_5, l'_4, l'_3, l'_2, l'_1] \sim_s [l'_4, l'_3, l'_2, l'_1, l'_6, l'_5] \sim_s [l'_2, l'_1, l'_6, l'_5, l'_4, l'_3]$ is the circuit contained in $(\bar{\gamma})^G$.

From Table 1 it is easy to see that the effect of permutation on circuit is same as to change the places of circuit accordingly. So if circuit in which at least two entries are same we change places of circuits according to permutation. As each circuit of length 6 contains 3 reduced numbers, so each cyclically equivalent class contains 12 reduced numbers. Since there are 60 cyclically equivalent classes each class contains 12 reduced numbers, so reduced numbers used in E_{T_1} are $12 \times 60 = 720$ equals to $[E_{T_1}]$.

Table 1. Cyclically Equivalent circuits Length 6

θ	$[l'_{(1)\theta}, l'_{(2)\theta}, l'_{(3)\theta}, l'_{(4)\theta}, l'_{(5)\theta}, l'_{(6)\theta}]$
$e = (1)(2)(3)(4)(5)(6)$	$[l'_1, l'_2, l'_3, l'_4, l'_5, l'_6]$
$\alpha = (1\ 2\ 3\ 4\ 5\ 6)$	$[l'_2, l'_3, l'_4, l'_5, l'_6, l'_1]$
$\alpha^2 = (1\ 3\ 5)(2\ 4\ 6)$	$[l'_3, l'_4, l'_5, l'_6, l'_1, l'_2]$
$\alpha^3 = (1\ 4)(2\ 5)(3\ 6)$	$[l'_4, l'_5, l'_6, l'_1, l'_2, l'_3]$
$\alpha^4 = (1\ 5\ 3)(2\ 6\ 4)$	$[l'_5, l'_6, l'_1, l'_2, l'_3, l'_4]$
$\alpha^5 = (1\ 6\ 5\ 4\ 3\ 2)$	$[l'_6, l'_1, l'_2, l'_3, l'_4, l'_5]$
$\beta = (1\ 6)(2\ 5)(3\ 4)$	$[l'_6, l'_5, l'_4, l'_3, l'_2, l'_1]$
$\alpha\beta = (1\ 5)(2\ 4)$	$[l'_5, l'_4, l'_3, l'_2, l'_1, l'_6]$
$\alpha^2\beta = (1\ 4)(2\ 3)(5\ 6)$	$[l'_4, l'_3, l'_2, l'_1, l'_6, l'_5]$
$\alpha^3\beta = (1\ 3)(4\ 6)$	$[l'_3, l'_2, l'_1, l'_6, l'_5, l'_4]$
$\alpha^4\beta = (1\ 2)(4\ 5)(3\ 6)$	$[l'_2, l'_1, l'_6, l'_5, l'_4, l'_3]$
$\alpha^5\beta = (3\ 5)(2\ 6)$	$[l'_1, l'_6, l'_5, l'_4, l'_3, l'_2]$

The following Table 2 is of considerable utility because it provides us with exact number of circuits of length 6 and hence the number of G-orbits of $Q^*(\sqrt{n})$.

Table 2. Enumeration of orbits

Cecl6	$ E_{T_i} $	Ncec	$ O_{T_i} $	Rdo
E_{T_1}	720	60	240	$\gamma^G, (-\bar{\gamma})^G, (\bar{\gamma})^G, (-\gamma)^G$
E_{T_2}	360	30	120	$\gamma^G, (-\bar{\gamma})^G, (\bar{\gamma})^G, (-\gamma)^G$
E_{T_3}	168	14	56	$\gamma^G, (-\bar{\gamma})^G, (\bar{\gamma})^G, (-\gamma)^G$
	12	2	4	$\gamma^G = (-\bar{\gamma})^G, (\bar{\gamma})^G = (-\gamma)^G$
E_{T_4}	120	10	40	$\gamma^G, (-\bar{\gamma})^G, (\bar{\gamma})^G, (-\gamma)^G$
	48	4	16	$\gamma^G, (-\bar{\gamma})^G, (\bar{\gamma})^G, (-\gamma)^G$
E_{T_5}	18	3	6	$\gamma^G = (-\bar{\gamma})^G, (\bar{\gamma})^G = (-\gamma)^G$
	18	3	6	$\gamma^G = (\bar{\gamma})^G, (-\bar{\gamma})^G = (-\gamma)^G$
	6	1	2	$\gamma^G = (-\gamma)^G, (\bar{\gamma})^G = (-\bar{\gamma})^G$
E_{T_6}	48	4	16	$\gamma^G, (-\bar{\gamma})^G, (\bar{\gamma})^G, (-\gamma)^G$
	12	2	4	$\gamma^G = (-\bar{\gamma})^G, (\bar{\gamma})^G = (-\gamma)^G$
E_{T_7}	24	2	8	$\gamma^G, (-\bar{\gamma})^G, (\bar{\gamma})^G, (-\gamma)^G$
	6	1	2	$\gamma^G = (-\bar{\gamma})^G, (\bar{\gamma})^G = (-\gamma)^G$
E_{T_8}	12	1	4	$\gamma^G, (-\bar{\gamma})^G, (\bar{\gamma})^G, (-\gamma)^G$
	6	1	2	$\gamma^G = (-\bar{\gamma})^G, (\bar{\gamma})^G = (-\gamma)^G$
E_{T_9}	6	1	2	$\gamma^G = (\bar{\gamma})^G, (-\gamma)^G = (-\bar{\gamma})^G$
	3	1	1	$\gamma^G = (-\bar{\gamma})^G = (\bar{\gamma})^G = (-\gamma)^G$
	6	1	2	$\gamma^G = (-\bar{\gamma})^G, (\bar{\gamma})^G = (-\gamma)^G$
$E_{T_{10}}$	6	1	2	$\gamma^G = (-\bar{\gamma})^G, (-\gamma)^G = (\bar{\gamma})^G$

Cecl6: Classes of equivalent circuits of length 6;

Ncec: Number of classes of cyclically equivalent circuits;

Rdo: Regarding distinct orbits

Corollary 3.6: There are 6 Equivalent circuits of length 6 in which 5 numbers are alike and one number is different.

Proof: By Lemma 2.4, the number of equivalent circuits of length 6 in which 5 numbers are alike is $\frac{6!}{5!1!} = 6$. These 6 circuits are $[l'_1, l'_1, l'_1, l'_1, l'_1, l'_2]$, $[l'_1, l'_1, l'_1, l'_1, l'_2, l'_1]$, $[l'_1, l'_1, l'_1, l'_1, l'_2, l'_1]$, $[l'_2, l'_1, l'_1, l'_1, l'_1, l'_1]$, $[l'_1, l'_1, l'_1, l'_2, l'_1, l'_1]$ and $[l'_1, l'_1, l'_2, l'_1, l'_1, l'_1]$. Clearly all these types are cyclically equivalent as well. Moreover by Lemma

2.1, $[l'_1, l'_1, l'_1, l'_1, l'_1, l'_2] \sim_s [l'_1, l'_2, l'_1, l'_1, l'_1, l'_1] \sim_s [l'_1, l'_1, l'_1, l'_2, l'_1, l'_1]$ is the circuit contained in $\gamma^G = (-\bar{\gamma})^G$ and $[l'_2, l'_1, l'_1, l'_1, l'_1, l'_1] \sim_s [l'_1, l'_1, l'_1, l'_1, l'_2, l'_1] \sim_s [l'_1, l'_1, l'_2, l'_1, l'_1, l'_1]$ is the circuit contained in $(\bar{\gamma})^G = (-\gamma)^G$. In this situation there is only one cyclically equivalent class namely $E^c_{[l'_1, l'_1, l'_1, l'_1, l'_1, l'_2]}$.

Following is the generalization of the above circuit T_{10} whose length is “2t” and its proof can be obtained by applying induction on “t”.

Note 3.6.1: If $[l'_1, l'_1, l'_1, l'_1, \dots, l'_1, l'_2]$ is circuit contained in the orbit of γ^G where l'_1 repeats “2t-1” times then $\gamma^G = (-\bar{\gamma})^G$.

Corollary 3.7: There are 15 equivalent circuits of length 6 in which 4 numbers are alike and 2 numbers are alike.

Proof: By Lemma 2.4, the number of equivalent circuits of length 6 in which 4 numbers are alike and 2 numbers are alike is $\frac{6!}{4!2!} = 15$. These possible 15 circuits are $[l'_1, l'_1, l'_1, l'_1, l'_2, l'_2]$, $[l'_1, l'_1, l'_1, l'_2, l'_1, l'_2]$, $[l'_1, l'_1, l'_1, l'_2, l'_2, l'_1]$, $[l'_1, l'_1, l'_2, l'_1, l'_1, l'_2]$, $[l'_1, l'_1, l'_2, l'_1, l'_2, l'_1]$, $[l'_1, l'_1, l'_2, l'_2, l'_1, l'_1]$, $[l'_1, l'_2, l'_1, l'_1, l'_1, l'_2]$, $[l'_1, l'_2, l'_1, l'_1, l'_2, l'_1]$, $[l'_1, l'_2, l'_1, l'_2, l'_1, l'_1]$, $[l'_1, l'_2, l'_2, l'_1, l'_1, l'_1]$, $[l'_2, l'_2, l'_1, l'_1, l'_1, l'_1]$, $(l'_2, l'_1, l'_1, l'_2, l'_1, l'_1)$, $[l'_2, l'_1, l'_1, l'_1, l'_2, l'_1]$, $[l'_2, l'_1, l'_1, l'_1, l'_2, l'_1]$ are equivalent circuits.

In the above circuits $[l'_1, l'_1, l'_1, l'_1, l'_2, l'_2]$, $[l'_1, l'_1, l'_1, l'_2, l'_2, l'_1]$, $[l'_1, l'_2, l'_2, l'_1, l'_1, l'_1]$, $[l'_2, l'_2, l'_1, l'_1, l'_1, l'_1]$, $[l'_1, l'_1, l'_2, l'_2, l'_1, l'_1]$, $[l'_2, l'_1, l'_1, l'_1, l'_2, l'_1]$ are cyclically equivalent. Moreover by Lemma 2.1,

$[l'_1, l'_1, l'_1, l'_1, l'_2, l'_2] \sim_s [l'_1, l'_1, l'_2, l'_2, l'_1, l'_1] \sim_s [l'_2, l'_2, l'_1, l'_1, l'_1, l'_1]$ is the circuit contained in $\alpha^G = (\bar{\alpha})^G$ and $[l'_2, l'_1, l'_1, l'_1, l'_1, l'_2] \sim_s [l'_1, l'_2, l'_2, l'_1, l'_1, l'_1] \sim_s [l'_1, l'_1, l'_1, l'_2, l'_2, l'_1]$ is the circuit contained in $(-\bar{\alpha})^G = (-\alpha)^G$.

Structure of G-orbit of remaining circuits can be found from Table 3.

Table 3.

$E^c_{T_9}$	Structure of orbits	Nce	$ O_{T_9} $
$E^c_{[l'_1, l'_1, l'_1, l'_1, l'_2, l'_2]}$	$\gamma^G = (\bar{\gamma})^G, (-\gamma)^G = (-\bar{\gamma})^G$	1	2
$E^c_{[l'_1, l'_2, l'_1, l'_2, l'_1, l'_1]}$	$\gamma^G = (-\bar{\gamma})^G, (-\gamma)^G = (\bar{\gamma})^G$	1	2
$E^c_{[l'_1, l'_1, l'_2, l'_1, l'_1, l'_2]}$	$\gamma^G = (-\bar{\gamma})^G = (-\gamma)^G = (\bar{\gamma})^G$	1	1

Nce: Number of classes of cyclically equivalent circuits;

Following is the generalizations of the above circuit T_9 whose length is “2t” and its proof can be obtained by applying induction on “t”.

Note 3.7.1: If $[l'_1, l'_1, l'_1, l'_1, \dots, l'_1, l'_2, l'_2]$ is circuit contained in the orbit of γ^G then $\gamma^G = (\bar{\gamma})^G$.

Note 3.7.2: If $[l'_1, l'_1, l'_1, \dots, l'_1, l'_2, l'_1, l'_1, l'_1, \dots, l'_1, l'_2]$ is circuit contained in the orbit of γ^G where l'_1 repeats “t-1” times and $2t \equiv 2 \pmod{4}$ then $\gamma^G = (\bar{\gamma})^G = (-\gamma)^G = (-\bar{\gamma})^G$.

Corollary 3.8: There are 18 equivalent circuits of length 6 in which 3 numbers are alike and 3 numbers are alike.

Proof: By Lemma 2.4, the number of possible equivalent circuits of length 6 in which 3 numbers are alike and 3 numbers are alike is $\frac{6!}{3!3!} = 20$. These possible 20 circuits are given by

$$\begin{bmatrix} l'_1, l'_1, l'_1, l'_2, l'_2, l'_2 \end{bmatrix}, \begin{bmatrix} l'_1, l'_1, l'_2, l'_1, l'_2, l'_2 \end{bmatrix}, \begin{bmatrix} l'_1, l'_1, l'_2, l'_2, l'_1, l'_2 \end{bmatrix}, \begin{bmatrix} l'_1, l'_1, l'_2, l'_2, l'_2, l'_1 \end{bmatrix}$$

$$\begin{bmatrix} l'_1, l'_2, l'_1, l'_1, l'_2, l'_2 \end{bmatrix}, \begin{bmatrix} l'_1, l'_2, l'_1, l'_2, l'_1, l'_2 \end{bmatrix}, \begin{bmatrix} l'_1, l'_2, l'_1, l'_2, l'_2, l'_1 \end{bmatrix}, \begin{bmatrix} l'_1, l'_2, l'_2, l'_1, l'_1, l'_2 \end{bmatrix}$$

$$\begin{bmatrix} l'_1, l'_2, l'_2, l'_1, l'_2, l'_1 \end{bmatrix}, \begin{bmatrix} l'_1, l'_2, l'_2, l'_2, l'_1, l'_1 \end{bmatrix}, \begin{bmatrix} l'_2, l'_2, l'_2, l'_1, l'_1, l'_1 \end{bmatrix}, \begin{bmatrix} l'_2, l'_2, l'_1, l'_2, l'_1, l'_1 \end{bmatrix}$$

$$\begin{bmatrix} l'_2, l'_2, l'_1, l'_1, l'_2, l'_1 \end{bmatrix}, \begin{bmatrix} l'_2, l'_2, l'_1, l'_1, l'_1, l'_2 \end{bmatrix}, \begin{bmatrix} l'_2, l'_1, l'_2, l'_2, l'_1, l'_1 \end{bmatrix}, \begin{bmatrix} l'_2, l'_1, l'_2, l'_1, l'_2, l'_1 \end{bmatrix}$$

$$\begin{bmatrix} l'_2, l'_1, l'_2, l'_1, l'_1, l'_2 \end{bmatrix}, \begin{bmatrix} l'_2, l'_1, l'_1, l'_2, l'_2, l'_1 \end{bmatrix}, \begin{bmatrix} l'_2, l'_1, l'_1, l'_2, l'_1, l'_2 \end{bmatrix}, \begin{bmatrix} l'_2, l'_1, l'_1, l'_1, l'_2, l'_2 \end{bmatrix}$$

are equivalent circuits.

In the above circuits $\begin{bmatrix} l'_1, l'_1, l'_1, l'_2, l'_2, l'_2 \end{bmatrix}, \begin{bmatrix} l'_1, l'_1, l'_2, l'_2, l'_2, l'_1 \end{bmatrix}, \begin{bmatrix} l'_1, l'_2, l'_2, l'_2, l'_1, l'_1 \end{bmatrix}, \begin{bmatrix} l'_2, l'_2, l'_2, l'_1, l'_1, l'_1 \end{bmatrix},$
 $\begin{bmatrix} l'_2, l'_2, l'_1, l'_1, l'_1, l'_2 \end{bmatrix}, \begin{bmatrix} l'_2, l'_1, l'_1, l'_1, l'_2, l'_2 \end{bmatrix}$

are cyclically equivalent.

Moreover by Lemma 2.1, $\begin{bmatrix} l'_1, l'_1, l'_1, l'_2, l'_2, l'_2 \end{bmatrix} \sim_s \begin{bmatrix} l'_1, l'_2, l'_2, l'_2, l'_1, l'_1 \end{bmatrix} \sim_s \begin{bmatrix} l'_2, l'_2, l'_1, l'_1, l'_1, l'_2 \end{bmatrix}$ is the circuit of $\alpha^G = (-\bar{\alpha})^G$ and $\begin{bmatrix} l'_2, l'_1, l'_1, l'_1, l'_2, l'_2 \end{bmatrix} \sim_s \begin{bmatrix} l'_1, l'_1, l'_2, l'_2, l'_2, l'_1 \end{bmatrix} \sim_s \begin{bmatrix} l'_2, l'_2, l'_2, l'_1, l'_1, l'_1 \end{bmatrix}$ is the circuit of $(\bar{\alpha})^G = (-\alpha)^G$.

Also

$$\begin{bmatrix} l'_1, l'_1, l'_2, l'_2, l'_1, l'_2 \end{bmatrix}, \begin{bmatrix} l'_1, l'_2, l'_1, l'_1, l'_2, l'_2 \end{bmatrix}, \begin{bmatrix} l'_2, l'_2, l'_1, l'_2, l'_1, l'_1 \end{bmatrix}, \begin{bmatrix} l'_1, l'_2, l'_2, l'_1, l'_2, l'_1 \end{bmatrix},$$

$$\begin{bmatrix} l'_2, l'_1, l'_2, l'_1, l'_1, l'_2 \end{bmatrix}, \begin{bmatrix} l'_2, l'_1, l'_1, l'_2, l'_2, l'_1 \end{bmatrix}, \begin{bmatrix} l'_1, l'_2, l'_1, l'_2, l'_2, l'_1 \end{bmatrix}, \begin{bmatrix} l'_1, l'_2, l'_2, l'_1, l'_1, l'_2 \end{bmatrix},$$

$$\begin{bmatrix} l'_2, l'_1, l'_1, l'_2, l'_1, l'_2 \end{bmatrix}, \begin{bmatrix} l'_2, l'_1, l'_2, l'_2, l'_1, l'_1 \end{bmatrix}, \begin{bmatrix} l'_2, l'_2, l'_1, l'_1, l'_2, l'_1 \end{bmatrix}, \begin{bmatrix} l'_1, l'_1, l'_2, l'_1, l'_2, l'_2 \end{bmatrix}$$

cyclically equivalent.

Moreover by Lemma 2.1, $\begin{bmatrix} l'_1, l'_1, l'_2, l'_2, l'_1, l'_2 \end{bmatrix} \sim_s \begin{bmatrix} l'_1, l'_2, l'_1, l'_1, l'_2, l'_2 \end{bmatrix} \sim_s \begin{bmatrix} l'_2, l'_2, l'_1, l'_2, l'_1, l'_1 \end{bmatrix}$ is the circuit of β^G , $\begin{bmatrix} l'_1, l'_2, l'_2, l'_1, l'_2, l'_1 \end{bmatrix} \sim_s \begin{bmatrix} l'_2, l'_1, l'_2, l'_1, l'_1, l'_2 \end{bmatrix} \sim_s \begin{bmatrix} l'_2, l'_1, l'_1, l'_2, l'_2, l'_1 \end{bmatrix}$ is the circuit contained in $(-\beta)^G$, $\begin{bmatrix} l'_1, l'_2, l'_1, l'_2, l'_2, l'_1 \end{bmatrix} \sim_s \begin{bmatrix} l'_1, l'_2, l'_2, l'_1, l'_1, l'_2 \end{bmatrix} \sim_s \begin{bmatrix} l'_2, l'_1, l'_1, l'_2, l'_1, l'_2 \end{bmatrix}$ is the circuit of $(-\bar{\beta})^G$, $\begin{bmatrix} l'_2, l'_1, l'_2, l'_2, l'_1, l'_1 \end{bmatrix} \sim_s \begin{bmatrix} l'_2, l'_2, l'_1, l'_1, l'_2, l'_1 \end{bmatrix} \sim_s \begin{bmatrix} l'_1, l'_1, l'_2, l'_1, l'_2, l'_2 \end{bmatrix}$ is the circuit contained in $(\bar{\beta})^G$.

Moreover, $\begin{bmatrix} l'_1, l'_2, l'_1, l'_2, l'_1, l'_2 \end{bmatrix}$ and $\begin{bmatrix} l'_2, l'_1, l'_2, l'_1, l'_2, l'_1 \end{bmatrix}$ are not possible by Lemma 2.3.

Table 4 summarizes all these information's

Table 4. summarizes all these information's

$E_{T_8}^c$	Nce	$ O_{T_8} $	Structure of orbits
$E_{[l'_1, l'_1, l'_1, l'_2, l'_2, l'_2]}^c$	1	2	$\gamma^G = (-\bar{\gamma})^G, (-\gamma)^G = (\bar{\gamma})^G$
$E_{[l'_1, l'_1, l'_2, l'_1, l'_2, l'_2]}^c$	1	4	$\gamma^G, (-\bar{\gamma})^G, (-\gamma)^G, (\bar{\gamma})^G$

Nce: Number of classes of cyclically equivalent circuits;

Following is the generalizations of the above circuit T_8 whose length is “2t” and its proof can be obtained by applying induction on “t”.

Note 3.7.1: If $\begin{bmatrix} l'_1, l'_1, l'_1, \dots, l'_1, l'_2, \dots, l'_2, l'_2, l'_2 \end{bmatrix}$ is circuit contained in the orbit of γ^G where l'_1 repeats “t” times l'_2 repeats “t” times and “t” is even integer then $\gamma^G = (\bar{\gamma})^G$.

Note 3.7.2: If $\begin{bmatrix} l'_1, l'_1, l'_1, \dots, l'_1, l'_2, \dots, l'_2, l'_2, l'_2 \end{bmatrix}$ is circuit contained in the orbit of γ^G where l'_1 repeats “t” times l'_2 repeats “t” times and “t” is odd integer then $\gamma^G = (-\bar{\gamma})^G$.

Corollary 3.9: There are 30 equivalent circuits of length 6 in which 4 numbers are alike and 2 numbers are different.

Proof: By Lemma 2.4, the number of equivalent circuits of length 6 in which 4 numbers are alike and 2 numbers are different is $\frac{6!}{4!1!1!} = 30$. These 30 circuits are $\begin{bmatrix} l'_1, l'_1, l'_1, l'_1, l'_2, l'_3 \end{bmatrix},$

$$\begin{aligned} & \left[l'_1, l'_1, l'_1, l'_2, l'_1, l'_3 \right], \left[l'_1, l'_1, l'_1, l'_2, l'_3, l'_1 \right], \left[l'_1, l'_1, l'_1, l'_3, l'_1, l'_2 \right], \left[l'_1, l'_1, l'_1, l'_1, l'_3, l'_2 \right], \\ & \left[l'_1, l'_1, l'_3, l'_2, l'_1, l'_1 \right], \left[l'_1, l'_1, l'_3, l'_1, l'_2, l'_1 \right], \left[l'_1, l'_1, l'_3, l'_1, l'_1, l'_2 \right], \left[l'_1, l'_1, l'_1, l'_3, l'_2, l'_1 \right], \\ & \left[l'_1, l'_1, l'_2, l'_3, l'_1, l'_1 \right], \left[l'_1, l'_1, l'_2, l'_1, l'_3, l'_1 \right], \left[l'_1, l'_1, l'_2, l'_1, l'_1, l'_3 \right], \left[l'_1, l'_2, l'_1, l'_1, l'_1, l'_3 \right], \\ & \left[l'_1, l'_2, l'_1, l'_1, l'_3, l'_1 \right], \left[l'_1, l'_2, l'_1, l'_3, l'_1, l'_1 \right], \left[l'_1, l'_2, l'_3, l'_1, l'_1, l'_1 \right], \left[l'_1, l'_3, l'_1, l'_1, l'_1, l'_2 \right], \\ & \left[l'_1, l'_3, l'_1, l'_1, l'_2, l'_1 \right], \left[l'_1, l'_3, l'_1, l'_2, l'_1, l'_1 \right], \left[l'_1, l'_3, l'_2, l'_1, l'_1, l'_1 \right], \left[l'_2, l'_1, l'_1, l'_1, l'_1, l'_3 \right], \\ & \left[l'_2, l'_1, l'_1, l'_1, l'_3, l'_1 \right], \left[l'_2, l'_1, l'_1, l'_3, l'_1, l'_1 \right], \left[l'_2, l'_1, l'_3, l'_1, l'_1, l'_1 \right], \left[l'_2, l'_3, l'_1, l'_1, l'_1, l'_1 \right], \\ & \left[l'_3, l'_1, l'_1, l'_1, l'_1, l'_2 \right], \left[l'_3, l'_1, l'_1, l'_1, l'_2, l'_1 \right], \left[l'_3, l'_1, l'_1, l'_2, l'_1, l'_1 \right], \left[l'_3, l'_1, l'_2, l'_1, l'_1, l'_1 \right], \\ & \left[l'_3, l'_2, l'_1, l'_1, l'_1, l'_1 \right] \end{aligned}$$

Structure of G-orbit of these 30 circuits can be found from Table 5.

Table 5. Structure of G-orbit of these 30 circuits

$E_{T_7}^c$	Nce	$ O_{T_7} $	Structure of orbits
$E_{[l'_1, l'_1, l'_2, l'_1, l'_1, l'_3]}^c$	1	2	$\gamma^G = (-\bar{\gamma})^G, (\bar{\gamma})^G = (-\gamma)^G$
$E_{[l'_1, l'_1, l'_1, l'_1, l'_2, l'_3]}^c, E_{[l'_1, l'_2, l'_1, l'_3, l'_1, l'_1]}^c$	2	8	$\gamma^G = (-\bar{\gamma})^G, (\bar{\gamma})^G = (-\gamma)^G$

Nce: Number of classes of cyclically equivalent circuits

Following is the generalization of the circuit T_7 whose length is “2t” and its proof can be obtained by applying induction on “t”.

Note 3.7.1: If $[l'_1, l'_1, l'_1, \dots, l'_1, l'_2, l'_3]$ is circuit contained in the orbit of γ^G where l'_1 repeats “2t–2” times then $\gamma^G, (-\gamma)^G, (\bar{\gamma})^G$ and $(-\bar{\gamma})^G$ are all distinct.

Corollary 3.10: There are 60 equivalent circuits of length 6 in which 3 numbers are alike, 2 numbers are alike and 1 number is different.

Proof: By Lemma 2.4, the number of equivalent length 6 in which 3 numbers are alike, 2 numbers are alike and 1 number is different is $\frac{6!}{3!2!1!} = 60$. These circuits are

$$\begin{aligned} & \left[l'_1, l'_1, l'_1, l'_2, l'_2, l'_3 \right], \left[l'_1, l'_2, l'_2, l'_3, l'_1, l'_1 \right], \left[l'_2, l'_3, l'_1, l'_1, l'_1, l'_2 \right], \left[l'_1, l'_1, l'_2, l'_2, l'_3, l'_1 \right], \\ & \left[l'_2, l'_2, l'_3, l'_1, l'_1, l'_1 \right], \left[l'_3, l'_1, l'_1, l'_1, l'_2, l'_2 \right], \left[l'_1, l'_3, l'_2, l'_2, l'_1, l'_1 \right], \left[l'_2, l'_2, l'_1, l'_1, l'_1, l'_3 \right], \\ & \left[l'_1, l'_1, l'_1, l'_3, l'_2, l'_2 \right], \left[l'_3, l'_2, l'_2, l'_1, l'_1, l'_1 \right], \left[l'_2, l'_1, l'_1, l'_1, l'_3, l'_2 \right], \left[l'_1, l'_1, l'_3, l'_2, l'_2 \right] \end{aligned}$$

so on.

Structure of G-orbit of these 60 circuits can be found from Table 6.

Table 6. Structure of G-orbit of these 60 circuits

$E_{T_6}^c$	Ncc	$ O_{T_6} $	Structure of orbits
$E_{[l'_1, l'_3, l'_2, l'_2, l'_1, l'_1]}^c, E_{[l'_1, l'_1, l'_2, l'_1, l'_2, l'_3]}^c, E_{[l'_1, l'_2, l'_2, l'_1, l'_3, l'_1]}^c, E_{[l'_1, l'_2, l'_1, l'_1, l'_2, l'_3]}^c$	4	16	$\gamma^G, (-\bar{\gamma})^G, (-\gamma)^G, (\bar{\gamma})^G$
$E_{[l'_1, l'_1, l'_1, l'_2, l'_3, l'_2]}^c, E_{[l'_1, l'_2, l'_1, l'_2, l'_1, l'_3]}^c$	2	4	$\gamma^G = (-\bar{\gamma})^G, (\bar{\gamma})^G = (-\gamma)^G$

Ncc: Number of classes of cyclically equivalent circuits;

Following is the generalization of the circuit T_6 whose length is “2t” and its proof can be obtained by applying induction on “t”.

Note 3.10.1: If $[l'_1, l'_1, l'_1, \dots, l'_1, l'_2, l'_2, l'_3]$ is circuit contained in the orbit of γ^G where l'_1 repeats “2t–3” times then $\gamma^G, (-\gamma)^G, (\bar{\gamma})^G$ and $(-\bar{\gamma})^G$ are all distinct.

Corollary 3.11: There are 90 equivalent circuits of length 6 in which 2 numbers are alike, 2 numbers are alike and 2 numbers are alike.

Proof: By Lemma 2.4, the number of equivalent circuits of length 6 in which 2 numbers are alike, 2 numbers are alike and 2 numbers are alike is $\frac{6!}{2!2!2!} = 90$. Some of these circuits are

$$\begin{aligned} & \left[l'_1, l'_1, l'_2, l'_2, l'_3, l'_3 \right], \left[l'_2, l'_2, l'_3, l'_3, l'_1, l'_1 \right], \left[l'_3, l'_3, l'_1, l'_1, l'_2, l'_2 \right], \left[l'_1, l'_2, l'_2, l'_3, l'_3, l'_1 \right] \\ & \left[l'_2, l'_3, l'_3, l'_1, l'_1, l'_2 \right], \left[l'_3, l'_1, l'_1, l'_2, l'_2, l'_3 \right], \left[l'_1, l'_3, l'_3, l'_2, l'_2, l'_1 \right], \left[l'_3, l'_2, l'_2, l'_1, l'_1, l'_3 \right] \\ & \left[l'_2, l'_1, l'_1, l'_3, l'_3, l'_2 \right], \left[l'_3, l'_3, l'_2, l'_2, l'_1, l'_1 \right], \left[l'_2, l'_2, l'_1, l'_1, l'_3, l'_3 \right], \left[l'_1, l'_1, l'_3, l'_3, l'_2, l'_2 \right] \\ & \left[l'_1, l'_1, l'_3, l'_2, l'_2, l'_3 \right], \left[l'_1, l'_3, l'_2, l'_2, l'_3, l'_1 \right], \left[l'_3, l'_2, l'_2, l'_3, l'_1, l'_1 \right], \left[l'_2, l'_2, l'_3, l'_1, l'_1, l'_3 \right] \\ & \left[l'_2, l'_3, l'_1, l'_1, l'_3, l'_2 \right], \left[l'_3, l'_1, l'_1, l'_3, l'_2, l'_2 \right] \end{aligned}$$

and so on.

Structure of G-orbit of these 90 circuits can be found from Table 7.

Table 7. Structure of G-orbit of these 90 circuits

$E_{T_5}^c$	Ncc	$ O_{T_5} $	Structure of orbits
$E_{[l'_1, l'_1, l'_2, l'_2, l'_3, l'_3]}^c, E_{[l'_1, l'_2, l'_2, l'_3, l'_3, l'_1]}^c, E_{[l'_1, l'_2, l'_1, l'_1, l'_2, l'_2]}^c, E_{[l'_1, l'_2, l'_2, l'_3, l'_3, l'_1]}^c, E_{[l'_2, l'_3, l'_3, l'_1, l'_1, l'_2]}^c, E_{[l'_3, l'_1, l'_1, l'_2, l'_2, l'_3]}^c, E_{[l'_1, l'_3, l'_3, l'_2, l'_2, l'_1]}^c, E_{[l'_3, l'_2, l'_2, l'_1, l'_1, l'_3]}^c, E_{[l'_2, l'_1, l'_1, l'_3, l'_3, l'_2]}^c, E_{[l'_3, l'_3, l'_2, l'_2, l'_1, l'_1]}^c, E_{[l'_2, l'_2, l'_1, l'_1, l'_3, l'_3]}^c, E_{[l'_1, l'_1, l'_3, l'_3, l'_2, l'_2]}^c, E_{[l'_1, l'_1, l'_3, l'_2, l'_2, l'_3]}^c, E_{[l'_1, l'_3, l'_2, l'_2, l'_3, l'_1]}^c, E_{[l'_3, l'_2, l'_2, l'_3, l'_1, l'_1]}^c, E_{[l'_2, l'_2, l'_3, l'_1, l'_1, l'_3]}^c, E_{[l'_2, l'_3, l'_1, l'_1, l'_3, l'_2]}^c, E_{[l'_3, l'_1, l'_1, l'_3, l'_2, l'_2]}^c$	4	16	$\gamma^G, (-\bar{\gamma})^G, (-\gamma)^G, (\bar{\gamma})^G$
$E_{[l'_3, l'_2, l'_1, l'_1, l'_3, l'_3]}^c, E_{[l'_1, l'_3, l'_2, l'_2, l'_3, l'_1]}^c, E_{[l'_1, l'_3, l'_2, l'_2, l'_3, l'_1]}^c, E_{[l'_1, l'_3, l'_2, l'_2, l'_3, l'_1]}^c$	3	12	$\gamma^G = (-\bar{\gamma})^G, (\bar{\gamma})^G = (-\gamma)^G$
$E_{[l'_1, l'_3, l'_2, l'_2, l'_3, l'_1]}^c, E_{[l'_1, l'_2, l'_2, l'_1, l'_1, l'_3]}^c, E_{[l'_1, l'_1, l'_2, l'_3, l'_3, l'_2]}^c$	3	12	$\gamma^G = (\bar{\gamma})^G, (-\bar{\gamma})^G = (-\gamma)^G$
$E_{[l'_1, l'_2, l'_3, l'_1, l'_1, l'_3]}^c$	1	2	$\gamma^G = (-\gamma)^G, (-\bar{\gamma})^G = (\bar{\gamma})^G$

Ncc: Number of classes of cyclically equivalent circuits

Following is the generalization of the circuit T_5 whose length is “2t” and its proof can be obtained by applying induction on “t”.

Note 3.11.1: If $[l'_1, l'_2, l'_3, \dots, l'_t, l'_1, l'_2, l'_3, \dots, l'_t]$ is circuit contained in the orbit of γ^G where “t ≥ 3” is odd integer then $\gamma^G = (-\gamma)^G$.

Corollary 3.12: There are 120 equivalent circuits of length 6 in which 3 numbers are alike and 3 numbers are different.

Proof: By Lemma 2.4, the number of equivalent circuits of length 6 in which 3 numbers are alike and 3 numbers are different is $\frac{6!}{3!1!1!1!} = 120$. Some of these circuits are

$$\begin{aligned} & \left[l'_1, l'_1, l'_1, l'_2, l'_3, l'_4 \right], \left[l'_1, l'_2, l'_3, l'_4, l'_1, l'_1 \right], \left[l'_3, l'_4, l'_1, l'_1, l'_1, l'_2 \right], \left[l'_1, l'_1, l'_2, l'_3, l'_4, l'_1 \right] \\ & \left[l'_2, l'_3, l'_4, l'_1, l'_1, l'_1 \right], \left[l'_4, l'_1, l'_1, l'_1, l'_2, l'_3 \right], \left[l'_1, l'_4, l'_3, l'_2, l'_1, l'_1 \right], \left[l'_3, l'_2, l'_1, l'_1, l'_1, l'_4 \right] \\ & \left[l'_1, l'_1, l'_1, l'_4, l'_3, l'_2 \right], \left[l'_4, l'_3, l'_2, l'_1, l'_1, l'_1 \right], \left[l'_2, l'_1, l'_1, l'_1, l'_4, l'_3 \right], \left[l'_1, l'_1, l'_4, l'_3, l'_2, l'_1 \right] \\ & \left[l'_1, l'_1, l'_1, l'_3, l'_2, l'_4 \right], \left[l'_1, l'_1, l'_3, l'_2, l'_4, l'_1 \right], \left[l'_1, l'_3, l'_2, l'_4, l'_1, l'_1 \right], \left[l'_3, l'_2, l'_4, l'_1, l'_1, l'_1 \right] \\ & \left[l'_2, l'_4, l'_1, l'_1, l'_1, l'_3 \right], \left[l'_4, l'_1, l'_1, l'_1, l'_3, l'_2 \right], \left[l'_1, l'_4, l'_2, l'_3, l'_1, l'_1 \right], \left[l'_4, l'_2, l'_3, l'_1, l'_1, l'_1 \right] \end{aligned}$$

so on.

Structure of G-orbit of 120 circuits can be found from Table 8.

Following is the generalization of the circuit T_4 whose length is “2t” and its proof can be obtained by applying induction on “t”.

Note 3.12.1: If $[l'_1, l'_1, l'_1, \dots, l'_1, l'_2, l'_3, l'_4, \dots, l'_{t+1}]$ is circuit contained in the orbit of γ^G where l'_1 repeats “t ≥ 2” times then $\gamma^G, (-\gamma)^G, (\bar{\gamma})^G$ and $(-\bar{\gamma})^G$ are all distinct.

Table 8. Structure of G-orbit of 120 circuits

$E_{T_4}^c$	Ncc	$ O_{T_4} $	Structure of orbits
$E^c_{[l'_1, l'_4, l'_3, l'_2, l'_1, l'_1]}$, $E^c_{[l'_1, l'_1, l'_1, l'_2, l'_4, l'_3]}$ $E^c_{[l'_1, l'_3, l'_4, l'_1, l'_2, l'_1]}$, $E^c_{[l'_1, l'_2, l'_4, l'_1, l'_1, l'_3]}$ $E^c_{[l'_1, l'_2, l'_1, l'_1, l'_4, l'_3]}$, $E^c_{[l'_1, l'_1, l'_4, l'_1, l'_2, l'_3]}$ $E^c_{[l'_1, l'_4, l'_1, l'_2, l'_1, l'_3]}$, $E^c_{[l'_1, l'_3, l'_2, l'_1, l'_1, l'_4]}$ $E^c_{[l'_1, l'_4, l'_2, l'_3, l'_1, l'_1]}$, $E^c_{[l'_1, l'_2, l'_4, l'_1, l'_3, l'_1]}$		40	$\gamma^G, (-\bar{\gamma})^G, (-\gamma)^G, (\bar{\gamma})^G$

Ncc: Number of classes of cyclically equivalent circuits;

Corollary 3.13: There are 180 equivalent circuits of length 6 in which 2 numbers are alike, 2 numbers are alike and 2 numbers are different.

Proof: By Lemma 2.4, the number of equivalent length 6 in which 2 numbers are alike, 2 numbers are alike and 2 numbers are different is $\frac{6!}{2!2!1!1!1!} = 180$. Some these circuits are

$$\left[\begin{matrix} l'_1, l'_1, l'_2, l'_2, l'_3, l'_4 \\ l'_1, l'_2, l'_2, l'_3, l'_4, l'_1 \\ l'_1, l'_4, l'_3, l'_2, l'_2, l'_1 \\ l'_4, l'_3, l'_2, l'_2, l'_1, l'_1 \\ l'_1, l'_1, l'_2, l'_3, l'_4, l'_2 \\ l'_1, l'_1, l'_3, l'_2, l'_2, l'_4 \\ l'_1, l'_1, l'_4, l'_2, l'_2, l'_3 \end{matrix} \right], \left[\begin{matrix} l'_2, l'_2, l'_3, l'_4, l'_1, l'_1 \\ l'_2, l'_3, l'_4, l'_1, l'_1, l'_2 \\ l'_3, l'_2, l'_2, l'_1, l'_1, l'_4 \\ l'_2, l'_2, l'_1, l'_1, l'_4, l'_3 \\ l'_1, l'_1, l'_2, l'_4, l'_2, l'_3 \\ l'_1, l'_1, l'_3, l'_2, l'_4, l'_2 \\ l'_1, l'_1, l'_2, l'_4, l'_2, l'_3 \end{matrix} \right], \left[\begin{matrix} l'_3, l'_4, l'_1, l'_1, l'_2, l'_2 \\ l'_4, l'_1, l'_1, l'_2, l'_2, l'_3 \\ l'_2, l'_1, l'_1, l'_4, l'_3, l'_2 \\ l'_1, l'_1, l'_4, l'_3, l'_2, l'_2 \\ l'_1, l'_1, l'_2, l'_4, l'_3, l'_2 \\ l'_1, l'_1, l'_3, l'_4, l'_2, l'_2 \\ l'_1, l'_1, l'_3, l'_4, l'_2, l'_2 \end{matrix} \right],$$

and so on.

Structure of G-orbit of these 180 circuits can be found from Table 9.

Table 9. Structure of G-orbit of these 180

$E_{T_3}^c$	Ncc	$ O_{T_3} $	Structure of orbits
$E^c_{[l'_1, l'_4, l'_3, l'_2, l'_1, l'_1]}$, $E^c_{[l'_1, l'_1, l'_2, l'_2, l'_4, l'_3]}$ $E^c_{[l'_1, l'_4, l'_2, l'_3, l'_2, l'_1]}$, $E^c_{[l'_1, l'_1, l'_2, l'_3, l'_4, l'_2]}$ $E^c_{[l'_1, l'_3, l'_2, l'_4, l'_2, l'_1]}$, $E^c_{[l'_1, l'_1, l'_3, l'_2, l'_2, l'_4]}$ $E^c_{[l'_1, l'_4, l'_3, l'_2, l'_1, l'_2]}$, $E^c_{[l'_1, l'_2, l'_1, l'_2, l'_4, l'_3]}$ $E^c_{[l'_1, l'_2, l'_1, l'_3, l'_2, l'_4]}$, $E^c_{[l'_1, l'_3, l'_1, l'_2, l'_2, l'_4]}$ $E^c_{[l'_1, l'_2, l'_2, l'_1, l'_3, l'_4]}$, $E^c_{[l'_1, l'_3, l'_2, l'_1, l'_2, l'_4]}$ $E^c_{[l'_1, l'_4, l'_2, l'_1, l'_3, l'_2]}$, $E^c_{[l'_1, l'_3, l'_2, l'_2, l'_1, l'_4]}$	14	56	$\gamma^G, (-\bar{\gamma})^G, (-\gamma)^G, (\bar{\gamma})^G$
$E^c_{[l'_1, l'_2, l'_4, l'_2, l'_1, l'_3]}$, $E^c_{[l'_2, l'_3, l'_2, l'_1, l'_1, l'_4]}$	2	4	$\gamma^G = (-\bar{\gamma})^G, (\bar{\gamma})^G = (-\gamma)^G$

Ncc: Number of classes of cyclically equivalent circuits

Following is the generalization of the circuit T_3 whose length is “2t” and its proof can be obtained by applying induction on “t”.

Note 3.13.1: If $|l'_1, l'_1, l'_1, \dots, l'_1, l'_2, l'_2, l'_2, \dots, l'_2, l'_3, l'_4|$ is circuit contained in the orbit of γ^G where l'_1 repeats “2t–1” times and l'_2 repeats “2t–1” times where “t ≥ 1” then $\gamma^G, (-\gamma)^G, (\bar{\gamma})^G$ and $(-\bar{\gamma})^G$ are all distinct.

Corollary 3.14: There are 360 equivalent circuits of length 6 in which 2 numbers are alike and 4 numbers are different.

Proof: By Lemma 2.4, the number of equivalent circuits of length 6 in which 2 numbers are alike and 4 numbers are different is $\frac{6!}{2111111} = 360$. Some of these circuits are given by

$$\begin{bmatrix} l'_1, l'_1, l'_2, l'_3, l'_4, l'_5 \\ l'_3, l'_4, l'_5, l'_1, l'_1, l'_2 \\ l'_2, l'_1, l'_1, l'_5, l'_4, l'_3 \end{bmatrix}, \begin{bmatrix} l'_2, l'_3, l'_4, l'_5, l'_1, l'_1 \\ l'_5, l'_1, l'_1, l'_2, l'_3, l'_4 \\ l'_5, l'_4, l'_3, l'_2, l'_1, l'_1 \end{bmatrix}, \begin{bmatrix} l'_4, l'_5, l'_1, l'_1, l'_2, l'_3 \\ l'_1, l'_5, l'_4, l'_3, l'_2, l'_1 \\ l'_3, l'_2, l'_1, l'_1, l'_5, l'_4 \end{bmatrix}, \begin{bmatrix} l'_1, l'_2, l'_3, l'_4, l'_5, l'_1 \\ l'_4, l'_3, l'_2, l'_1, l'_1, l'_5 \\ l'_1, l'_1, l'_5, l'_4, l'_3, l'_2 \end{bmatrix}$$

and so on.

Structure of G-orbit of these 360 circuits can be found from Table 10.

Table 10. Structure of G-orbit of these 360 circuits

$E^c_{T_2}$	Nce	$ O_{T_2} $	Structure of orbits
$E^c_{(l'_1, l'_1, l'_2, l'_3, l'_4, l'_5)}$, $E^c_{(l'_1, l'_1, l'_2, l'_3, l'_5, l'_4)}$, $E^c_{(l'_1, l'_1, l'_2, l'_4, l'_3, l'_5)}$, $E^c_{(l'_1, l'_1, l'_2, l'_4, l'_5, l'_3)}$, $E^c_{(l'_1, l'_1, l'_2, l'_5, l'_3, l'_4)}$, $E^c_{(l'_1, l'_1, l'_2, l'_5, l'_4, l'_3)}$, $E^c_{(l'_1, l'_1, l'_3, l'_2, l'_5, l'_4)}$, $E^c_{(l'_1, l'_1, l'_3, l'_2, l'_4, l'_5)}$, $E^c_{(l'_1, l'_1, l'_3, l'_4, l'_2, l'_5)}$, $E^c_{(l'_1, l'_1, l'_3, l'_5, l'_2, l'_4)}$, $E^c_{(l'_1, l'_1, l'_4, l'_2, l'_3, l'_5)}$, $E^c_{(l'_1, l'_1, l'_4, l'_3, l'_2, l'_5)}$, $E^c_{(l'_1, l'_4, l'_1, l'_2, l'_3, l'_5)}$, $E^c_{(l'_2, l'_1, l'_4, l'_3, l'_5, l'_1)}$, $E^c_{(l'_2, l'_1, l'_5, l'_4, l'_3, l'_1)}$, $E^c_{(l'_4, l'_3, l'_2, l'_1, l'_5, l'_1)}$, $E^c_{(l'_3, l'_4, l'_1, l'_2, l'_5, l'_1)}$, $E^c_{(l'_4, l'_2, l'_5, l'_1, l'_3, l'_1)}$, $E^c_{(l'_2, l'_1, l'_3, l'_5, l'_4, l'_1)}$, $E^c_{(l'_3, l'_2, l'_1, l'_5, l'_4, l'_1)}$, $E^c_{(l'_5, l'_3, l'_1, l'_2, l'_4, l'_1)}$, $E^c_{(l'_5, l'_1, l'_4, l'_2, l'_3, l'_1)}$, $E^c_{(l'_3, l'_5, l'_2, l'_1, l'_4, l'_1)}$, $E^c_{(l'_5, l'_4, l'_2, l'_1, l'_5, l'_1)}$, $E^c_{(l'_2, l'_5, l'_4, l'_1, l'_3, l'_1)}$, $E^c_{(l'_5, l'_2, l'_3, l'_1, l'_4, l'_1)}$, $E^c_{(l'_5, l'_1, l'_3, l'_4, l'_2, l'_1)}$, $E^c_{(l'_5, l'_2, l'_1, l'_3, l'_4, l'_1)}$, $E^c_{(l'_5, l'_3, l'_1, l'_4, l'_2, l'_1)}$, $E^c_{(l'_5, l'_4, l'_1, l'_2, l'_3, l'_1)}$	30	120	$\gamma^G, (-\bar{\gamma})^G, (-\gamma)^G, (\bar{\gamma})^G$

Nce: Number of classes of cyclically equivalent circuits

Following is the generalization of the circuit T_2 whose length is “2t” and its proof can be obtained by applying induction on “t”.

Note 3.14.1: If $(l'_1, l'_1, l'_2, l'_3, l'_4, \dots, l'_{2t-3}, l'_{2t-2}, l'_{2t-1})$ is circuit contained in the orbit of γ^G then $\gamma^G, (-\gamma)^G, (\bar{\gamma})^G$ and $(-\bar{\gamma})^G$ are all distinct.

It was proved in⁽⁹⁾ that any circuit $(l'_1, l'_2, l'_3, l'_4, l'_5, l'_6)$ of length 6 corresponds to the orbit contained in $Q^* \left(\sqrt{f(l'_1, l'_2, l'_3, l'_4, l'_5, l'_6)} \right)$.

Corollary 3.15: All the circuits in $E^c_{(l'_1, l'_2, l'_3, l'_4, l'_5, l'_6)}$ corresponds to the orbits contained in $Q^* (\sqrt{n_1})$.

Proof: To prove this result, it enough to show that $f(l'_1, l'_2, l'_3, l'_4, l'_5, l'_6)$ is unchanged for $\theta \in D_6$. Where

$$\begin{aligned} f(l'_1, l'_2, l'_3, l'_4, l'_5, l'_6) &= (l'_1 (l'_6 + l'_4 + l'_2) + l'_3 (l'_6 + l'_4) + l'_5 l'_6 + l'_2 l'_1 (l'_4 l'_3 + l'_6 l'_5) \\ &\quad + l'_4 l'_5 l'_6 (l'_3 + l'_1) + l'_1 l'_6 l'_3 l'_2 (1 + l'_4 l'_5) - l'_2 (l'_5 + l'_3) - l'_5 l'_4 (1 + l'_2 l'_3))^2 \\ &\quad + 4 (l'_5 (1 + l'_1 l'_2 + l'_3 l'_4 + l'_1 l'_4 + l'_1 l'_2 l'_3 l_4) + l'_3 + l'_1 \\ &\quad + l'_3 l'_2 l'_1) \left((1 + l'_6 l'_5) (l'_4 + l'_2 + l'_2 l'_3 l_4) + l'_6 (1 + l'_2 l'_3) \right) \end{aligned}$$

It is easy to see that after applying $\theta \in D_6$ and simplifying the expression for $f(l'_1, l'_2, l'_3, l'_4, l'_5, l'_6)$ is unaltered. Since there are 60 cyclically equivalent classes, so we have sixty n’s namely $n_1, n_2, n_3, \dots, n_{60}$.

Example 3.16:

We verify above results by considering the circuits [1, 2, 3, 4, 5, 6], [2, 3, 4, 5, 6, 1], [1, 6, 5, 4, 3, 2] and [6, 5, 4, 3, 2, 1]. These four circuits correspond to the orbits $\gamma^G, (-\gamma)^G, (-\bar{\gamma})^G, (\bar{\gamma})^G$ and these four circuits can be seen in Figures 1, 2, 3 and 4 respectively.

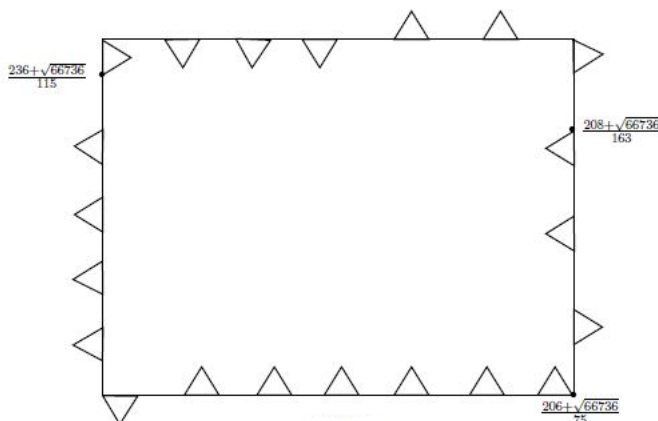


Fig 1.

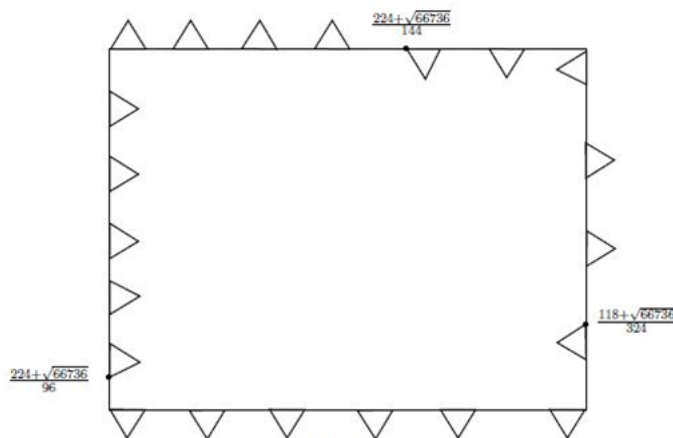


Fig 2.

Figure 1 shows location of reduced numbers and describes that $[1, 2, 3, 4, 5, 6] \sim_s [3, 4, 5, 6, 1, 2] \sim_s [5, 6, 1, 2, 3, 4]$

Figure 2 shows the location of reduced numbers and describes that $[2, 3, 4, 5, 6, 1] \sim_s [4, 5, 6, 1, 2, 3] \sim [6, 1, 2, 3, 4, 5]$

Figure 3 shows the location of reduced numbers and describes that $[1, 6, 5, 4, 3, 2] \sim_s [5, 4, 3, 2, 1, 6] \sim_s [3, 2, 1, 6, 5, 4]$

Figure 4 shows the location of reduced numbers and describes that $[6, 5, 4, 3, 2, 1] \sim_s [4, 3, 2, 1, 6, 5] \sim_s [2, 1, 6, 5, 4, 3]$.

Now in all above Figures containing 3 reduced numbers. So, $E_{[1, 2, 3, 4, 5, 6]}^c$ contains 12 reduced numbers.

4 Conclusion

The idea to classify G-circuits of G-orbits on quadratic field by modular group, which is given in this paper, is new and original. We have classify G-circuits into the distinct equivalence classes of equivalent circuits by using partition function and they are precisely $p(q) - 1$ in number when $q \geq 2$, Particularly for circuits of length 6 we have $p(6) - 1 = 10$ equivalence classes of equivalent circuits, i.e $E_6 = \bigcup_{i=1}^{10} E_{T_i}$. We further classify equivalence classes of equivalent circuits into cyclically equivalence classes and this study help us to determine exact number of G-orbits corresponding to each cyclically equivalence class. Furthermore, we describe some generalized G-circuits of length $2t$ corresponding to each of these ten equivalence classes and the structure of these G-circuits with conditions on t . All results are verified by suitable example.

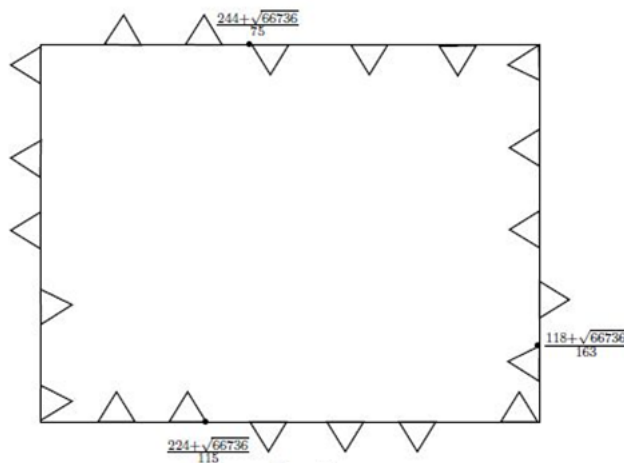


Fig 3.

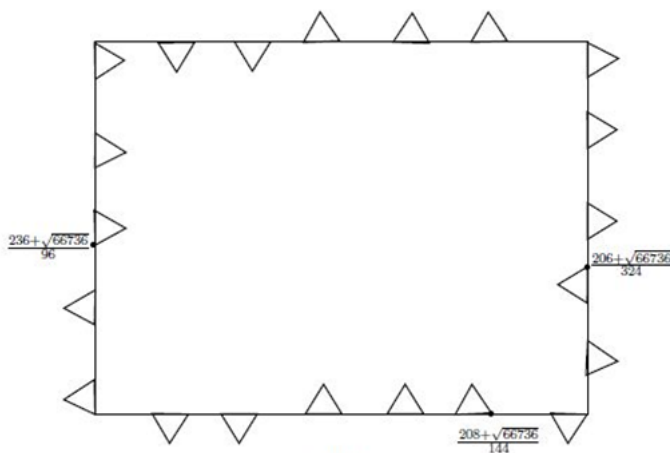


Fig 4.

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