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RESERACH ARTICLE

Beta Lehmann-2 power function distribution with application to bladder cancer susceptibility and failure times of air-conditioned system

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Abstract

Objectives: Probability distributions have great use in reliability engineering where the researchers try to find the distribution of the different processes. To meet the needs of the reliability engineers, we have proposed a simple probability distribution named as Beta Lehman-2 which may be proved more useful as compared to already existing models of the probability distributions. The aim of the study is to show the performance of the proposed distribution over already existing distributions. **Methods:** In this study, a new Beta Lehmann-2 Power function distribution (BL2PFD) is proposed. We suggest a new generator that will modify the Power function distribution called Beta Lehmann-2 generator (BL2-G). **Findings:** The various properties of the new distribution have been discussed in detail such as moments, vitality function, conditional moments and order statistics etc. We have also characterized the BL2PFD based on conditional variance. This distribution can be used for approximately symmetric data (normal data), positive and negative skewed data. **Application:** The application of this distribution is illustrated by using data sets from medical and engineering sources. The shape of the new distribution has been studied for applied sciences. After analyzing data, we conclude that the proposed model BL2PFD perform better in all the data sets while compared to different competitor models.

Keywords: Beta Lehmann-2 Power function distribution; Characterization of truncated distribution; Lehmann alternatives; Percentile estimator; Power function distribution

1 Introduction

The researchers in Engineering sciences mostly study the reliability of different components by taking the help from probability distributions that are simple in mathematical expression instead of using mathematically complex probability distributions. In Dallas[(1)] introduced the power function as the inverse of Pareto distribution. Meniconi and Barry[(2)] showed that power function distribution is better to fit for failure data over exponential, log normal and Weibull because it provides a better fit.

More studies about the application of this distribution and its applications can be found in Ahsanullah et al.[3], Chang[4], van Dorp and Kotz[5]. For modeling heterogeneous population, Saleem et al.[6] talked about the two component mixture of one-parameter Power function distribution. Estimation of the parameters of the two-parameter Power function distribution was studied by Zaka and Akhter[7] through the methods of the least squares, relative least squares and ridge regression. According to its applicability in real life situations for modeling survival data, Tahir et al.[8] proposed the modification of the Power function distribution as Weibull-Power function distribution. By using the Bayesian inference, Hanif et al.[9] estimated the parameter of the one-parameter Power function distribution. In Shahzad and Asghar[10] introduced the Transmuted Power function distribution by following Shaw and Buckley Shaw and Buckley[11]. In Okorie et al.[12] proposed the modification of the Power function distribution by using Marshall and Olkin Marshall and Olkin[13] technique. In Haq et al.[14] proposed the McDonald Power function distribution and Ibrahim[15] proposed the Kumaraswamy Power function distribution. In Jabeen and Zaka[16] discussed the parameters estimation for continuous uniform distribution using modified percentile estimators. Further Zaka et al.[17] introduced the exponentiated generalized class of power function distribution.

2 Materials and Methods

Lehmann alternatives were introduced by Lehman[18] in the two-sample hypothesis testing context and are useful in survival analysis.

$\varnothing(x) = 1-(1-G(x)\}^{\alpha}$ (Lehmann2 relationship)

In Eugene et al.[19] proposed the Beta generator (Beta-G).

$$F(x) = \frac{B_{\varnothing(x)}(a,b)}{B(a,b)}$$

Then the mixture of these two techniques is known as Beta Lehmann-2 generator (BL2-G). The probability density function (pdf) and cumulative distribution function (cdf) of the BL2-G are given as

$$F(x) = \frac{B_{1-\{1-G(x)\}^{\alpha}}(a,b)}{B(a,b)} \tag{1}$$

And

$$f(x) = \frac{\left(1-(1-G(x))^{\alpha}\right)^{a-1}\left((1-G(x))^{\alpha}\right)^{b-1}\alpha(1-G(x))^{\alpha-1}g(x)}{B(a,b)} \tag{2}$$

Where $G(x): cdf\ and\ g(x): pdf\ of\ any\ probability\ distribution$

In this work, we suggest a new distribution that will generalize the Power function distribution (PFD) by using the above mentioned technique. We have derived some of the main structural properties of this distribution. We have also characterized the distribution by conditional moments (Right and Left Truncated mean), doubly truncated mean (DTM) and conditional variance. Maximum likelihood method (MLM) and Percentile estimation (P.E) method are used to estimate the shape and scale parameters of BL2PFD. The application of this distribution is illustrated by using data sets from medical and engineering sources.

2.1 Model Identification for Beta Lehmann-2 Power function distribution (BL2PFD)

The pdf and cdf of Power function distribution are given as follows

$$g(x) = \frac{\gamma x^{\gamma-1}}{\beta^{\gamma}}; \quad 0 < x < \beta, \quad \gamma > 0 \tag{3}$$

and

$$G(x) = \left(\frac{x}{\beta}\right)^{\gamma} \tag{4}$$

Where γ and β are the shape and scale parameters.

Following the generator (1), the BL2PFD is obtained by putting (3) and (4) in (2) and simplifying, we get

$$f(x) = \frac{\left(1-\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right)^{\alpha}\right)^{a-1}\left(\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right)^{\alpha}\right)^{b-1}\alpha\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right)^{\alpha-1}\frac{\gamma x^{\gamma-1}}{\beta^{\gamma}}}{B(a,b)}\ ;\ \ 0<x<\beta \tag{5}$$

and associated cdf is obtained by putting (4) in (1) as

$$F(x) = \frac{B_{\left\{1-\left\{1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\alpha}\right\}}(a,b)}{B(a,b)} \tag{6}$$

We may observe α, a and b are the tuning parameters. γ as the shape and β as scale parameter.

By definition, the survival function is

$$S(x) = 1-F(x) = 1-\left\{\frac{B_{\left\{1-\left\{1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\alpha}\right\}}(a,b)}{B(a,b)}\right\}$$

And the Hazard Rate Function (HRF) of probability distribution is given as

2.2 Asymptotic behavior

The behavior of the pdf, cdf, hazard and survival functions of BL2PFD are being investigated as x $\rightarrow$ 0 and x $\rightarrow \infty$.

i. $\lim_{x\rightarrow 0} f(x) = 0; \forall$ possible values of α, a, b, γ and

ii. $\lim_{x\rightarrow \infty} f(x) = \infty; \forall$ possible values of α, a, b, γ and

iii. $\lim_{x\rightarrow 0} F(x) = 0; \forall$ possible values of α, a, b, γ and

iv. $\lim_{x\rightarrow \infty} F(x) = 1;$ if $x = \beta$ and $\forall$ possible values of α, a, b, γ and β

v. $\lim_{x\rightarrow \infty} F(x) = 0;$ if $x \neq \beta$ if $\gamma = 0$ and $\alpha \neq 0$

vi. $\lim_{x\rightarrow 0} F(x) = 1;$ if $x \neq \beta$ if $\gamma > 0$ and $\alpha = 0$

vii. $\lim_{x\rightarrow 0} S(x) = 1;$ if $x \neq \beta$ if $\gamma = 0$ and $\alpha \neq 0$

viii. $\lim_{x\rightarrow \infty} S(x) = 0;$ if $x \neq \beta$ if $\gamma > 0$ and $\alpha = 0$

ix. $\lim_{x\rightarrow 0} H(x) = 0; \forall$ possible values of $\alpha, \beta, \gamma, \varphi$ and θ

x. $\lim_{x\rightarrow \infty} H(x) = \infty; \forall$ possible values of $\alpha, \beta, \gamma, \varphi$ and θ

2.3 Characteristics of hazard function using glaser method

In Glaser [20] had defined the conditions of increasing, decreasing, and upside-down bathtub-shaped failure rate. We use these conditions in our proposed distribution.

$$\eta(x) = -\frac{f'(x)}{f(x)}$$

$$\eta(x) = -\beta^{\gamma}\left(\frac{(\gamma-1)}{x} - (\alpha b-1)\left(\frac{\frac{\gamma x^{\gamma-1}}{\beta^{\gamma}}}{\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right)}\right) + \alpha(a-1)\left(\frac{\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right)^{\alpha-1}}{\left(1-\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right)^{\alpha}\right)}\right)\frac{\gamma x^{\gamma-1}}{\beta^{\gamma}}\right)$$

If x > 0, then the values of $\acute{\eta}(x)$ under the following conditions are given in Table 1 .

The above conditions shows that the hazard function of BL2PFD is increasing but if $(a, b\ or\ \gamma) \rightarrow 0$, then it will be decreasing function. (See Figure 1)

Table 1. Values of $\acute{\eta}\,(x)$ under the following conditions

a	b	γ	α	β	$\eta\,(x)$
1	1	1	≥1	≥1	0
≥2	1	1	≥1	≥1	>0
1	≥2	1	≥1	≥1	>0
1	1	≥2	≥1	≥1	>0
0	1	1	≥1	≥1	<0
1	0	1	≥1	≥1	<0
1	1	0	≥1	≥1	<0

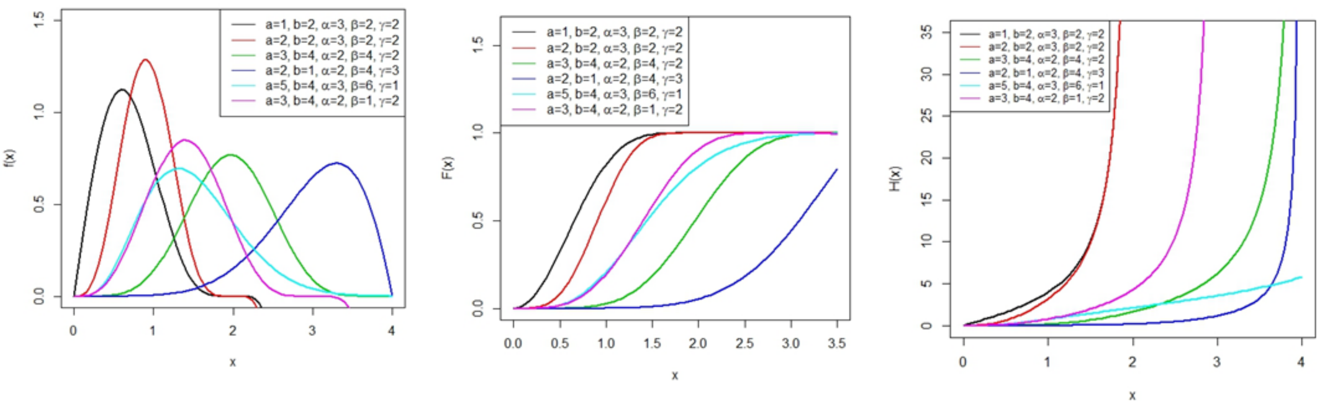

Fig 1. Plots of PDF, CDF and HRF of BL2PFD

2.4 Shapes

The BL2PFD can be approximately Normal Curve, whereas the HRF can be bathtub, monotonically increasing and decreasing shapes. (See Figure 1)

2.5 Moments about zero

The r^th moments about zero of any distribution is described below

$\mu_r^{'} = \int_0^{\beta} x^r f(x)dx$

By solving we get

$\mu_r^{'} = \frac{a_j\, a_i\, a_l\, \beta^r}{B(a,b)\,(a+l+j)}$

$$where\ a_j = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)\ j!},\ a_i = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(\frac{r}{\gamma}+1)}{\Gamma\left(\frac{r}{\gamma}+1-i\right)\ i!}\ and\ a_l = \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(\frac{i}{\alpha}+1)}{\Gamma\left(\frac{i}{\alpha}+1-l\right)\ l!}$$

2.6 Moment generating function

Apart from generating functions, the moment generating function can be utilized to describe the characteristic of the random variable.

$$M_o\,(t) = \int_0^{\beta} e^{tx} f(x)dx$$

If X follows BL2PFD, the moment generating function may be derived as,

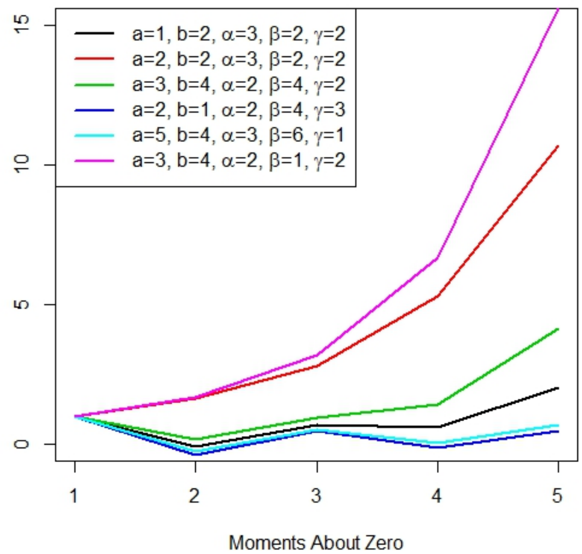

Fig 2. Plots of moments underdifferent parametric values of BL2PFD

$$M_o(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{a_j\, a_i\, a_l\, \beta^r}{B(a,b)\ (a+l+j)}$$

$$where\ a_j = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)\ j!},\ a_i = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(\frac{r}{\gamma}+1)}{\Gamma\left(\frac{r}{\gamma}+1-i\right)\ i!}\ and\ a_l = \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(\frac{i}{\alpha}+1)}{\Gamma\left(\frac{i}{\alpha}+1-l\right)\ l!}$$

2.7 Random number generator

The random number of BL2PFD may be obtained from

$$F(x) = \frac{B_{\left(1-\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\alpha}\right\}}(a,b)}{B(a,b)}$$

After simplifying we get,

$$x = \beta\left(1-\left(1-rbeta(n,a,b)\right\}^{\frac{1}{\alpha}}\right\}^{\frac{1}{\gamma}}$$

Where “$rbeta(n,a,b)$” is the random numbers generated from Beta distribution.

2.8 Inverse moments

By definition Inverse moments may be obtained as

$$\mu'_{-r} = \int_0^{\beta} x^{-r} f(x) dx$$

We get inverse moments for BL2PFD as

$$\mu'_{-r} = \frac{a_j\, a_i\, a_l\, \beta^{-r}}{B(a,b)\ (a+l+j)}$$

$$where\ a_j = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)\ j!},\ a_i = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(\frac{-r}{\gamma}+1)}{\Gamma\left(\frac{-r}{\gamma}+1-i\right)\ i!}\ and\ a_l = \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(\frac{i}{\alpha}+1)}{\Gamma\left(\frac{i}{\alpha}+1-l\right)\ l!}$$

2.9 Vitality function

The vitality function is obtained for BL2PFD as

$$V(x) = \frac{1}{S(x)} \int_x^{\beta} x\, f(x)\, dx$$

That may be obtained as

$$V(x) = \frac{\dfrac{\frac{\beta}{B(a,b)}\, a_j\, a_i\, a_l \left(1-\left(1-\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\alpha}\right\}^{(a+l+j)}\right]}{(a+l+j)}}{1-\left(\dfrac{B_{\left(1-\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\alpha}\right\}}(a,b)}{B(a,b)}\right\}}$$

$$where\ a_j = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)\ j!},\ a_i = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(\frac{1}{\gamma}+1)}{\Gamma\left(\frac{1}{\gamma}+1-i\right)\ i!}\ and\ a_l = \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(\frac{i}{\alpha}+1)}{\Gamma\left(\frac{i}{\alpha}+1-l\right)\ l!}$$

2.10 Information function

The Information Function is given as

$IF = \int_0^{\beta} (f(x)\}^s dx$

For BL2PFD the information function is given as

$$IF = \frac{(\alpha\gamma)^{s-1} \beta^{(\gamma-1)(s-1)} a_j a_i a_l}{\beta^{(\gamma s-\gamma)} B(a,b)\, (s(a-1)+j+l)}$$

$$where\ a_j = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma\left(\frac{\alpha(sb-1)-(s-1)}{\alpha}+1\right)}{\Gamma\left(\frac{\alpha(sb-1)-(s-1)}{\alpha}+1-j\right)\ j!},\ a_i = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma\left(\frac{(\gamma-1)-(s-1)}{\gamma}+1\right)}{\Gamma\left(\frac{(\gamma-1)-(s-1)}{\gamma}+1-i\right)\ i!}$$

$$and\ a_l = \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(\frac{i}{\alpha}+1)}{\Gamma\left(\frac{i}{\alpha}+1-l\right)\ l!}$$

2.11 Order statistics

The pdf of the order statistic may be written as

$$f_{1:n}(x) = \frac{1}{B(1,n)} f(x)\, (1-F(x)\}^{n-1}$$

For BL2PFD, we may write the lower and upper order statistics as

$$f_{1:n}(x) = \frac{1}{B(1,n)} \left(\frac{\left(1-\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right)^{\alpha}\right)^{a-1} \left(\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right)^{\alpha}\right)^{b-1} \alpha\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right)^{\alpha-1} \frac{\gamma x^{\gamma-1}}{\beta^{\gamma}}}{B(a,b)}\right\} * \left(1-\left(\frac{B_{\left(1-\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\alpha}\right\}}(a,b)}{B(a,b)}\right\}\right]^{n-1}$$

and

$$f_{n:n}(x) = \frac{1}{B(1,n)} f(x)\,(F(x)\}^{n-1}$$

$$f_{n:n}(x) = \frac{1}{B(1,n)} \left(\frac{\left(1-\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right)^{\alpha}\right)^{a-1} \left(\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right)^{\alpha}\right)^{b-1} \alpha\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right)^{\alpha-1} \frac{\gamma x^{\gamma-1}}{\beta^{\gamma}}}{B(a,b)} \right\} * \left(\frac{B_{\left(1-\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\alpha}\right\}}(a,b)}{B(a,b)} \right]^{n-1}$$

2.12 Incomplete moments

The incomplete moments are given as

$$\mu_{X|(a,b,\alpha,\beta,\gamma);r}(p) = \int_0^P x^r f(x)\,dx$$

By simplifying for BL2PFD we get

$$\mu_{X|(\alpha,\beta,\gamma,\varphi,\theta);r}(p) = \frac{a_j\, a_i\, a_l\, \beta^r \left(1-\left(1-\left(\frac{p}{\beta}\right)^{\gamma}\right\}^{\alpha}\right]^{(a+l+j)}}{B(a,b)\,(a+l+j)}$$

$$where\ a_j = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)\ j!},\ a_i = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(\frac{r}{\gamma}+1)}{\Gamma\left(\frac{r}{\gamma}+1-i\right)\ i!}\ and\ a_l = \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(\frac{i}{\alpha}+1)}{\Gamma\left(\frac{i}{\alpha}+1-l\right)\ l!}$$

2.13 Conditional moments

The conditional moments may be obtained as

$$E\left(X^r|X>t\right] = \frac{1}{\bar{F}(t)} \int_t^{\beta} x^r f(x)dx$$

The conditional moments for BL2PFD may be obtained by using above expression as

$$E\left(X^r|X>t\right] = \frac{1}{\bar{F}(t)} \frac{a_j\, a_i\, a_l\, \beta^r \left(\left(1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\alpha}\right]^{(a+l+j)}}{B(a,b)\,(a+l+j)}$$

$$where\ a_j = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)\ j!},\ a_i = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(\frac{r}{\gamma}+1)}{\Gamma\left(\frac{r}{\gamma}+1-i\right)\ i!}\ and\ a_l = \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(\frac{i}{\alpha}+1)}{\Gamma\left(\frac{i}{\alpha}+1-l\right)\ l!}$$

2.14 Lorenz and Bonferroni curve

The Lorenz and Bonferroni curve may be obtained as

$$L(p) = \frac{1}{\mu} \int_0^q x \sum_{l=0}^{\infty} t_l h_{l+1}(x)dx$$

$$L(p) = \frac{1}{\mu} \frac{a_j\, a_i\, a_l\, \beta \left(1-\left(1-\left(\frac{q}{\beta}\right)^{\gamma}\right\}^{\alpha}\right]^{(a+l+j)}}{B(a,b)\,(q+l+j)}$$

$$where\ a_j = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)\ j!},\ a_i = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(\frac{1}{\gamma}+1)}{\Gamma\left(\frac{1}{\gamma}+1-i\right)\ i!}\ and\ a_l = \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(\frac{i}{\alpha}+1)}{\Gamma\left(\frac{i}{\alpha}+1-l\right)\ l!}$$

$$B(p) = \frac{1}{P\mu} \frac{a_j\, a_i\, a_l\, \beta \left(1-\left(1-\left(\frac{q}{\beta}\right)^{\gamma}\right\}^{\alpha}\right]^{(a+l+j)}}{B(a,b)\,(a+l+j)}$$

2.15 Characterization of BL2PFD

Let "X" be Beta-Lehmann2- Power function variable with Probability density function

$$f(x)=\frac{\left(1-\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right)^{\alpha}\right)^{a-1}\left(\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right)^{\alpha}\right)^{b-1}\alpha\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right)^{\alpha-1}\frac{\gamma x^{\gamma-1}}{\beta^{\gamma}}}{B(a,b)};\quad 0<x<\beta$$

And let $\bar{F}(x)$ be the survival function respectively. Then the random variable "X" has BL2PFD if and only if

$$\mathrm{V(X\mid x\leq t)}=\frac{\mathrm{a_j a_h a_1}\beta^2}{\mathrm{F(t)B(a,b)}}\left[\frac{1-\left\{1-\left(\frac{\mathrm{t}}{\beta}\right)^{\gamma}\right\}}{\mathrm{a+j+1}}\right]^{\alpha}\Bigg]^{\mathrm{a+j+1}}-$$

$$\left[\frac{\mathrm{a_j a_i a_1}\beta}{\mathrm{F(t)B(a,b)}}\left[\frac{1-\left\{1-\left(\frac{\mathrm{t}}{\beta}\right)^{\gamma}\right\}^{\alpha}}{\mathrm{a+j+1}}\right]^{\mathrm{a+j+1}}\right]^2$$

$$where\quad \mathrm{V(X\mid x\leq t)}: Conditional variance$$

$$Also\ a_j=\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(b)}{\Gamma(b-j)\,j!},\ a_i=\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma\left(\frac{1}{\gamma}+1\right)}{\Gamma\left(\frac{1}{\gamma}+1-i\right)i!},\ a_l=\sum_{l=0}^{\infty}\frac{(-1)^l\Gamma\left(\frac{i}{\alpha}+1\right)}{\Gamma\left(\frac{i}{\alpha}+1-l\right)l!}\ and$$

$$a_h=\sum_{i=0}^{\infty}\frac{(-1)^i\Gamma\left(\frac{2}{\gamma}+1\right)}{\Gamma\left(\frac{2}{\gamma}+1-i\right)i!}$$

Proof:

Necessary part:

$$\mathrm{E(X^r\mid x\leq t)}=\frac{1}{\mathrm{F(t)}}\int_0^{\mathrm{t}}\mathrm{x^r}\frac{\left(1-\left(1-\left(\frac{\mathrm{x}}{\beta}\right)^{\gamma}\right)^{\alpha}\right)^{\mathrm{a-1}}\left(\left(1-\left(\frac{\mathrm{x}}{\beta}\right)^{\gamma}\right)^{\alpha}\right)^{\mathrm{b-1}}\alpha\left(1-\left(\frac{\mathrm{x}}{\beta}\right)^{\gamma}\right)^{\alpha-1}\frac{\gamma\mathrm{x}^{\gamma-1}}{\beta^{\gamma}}}{\mathrm{B(a,b)}}\mathrm{dx}$$

$$Put\ 1-\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right)^{\alpha}=z$$

$$\mathrm{E(X^r\mid x\leq t)}=\frac{1}{\mathrm{F(t)B(a,b)}}\left[\int_0^{1-\left(1-\left(\frac{\mathrm{t}}{\beta}\right)^{\gamma}\right)^{\alpha}}\beta^{\mathrm{r}}\left\{1-(1-\mathrm{z})^{1/\alpha}\right\}^{\mathrm{r}/\gamma}(\mathrm{z})^{\mathrm{a-1}}(1-\mathrm{z})^{\mathrm{b-1}}\mathrm{dz}\right]$$

$$\mathrm{E(X^r\mid x\leq t)}=\frac{\mathrm{a_j a_i a_1}\beta^{\mathrm{r}}}{\mathrm{F(t)B(a,b)}}\left[\frac{1-\left\{1-\left(\frac{\mathrm{t}}{\beta}\right)^{\gamma}\right\}^{\alpha}}{\mathrm{a+j+1}}\right]^{\mathrm{a+j+1}}$$

$$\mathrm{E(X\mid x\leq t)}=\frac{\mathrm{a_j a_i a_1}\beta}{\mathrm{F(t)B(a,b)}}\left[\frac{1-\left\{1-\left(\frac{\mathrm{t}}{\beta}\right)^{\gamma}\right\}^{\alpha}}{\mathrm{a+j+1}}\right]^{\mathrm{a+j+1}}$$

$$Where\ a_j = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)\ j!},\ a_i = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(\frac{1}{\gamma}+1)}{\Gamma\left(\frac{1}{\gamma}+1-i\right)\ i!}\ and\ a_l = \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(\frac{i}{\alpha}+1)}{\Gamma\left(\frac{i}{\alpha}+1-l\right)\ l!}$$

Put r=2

$$\mathrm{E}\left(\mathrm{X}^2 \mid \mathrm{x} \le \mathrm{t}\right) = \frac{\mathrm{a_j a_h a_l} \beta^2}{\mathrm{F(t)B(a,b)}} \left[\frac{1-\left\{1-\left(\frac{\mathrm{t}}{\beta}\right)^{\gamma}\right\}^{\alpha}}{\mathrm{a+j+1}}\right]^{\mathrm{a+j+1}}$$

$$Where\ a_j = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)\ j!},\ a_h = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(\frac{2}{\gamma}+1)}{\Gamma\left(\frac{2}{\gamma}+1-i\right)\ i!}\ and\ a_l = \sum_{l=0}^{\infty} \frac{(-1)^l \Gamma(\frac{i}{\alpha}+1)}{\Gamma\left(\frac{i}{\alpha}+1-l\right)\ l!}$$

$$\mathrm{V(X \mid x \le t)} = \frac{\mathrm{a_j a_h a_l} \beta^2}{\mathrm{F(t)B(a,b)}} \left[\frac{1-\left\{1-\left(\frac{\mathrm{t}}{\beta}\right)^{\gamma}\right\}^{\alpha}}{\mathrm{a+j+1}}\right]^{\mathrm{a+j+1}} - \left[\frac{\mathrm{a_j a_i a_l} \beta}{\mathrm{F(t)B(a,b)}} \left[\frac{1-\left\{1-\left(\frac{\mathrm{t}}{\beta}\right)^{\gamma}\right\}^{\alpha}}{\mathrm{a+j+1}}\right]^{\mathrm{a+j+1}}\right]^2 \tag{7}$$

Also Sufficient part

$$\mathrm{V(X \mid x \le t)} = \frac{1}{\mathrm{F(t)}} \int_0^{\mathrm{t}} \mathrm{x^2 dx} - \left\{\frac{1}{\mathrm{F(t)}} \int_0^{\mathrm{t}} \mathrm{x dx}\right\}^2$$

$$\mathrm{V(X \mid x \le t)} = \mathrm{t}^2 - 2\int_0^{\mathrm{t}} \frac{\mathrm{xF(x)}}{\mathrm{F(t)}} \mathrm{dx} - \left\{\mathrm{t} - \int_0^{\mathrm{t}} \frac{\mathrm{F(x)}}{\mathrm{F(t)}} \mathrm{dx}\right\}^2 \tag{8}$$

Equate (7) and (8), we get

$$t^2 - 2\int_0^t \frac{xF(x)}{F(t)} dx - \left(t - \int_0^t \frac{F(x)}{F(t)} dx\right\}^2 = \frac{a_j\ a_h\ a_l\ \beta^2}{F(t)B(a,b)} \left(\frac{1-\left(1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\alpha}}{a+j+l}\right]^{a+j+l} - \left(\frac{a_j\ a_i\ a_l\ \beta}{F(t)B(a,b)} \left(\frac{1-\left(1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\alpha}}{a+j+l}\right]^{a+j+l}\right]^2$$

$$t - \int_0^t \frac{F(x)}{F(t)} dx = \frac{a_j\ a_i\ a_l\ \beta}{F(t)B(a,b)} \left(\frac{1-\left(1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\alpha}}{a+j+l}\right]^{a+j+l}$$

Therefore

$$\mathrm{t}^2 - 2\int_0^{\mathrm{t}} \frac{\mathrm{xF(x)}}{\mathrm{F(t)}} \mathrm{dx} \frac{\mathrm{a_j a_h a_l} \beta^2}{\mathrm{F(t)B(a,b)}} \left[\frac{1-\left\{1-\left(\frac{\mathrm{t}}{\beta}\right)^{\gamma}\right\}^{\alpha}}{\mathrm{a+j+1}}\right]^{\mathrm{a+j+1}}$$

Differentiate w.r.t "t"

$$t^2 f(t) = \frac{a_j\ a_h\ a_l\ \beta^2}{B(a,b)} \left(1-\left(1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\alpha}\right]^{a+j+l-1} \alpha\left(1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\alpha-1} \frac{\gamma t^{\gamma-1}}{\beta^{\gamma}}$$

As

$$a_j\, a_h\, a_l \left(1-\left(1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\alpha}\right]^{a+j+l-1} = \left(1-\left(1-\left(\frac{t}{\beta}\right)^{\gamma}\right)^{\alpha}\right)^{a-1}\left(\left(1-\left(\frac{t}{\beta}\right)^{\gamma}\right)^{\alpha}\right)^{b-1}\left(\frac{t}{\beta}\right)^{2}$$

Therefore

$$t^2 f(t) = \frac{\beta^2}{B(a,b)}\left(1-\left(1-\left(\frac{t}{\beta}\right)^{\gamma}\right)^{\alpha}\right)^{a-1}\left(\left(1-\left(\frac{t}{\beta}\right)^{\gamma}\right)^{\alpha}\right)^{b-1}\left(\frac{t}{\beta}\right)^{2}\alpha\left(1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\alpha-1}\frac{\gamma t^{\gamma-1}}{\beta^{\gamma}}$$

$$f(t) = \frac{1}{B(a,b)}\left(1-\left(1-\left(\frac{t}{\beta}\right)^{\gamma}\right)^{\alpha}\right)^{a-1}\left(\left(1-\left(\frac{t}{\beta}\right)^{\gamma}\right)^{\alpha}\right)^{b-1}\alpha\left(1-\left(\frac{t}{\beta}\right)^{\gamma}\right\}^{\alpha-1}\frac{\gamma t^{\gamma-1}}{\beta^{\gamma}}$$

The pdf of BL2PFD

3 Results

3.1 Maximum Likelihood Method (MLM)

Let x_1, x_2 ,..., x_n be a random sample of size “n” from the BL2PFD. The log-likelihood function for the BL2PFD is given by

$$L(a,b,\alpha,\ \beta,\gamma) = nln\left(\frac{\alpha\gamma}{\beta^{\gamma}}\right) + n(a-1)\,ln\left(1-\left(1-\left(\frac{x_i}{\beta}\right)^{\gamma}\right)^{\alpha}\right\} + n(\alpha b-1)\,ln\left(1-\left(\frac{x_i}{\beta}\right)^{\gamma}\right) + n(\gamma-1)\,lnx_i$$

The score vector is

$$U_a(a,b,\alpha,\ \beta,\gamma) = \frac{\partial}{\partial a}L(a,b,\alpha,\ \beta,\gamma)$$
$$U_b(a,b,\alpha,\ \beta,\gamma) = \frac{\partial}{\partial b}L(a,b,\alpha,\ \beta,\gamma)$$
$$U_\alpha(a,b,\alpha,\ \beta,\gamma) = \frac{\partial}{\partial \alpha}L(a,b,\alpha,\ \beta,\gamma)$$

$$U_\beta(a,b,\alpha,\ \beta,\gamma) = \frac{\partial}{\partial \beta}L(a,b,\alpha,\ \beta,\gamma)$$

$$U_\gamma(a,b,\alpha,\ \beta,\gamma) = \frac{\partial}{\partial \gamma}L(a,b,\alpha,\ \beta,\gamma)$$

The parameters of BL2PFD can be obtained by solving the above equations resulting from setting the five partial derivatives of $L(a,b,\alpha,\beta,\gamma)$ equals to zero.

3.2 Estimation of BL2PFD Parameters from 'common percentiles' (P, E)

In Dubey[(21)] proposed a percentile estimator of the shape parameter, based on any two sample percentiles. After Dubey[(21)], Marks[(22)] also discussed it, in which he estimated the parameters of Weibull distribution with the help of percentiles.

Let x_1 ,x_2,x_3,..., x_n be a random sample of size n drawn from Probability density function of BL2PFD. The cumulative distribution function of BL2PFD with shape and scale parameters γ *and* β , respectively

$$F(x) = \left(\frac{B\left(1-\left(1-\left(\frac{x}{\beta}\right)^{\gamma}\right\}^{\alpha}\right\}^{(a,b)}}{B(a,b)}\right\}$$

By solving we get

$$x = \beta\left(1-(1-rbeta(n,a,b))^{\frac{1}{\alpha}}\right)^{\frac{1}{\gamma}} \tag{9}$$

Where "$rbeta(n,a,b)$" is the random numbers generated from Beta distribution.

Let P_{75} and P_{25} are the 75^{th} and 25^{th} Percentiles, $therefore\ (9)\ becomes$

$$P_{75} = \beta\left(1-(1-0.75\}^{\frac{1}{\alpha}}\right\}^{\frac{1}{\gamma}} \tag{10}$$

$$P_{25} = \ \beta\left(1-(1-0.25\}^{\frac{1}{\alpha}}\right\}^{\frac{1}{\gamma}} \tag{11}$$

Solving the above equations, we get

$$\left(\frac{P_{75}}{P_{25}}\right)^{\gamma} = \left(\frac{1-(1-0.75\}^{\frac{1}{\alpha}}}{\left(1-(1-0.25\}^{\frac{1}{\alpha}}\right\}}\right\}$$

$$\gamma ln\left(\frac{P_{75}}{P_{25}}\right) = \ ln\left(\frac{1-(1-0.75\}^{\frac{1}{\alpha}}}{\left(1-(1-0.25\}^{\frac{1}{\alpha}}\right\}}\right\}$$

$$\widehat{\gamma} = \frac{ln\left(\frac{1-(1-0.75\}^{\frac{1}{\alpha}}}{1-(1-0.25\}^{\frac{1}{\alpha}}}\right\}}{ln\left(\frac{P_{75}}{P_{25}}\right)}$$

$$and \qquad \widehat{\beta} = \frac{P_{75}}{\left(1-(1-0.75\}^{\frac{1}{\alpha}}\right]^{\frac{1}{\gamma}}}$$

$$generally \quad \widehat{\gamma} = \frac{ln\left(\frac{1-(1-H\}^{\frac{1}{\alpha}}}{1-(1-L\}^{\frac{1}{\alpha}}}\right\}}{ln\left(\frac{P_H}{P_L}\right)}$$

$$and \qquad \widehat{\beta} = \frac{P_H}{\left(1-(1-H\}^{\frac{1}{\alpha}}\right]^{\frac{1}{\gamma}}}$$

Where H= Maximum Percentage, L= Minimum Percentage and P = Percentile

A simulation study is used in order to compare the performance of the proposed estimation methods. We carry out this comparison taking the samples of sizes as n = 40 and 150 with pairs of $(\beta,\gamma) = \{(1,2),(2,1)$ and $(1.5,1.5)\}$. We generated random samples of different sizes by observing that if R_i is random number taking (0, 1), then $x_i = \ \beta\left(1-(1-rbeta(n,a,b)\}^{\frac{1}{\alpha}}\right\}^{\frac{1}{\gamma}}$ is the random number generation from BL2PFD with ($a,\ b, \alpha,\ \beta\ and\ \gamma$) parameters. All results are based on 5000 replications.

Such generated data have been used to obtain estimates of the unknown parameters. The results obtained from parameters estimation of the 2-parameters (shape and scale parameters) of BL2PFD using different sample sizes and different values of parameters with mean square error MSE.

$$\text{M. S.E }(\widehat{\beta}) = \text{E}\left[(\widehat{\beta}-\beta)^2\right],\ \text{M. S. E}(\hat{\gamma}) = \text{E}\left[(\hat{\gamma}-\gamma)^2\right]$$

Table 2. Estimates for the parameters of BL2PFD with different estimation methods under the sample size 40 when $a = 1,\ b = 2\ and\ \alpha = 3$

METHODS	True Values		Estimated Values		M.S.E	
	β	γ	$\widehat{\beta}$	$\widehat{\gamma}$	$\widehat{\beta}$	$\widehat{\gamma}$
MLM	1	2	0.9434932	1.9464116	0.1030683	0.059389
	2	1	2.3387613	0.8781634	0.5333875	0.0111197
	1.5	1.5	1.401726	1.440550	0.2055521	0.0289
P.E	1	2	0.753597	2.039228	1.56648	0.1911323
	2	1	1.572147	1.012443	0.8186342	0.04862102
	1.5	1.5	1.33692	1.518119	0.9712043	0.09801185

Table 3. Estimates for the parameters of BL2PFD with different estimation methods under the sample size 150 when $a = 1,\ b = 2\ and\ \alpha = 3$

METHODS	True Values		Estimated Values		M.S.E	
	β	γ	$\widehat{\beta}$	$\widehat{\gamma}$	$\widehat{\beta}$	$\widehat{\gamma}$
MLM	1	2	1.022541	1.896632	0.05892636	0.014971
	2	1	2.033640	1.055156	0.19859548	0.0155028
	1.5	1.5	1.485175	1.429680	0.11660383	0.01384723
P.E	1	2	0.7660154	1.919106	1.52632	0.04705253
	2	1	1.680933	0.961805	0.7078477	0.01222795
	1.5	1.5	1.351793	1.43825	0.9112946	0.02562018

If we study the results of the Tables 2 and 3, in which sample sizes are (40, and 150) and the combinations of the values of (β, γ) = {(1, 2), (2, 1) and (1.5, 1.5)}. Then we get the results that MLM is the best for the estimation of β and γ. After MLM, the P.E is best for the estimation of scale and shape parameters of the BL2PFD.

4 Application and Discussion

In this section, we have analyzed two real life data sets to demonstrate the performance of BL2PFD. The comparison of the Probability distributions has been made in all the data sets on the basis of Akaike information criterion (AIC), the correct Akaike information criterion (CAIC), Bayesian information criterion (BIC) and Hannan-Quinn information criterion (HQIC).

Finally, using the above mentioned criteria's, our proposed BL2PFD is better than the different competitor models for the same data sets.

4.1 Bladder Cancer Data

We have adopted the data set consisting the remission time of 128 bladder cancer patients to demonstrate the performance of our proposed BL2PFD. These data were also studied by Zea et al.[23] and Lee and Wang[24]. The remission times in months are given: 0.08, 0.20, 0.40, 0.50, 0.51, 0.81, 0.90, 1.05, 1.19, 1.26, 1.35, 1.40, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.25, 3.31, 3.36, 3.36, 3.48, 3.52, 3.57, 3.64, 3.70, 3.82, 3.88, 4.18, 4.23, 4.26, 4.33, 4.34, 4.40, 4.50, 4.51, 4.87, 4.98, 5.06, 5.09, 5.17, 5.32, 5.32, 5.34, 5.41, 5.41, 5.49, 5.62, 5.71, 5.85, 6.25, 6.54, 6.76, 6.93, 6.94, 6.97, 7.09, 7.26, 7.28, 7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 7.87, 7.93, 8.26, 8.37, 8.53, 8.65, 8.66, 9.02, 9.22, 9.47, 9.74, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 25.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01,46.12 ,79.05.

We have compared our proposed BL2PFD with the Beta Exponentiated Pareto distribution (BEPD) by McDonald's Power function distribution (McPFD) by Haq et al.[14], Kumaraswamy Power function distribution (KPFD) by Ibrahim[15], Beta exponentiated Pareto (BEPD) by Zea et al.[23], Marshall-Olkin Power Lomax distribution (MOPLx) by Haq et al.[25], and Power function distribution (PFD).

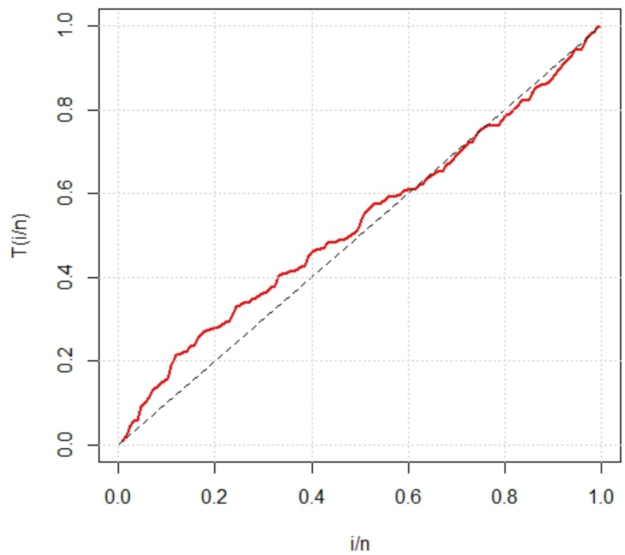

Fig 3. TTT Plot for Bladder Cancer Data

The TTT-plot of the remission time(in month) for bladder cancer patients is exhibited in Figure 3 , we may see that the Hazard rate function has little bit bathtub shape, So, we may easily fit BL2PFD on the bladder cancer data.

Table 4. "Statistics of bladder cancer data"

Models	-logL	AIC	BIC	CAIC
BL2PFD	**401.2683**	**810.5365**	**822.586**	**810.8644**
McPFD	811.5785	821.9553	811.9064	816.2008
KPFD	814.0711	822.6037	814.2662	817.5378
MOPLx	827.075	832.483	825.5162	847.3287
BEPD	826.1318	837.5085	826.4596	830.7540
PFD	942.4546	945.2988	942.4866	943.6102

From Table 4 , we may see that BL2PFD provides better fit for the above data set as it provides minimum AIC, BIC, CAIC, HQIC.

4.2 Failure Times Data of Air-Conditioned System

The 2nd data set is reported by Aarset[26] Dallas[1], which corresponds to the 30 failure times of air-conditioned system of an airplane. The data are as follows: 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, and 95.

We have compared BL2PFD with the alpha power transformed inverse exponential (APTIE) distribution by Dey et al.[27], Marshall Olkin length biased exponential (MOLBE) distribution by ul Haq et al.[28], APT inverted Weibull (APTIW) distribution by Ramadan and Magdy[29], APT Pareto (APTP) distribution by Ihtisham et al.[30] and Alpha Power Transformed Inverse Lomax distribution (APTIL) by ZeinEldin et al.[31].

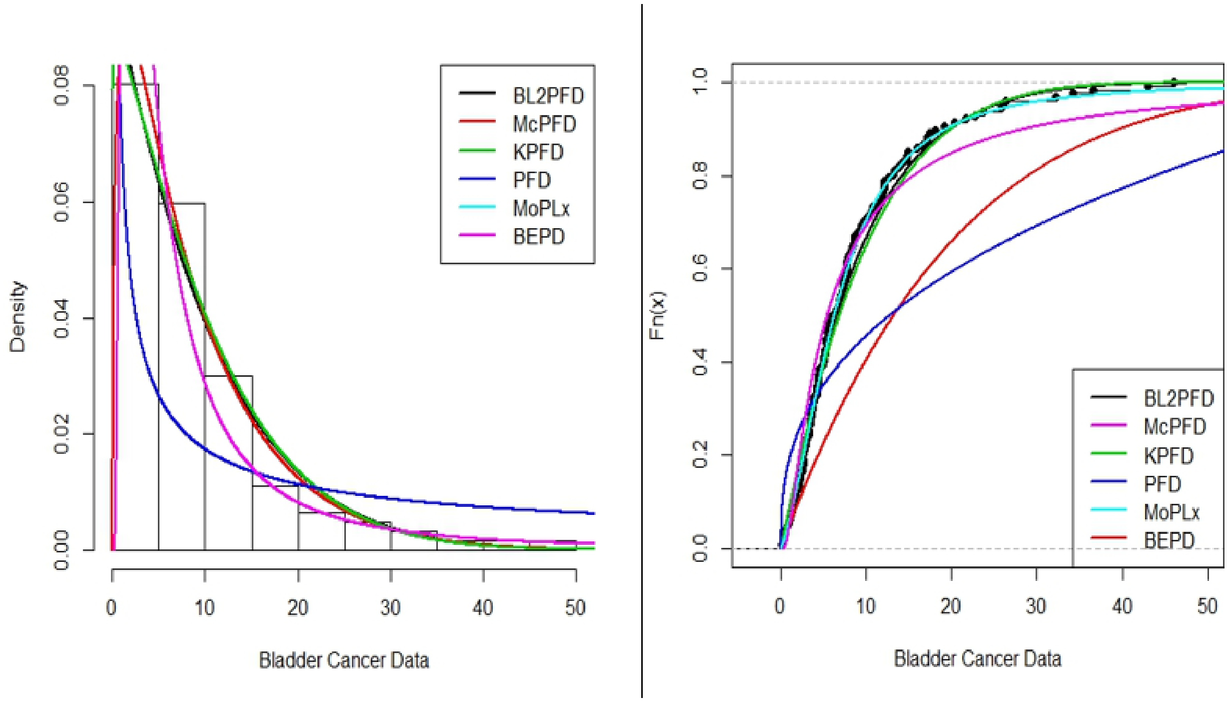

Fig 4. Estimated pdf and cdf curves for Bladder CancerData

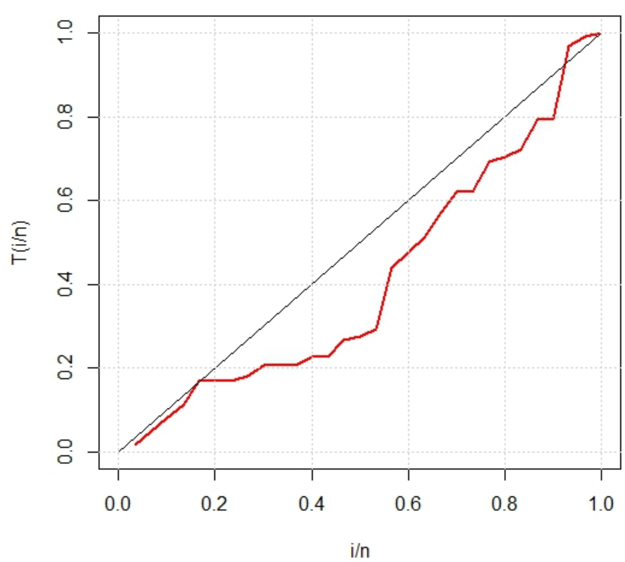

Fig 5. TTT Plot for Failure Times Data of Air-Conditioned System

The TTT-plot is displayed in Figure 5 , which indicates that the HRF associated with the data set has a bathtub shape, since the plot shows a first concave curvature. So, we can easily fit BL2PFD on the failure time's data of air-conditioned system.

Table 5. "Statistics of air-conditioned system"

Distribution	-logL	AIC	BIC	CAIC
BL2PFD	**143.0891**	**294.735**	**300.6473**	**296.5248**
APTIL	151.910	309.819	314.023	311.652
APTIW	153.147	312.293	316.497	314.653
APTIE	153.372	310.744	313.546	312.847
APTP	156.025	314.169	316.972	316.235
MOLBE	155.336	314.673	317.475	317.984

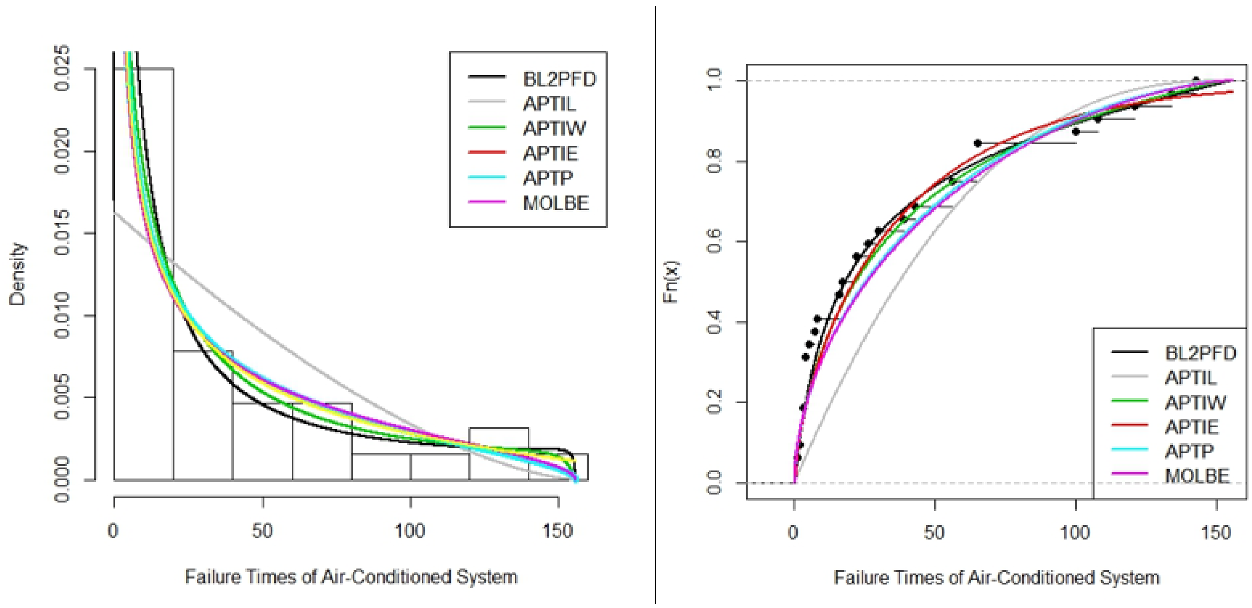

Fig 6. Estimated DensityPlot for Failure Times Data of Air-Conditioned System

From Table 5 , we may see that BL2PFD provides better fit for the above data set as it provides minimum AIC, BIC, CAIC, HQIC.

5 Conclusion

We have proposed a new distribution called Beta Lehmann-2 Power function distribution (BL2PFD). This distribution can have applications in the fields of reliability, economics, actuaries and survival analysis. We have studied the properties of the new distribution including moments, survival function, hazard function, inverse moments, conditional moments, Lorenz curve, incomplete moments and order Statistics. We have also characterized the distribution by conditional variance. Data sets from different scenarios of applied sciences are used to show the efficiency of the proposed model over the already available models. It is hoped that the findings of this study will be useful for researchers in different field of applied sciences.

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