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On Study of Generalized Novikov Equation by Reduced Differential Transform Method

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Abstract

Objectives: The object of the work is essentially to examine the generalization of NovikovPartial Differential Equations through differential transform algorithm. This work also shows that the method can allow us to construct explicit solutions highly nonlinear equations. We have also plotted the constructed solutions. **Methods:** We have constructed the approximate solutions of mentioned equation using a relatively new algorithm, known as reduced differential transform algorithm. **Findings:** It turns out that our solutions agree with the abstract findings known in key papers that we followed. **Applications:** Generalization of Novikov Partial Differential Equations models several physical phenomena such as shallow water flow, dynamics of enzymes in the human cells etc.

Keywords: Nonlinear Equations, Novikov Partial Differential Equations, Partial Differential Equations (PDE)

1. Introduction

This paper is intended to approximate an explicit solution to the following initial value problem,

$$\left(1-\partial_x^2\right)u_t = (1+\partial_x)(2u_x^2 u_{xx} - uu_x u_{xx} - u^3 - u^2 u_{xx} - uu_x^2 + 2u^2 u_x)$$

$$u\left(x,0\right) = u_0\left(x\right). \tag{1}$$

Where $t \in [0,\infty)$, $x \in \mathbb{R}$ and initial data will be suitably chosen from Sobelov space $H^1(\mathbb{R})$. There are several ways we can look at the above problem as generalized Nikov equation or as particular manifestation of following,

$$\left(1-\partial_x^2\right)u_t = F\left(u, u_x, u_{xx}, u_{xxx}\right),$$

where F is a homogeneous polynomial. This study gets its motivation from[1] where the authors studied the abstract well-posedness global weak solution of the above problem by arguing through viscosity vanishing method. Also the stability of weak solutions was proved in the case when the solutions have higher integrability. The equation was presented by[2] one of typical application of the Novikov equation is that it models shallow water flow. Moreover, it possesses a bi-Hamiltonian structure and has ce^{-x-ct} form of solution. Hamiltonian systems are the systems, admitting a complete sequence of first integrals. Bi Hamiltonian properties were first formulated in[3] the equation exhibits Bi-Hamiltonian structure, which means it is totallyintegrable like the Novikov equation. For more details on Novikov, CH equations and equations with Bi-Hamiltonian structure we refer to[4-13]. Now we give a breakdown of paper.

2. Description of Differential Transform Algorithm

This section has been devoted to give a precise description of the Reduced Differential Transform algorithm and how it works.Assume that we have a function $u(x,t)$ with arguments x and t, that can expressed as the product two functions of x and t i.e., $u(x,t) = f(x)g(t)$. Then differential transform of the function $u(x,t)$ can be explicitly written as,

$$u(x,t) = \left(\sum_{i=0}^{\infty} F(i)x^i\right)\left(\sum_{i=0}^{\infty} G(j)t^j\right) = \sum_{k=0}^{\infty} U_k(x)t^k, \tag{2}$$

Where $U_k(x)$ is transformed function in x. The more careful and precise definitions of transform of function $u(x,t)$ is following, (cf.[14,15])

Definition: Consider a function $u(x,t)$ is C^k –class with respect to time $t \geq 0$ and space $x \in \mathbb{R}$. Then define the transform of $u(x,t)$ as:

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}.$$

Where the $U_k(x)$ can be treated as transformed $u(x,t)$, and is essentially analogous the Taylor's coefficient in the 2D Taylor expansion. To recover the function $u(x,t)$ from transformed functions $U_k(x)$, we define the following inverse of differential transform in the following manner.

Definition: Consider a function $u(x,t)$ is C^k –class with respect to time $t \geq 0$ and space $x \in \mathbb{R}$. Then define the transform of $U_k(x)$, as:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k. \tag{3}$$

Or more explicitly,

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} t^k.$$

Next we discuss that how the above described transformation can be applied to solve the concrete nonlinear partial differential equations. Consider a nonlinear PDE in its generalized form,

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \tag{4}$$

Subject to the initial condition: $u(x,0) = f(x)$.

Here L denotes operator $\frac{\partial}{\partial t}$, $Ru(x,t)$ denotes the linear part of PDE that contains the linear expressions of u and its derivatives, $Nu(x,t)$ denotes the operator/expression containing the nonlinear terms involving u and its derivatives operator, $g(x,t)$ stands for an in-homogeneous term that can be treated a forcing factor in the model. Taking the differential transform of the eq. (4) leads to following recursive relation,

$$(k+1)U_{k+1}(x) = G_k(x) - RU_k(x) - NU_k(x), \tag{5}$$

Where $U_k(x)$, $RU_k(x)$, $NU_k(x)$ and $G_k(x)$ denotes the differential transformation of $Lu(x,t)$, $Ru(x,t)$, $Nu(x,t)$ and $g(x,t)$ respectively. Hence the key computation that one need to is the computation of functions $U_1, U_2, U_3 \ldots$ through recursive relation (5), by choosing

$$U_0(x) = f(x).$$

Once $U_1, U_2, U_3 \supset U_n$ are found then we can write n -term approximate solution of PDE (4) as follows:

$$\tilde{u}_n(x,t) = \sum_{k=0}^{n} U_k(x)t^k. \tag{6}$$

Thus by increasing n more and more we get an exact solution of nonlinear PDE (4)

$$u(x,t) = \lim_{n\to\infty} \tilde{u}_n(x,t). \tag{7}$$

Table 1. Reduced differential transformation

Functional form	**Transformed form**
$u(x,t)$	$U_k(x) = \frac{1}{k!}[\frac{\partial^k}{\partial t^k}u(x,t)]_{t=0}$
$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = U_k(x) \pm V_k(x)$
$w(x,t) = \alpha u(x,t)$	$W_k(x) = \alpha U_k(x)$ (α is a constant)
$w(x,t) = x^m t^m$	$W_k(x) = x^m \delta(k-n)$, $\quad \delta(k) = \begin{cases} 1, & k=0 \\ 0, & k \neq 0 \end{cases}$
$w(x,t) = x^m t^m u(x,t)$	$W_k(x) = x^m U_{k-n}(x)$
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} V_r(x)U_{k-r}(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x)$
$w(x,t) = \frac{\partial^r}{\partial t^r}u(x,t)$	$W_k(x) = (k+1)\cdots(k+r)U_{k+r}(x) = \frac{(k+1)!}{k!}U_{k+r}(x)$
$w(x,t) = \frac{\partial}{\partial x}u(x,t)$	$W_k(x) = \frac{\partial}{\partial x}U_k(x)$

Based on definition of the reduced differential transform algorithm following Table 1 of transformations can be proved. For the readers interested in the proofs we refer to[14-15].

3. Solution of Generalized form of Novikov Equation by RDTM

For following Novikov equation, we are applying RDTM method to get an approximate solution:

$$\left(1-\partial_x^2\right)u_t=\left(1+\partial_x\right)\left(2u_x^2u_{xx}-uu_xu_{xx}-u^3-u^2u_{xx}-uu_x^2+2u^2u_x\right) \tag{8}$$

With initial condition chosen from $H^1(\mathbb{R})$,

$$u(x,0)=-e^{-x} \tag{8}$$

On simplifying equation (8) we have

$$\begin{aligned}\left(1-\partial_x^2\right)u_t &=\left(1+\partial_x\right)\left(2u_x^2u_{xx}-uu_xu_{xx}-u^3-u^2u_{xx}-uu_x^2+2u^2u_x\right)\\ &=u^2u_{xx}-2u_x^2u_{xx}-5uu_xu_{xx}-2u_x^3+3uu_x^2+2u^2u_x-uu_xu_{xxx}\\ &-uu_{xx}^2-u^2u_{xx}+2u_x^2u_{xxx}+4u_xu_{xx}^2\end{aligned} \tag{9}$$

By applying RDTM to equation (10), we have:

$$\begin{aligned}(k+1)u_{k+1}=(k+1)\frac{\partial^2}{\partial x^2}u_{k+1}+A_k-5B_k\\ -2Z_k-2D_k+3E_k+2F_k+2G_k+4H_k\\ -I_k-J_k-L_k\end{aligned}$$

$u_k(x)$ is transformed function and dimensional spectrum function is t,

$$A_k=\sum_{k=0}^{h}\sum_{s=0}^{k}u_su_{h-s}\frac{\partial^2}{\partial x^2}u_{h-k} \tag{10}$$

$$B_k=\sum_{k=0}^{h}\sum_{s=0}^{k}u_s\frac{\partial}{\partial x}u_{h-s}\frac{\partial^2}{\partial x^2}u_{h-k} \tag{11}$$

$$Z_k=\sum_{k=0}^{h}\sum_{s=0}^{k}\frac{\partial}{\partial x}u_s\frac{\partial}{\partial x}u_{h-s}\frac{\partial^2}{\partial x^2}u_{h-k} \tag{12}$$

$$D_k=\sum_{k=0}^{h}\sum_{s=0}^{k}\frac{\partial}{\partial x}u_s\frac{\partial}{\partial x}u_{h-s}\frac{\partial}{\partial x}u_{h-k} \tag{13}$$

$$E_k=\sum_{k=0}^{h}\sum_{s=0}^{k}u_s\frac{\partial}{\partial x}u_{h-s}\frac{\partial}{\partial x}u_{h-k} \tag{14}$$

$$F_k=\sum_{k=0}^{h}\sum_{s=0}^{k}u_su_{h-s}\frac{\partial}{\partial x}u_{h-k} \tag{15}$$

$$G_k=\sum_{k=0}^{h}\sum_{s=0}^{k}\frac{\partial}{\partial x}u_s\frac{\partial}{\partial x}u_{h-s}\frac{\partial^3}{\partial x^3}u_{h-k} \tag{16}$$

$$H_k=\sum_{k=0}^{h}\sum_{s=0}^{k}\frac{\partial}{\partial x}u_s\frac{\partial^2}{\partial x^2}u_{h-s}\frac{\partial^2}{\partial x^2}u_{h-k} \tag{17}$$

$$I_k=\sum_{k=0}^{h}\sum_{s=0}^{k}u_s\frac{\partial}{\partial x}u_{h-s}\frac{\partial^3}{\partial x^3}u_{h-k} \tag{18}$$

$$J_k=\sum_{k=0}^{h}\sum_{s=0}^{k}u_s\frac{\partial^2}{\partial x^2}u_{h-s}\frac{\partial^2}{\partial x^2}u_{h-k} \tag{19}$$

$$L_k=\sum_{k=0}^{h}\sum_{s=0}^{k}u_su_{h-s}\frac{\partial^3}{\partial x^3}u_{h-k}. \tag{20}$$

From the initial condition eq. (9), we solve the first, second and third partial derivatives of eq. (9), we have:

$$u(x,0)=-e^{-x},\ \frac{\partial}{\partial x}u_0=e^{-x},\ \frac{\partial^2}{\partial x^2}u_0=-e^{-x},\ \frac{\partial^3}{\partial x^3}u_0=e^{x} \tag{21}$$

Now taking $k=0$ in Equation (11) and plugging partial derivatives of Eq. (21), A_0 of Equation (11), B_0 of Eq. (12), Z_0 of Eq. (13), D_0 of Eq. (14), E_0 of Eq. (15), F_0 of Equation (16), G_0 of Eq. (17), H_0 of Eq. (18), I_0 of Eq. (19), J_0 of Eq. (20), L_0 of Equation (21) in Equation (11).

$$\begin{aligned}(k+1)u_{k+1}=(k+1)\frac{\partial^2}{\partial x^2}u_{k+1}+A_k-5B_k-2Z_k-2D_k+3E_k\\ +2F_k+2G_k+4H_k-I_k-J_k-L_k\end{aligned}$$

$$\begin{aligned}u_1&=\frac{\partial^2}{\partial x^2}u_1+A_0-5B_0-2Z_0-2D_0\\ &+3E_0+2F_0+2G_0+4H_0-I_0-J_0-L_0\\ &=\frac{\partial^2}{\partial x^2}u_1+(-(e^{-x})^3)-5(e^{-x})^3-2(-(e^{-x})^3)\\ &-2(e^{-x})^3+3(-(e^{-x})^3)+2(e^{-x})^3\\ &+2(e^{-x})^3+4(e^{-x})^3-(-(e^{-x})^3)\\ &-(-(e^{-x})^3)-(e^{-x})^3\end{aligned}$$

$$u_1=\frac{\partial^2}{\partial x^2}u_1-0 \tag{22}$$

$$=c_1e^{x}+c_2e^{-x} \tag{23}$$

Taking $c_1=c_2=1$.

$$u_1=e^{x}+e^{-x} \tag{24}$$

Taking partial derivative of Eq. (25) and using in, A_1 of Eq. (11), B_1 of equation (12), Z_1 of eq. (13), E_1 of Eq. (14), F_1 of Eq. (15), G_1 of Eq. (16), H_1 of Eq. (17), I_1 of Eq. (18), J_1 of Eq. (19), L_1 of Eq. (21) in Eq. (11),

$$A_1 = u_0 u_1 \frac{\partial^2}{\partial x^2} u_0^2 + u_0 u_1 \frac{\partial^2}{\partial x^2} u_0^2$$
$$= 2(e^{-x})^2 (e^x + e^{-x}),$$

$$B_1 = u_0 \frac{\partial}{\partial x} u_1 \frac{\partial^2}{\partial x^2} u_0 + u_1 \frac{\partial}{\partial x} u_0 \frac{\partial^2}{\partial x^2} u_0$$
$$= -(e^{-x})(e^x - e^{-x})(-e^{-x}),$$

$$Z_1 = 2 \frac{\partial}{\partial x} u_0 \frac{\partial}{\partial x} u_1 \frac{\partial^2}{\partial x^2} u_0 = -2(e^{-x})^2 (e^x - e^{-x})$$

$$D_1 = 2\left(\frac{\partial}{\partial x} u_0\right)^2 \frac{\partial}{\partial x} u_1 = 2(e^{-x})^2 (e^x - e^{-x})$$

$$E_1 = u_0 \frac{\partial}{\partial x} u_1 \frac{\partial}{\partial x} u_0 + u_1 \left(\frac{\partial}{\partial x} u_0\right)^2$$
$$= -(e^{-x})^2 (e^x - e^{-x}) + (e^{-x})(e^x + e^{-x})$$

$$F_1 = 2u_0 u_1 \frac{\partial}{\partial x} u_0 = -2(e^{-x})^2 (e^x + e^{-x})$$

$$G_1 = 2 \frac{\partial}{\partial x} u_0 \frac{\partial}{\partial x} u_1 \frac{\partial^3}{\partial x^3} u_0 = 2(e^{-x})^2 (e^x - e^{-x})$$

$$H_1 = \frac{\partial}{\partial x} u_0 \frac{\partial^2}{\partial x^2} u_1 \frac{\partial^2}{\partial x^2} u_0 + \frac{\partial}{\partial x} u_1 \left(\frac{\partial^2}{\partial x^2} u_0\right)^2$$
$$= -(e^{-x})^2 (e^x + e^{-x}) + (e^{-x})^2 (e^x - e^{-x})$$

$$I_1 = u_0 \frac{\partial}{\partial x} u_1 \frac{\partial^3}{\partial x^3} u_0 + u_1 \frac{\partial}{\partial x} u_0 \frac{\partial^3}{\partial x^3} u_0$$
$$= -(e^{-x})^2 (e^x - e^{-x}) + (e^{-x})^2 (e^x + e^{-x})$$

$$J_1 = u_0 \frac{\partial^2}{\partial x^2} u_1 \frac{\partial^2}{\partial x^2} u_0 + u_1 \left(\frac{\partial^2}{\partial x^2} u_0\right)^2 = 2(e^{-x})^2 (e^x + e^{-x})$$

$$L_1 = 2u_0 u_1 \frac{\partial^3}{\partial x^3} u_0 = -2(e^{-x})^2 (e^x + e^{-x})$$

Substituting the value of $k=1$ in Eq. (11) and substituting the values of A_1, B_1, C_1 $D_1, E_1, F_1, G_1, H_1, I_1, J_1$ and L_1 in Eq. (11).

$$2u_2 = 2\frac{\partial^2}{\partial x^2} u_2 + A_1 - 5B_1 - 2Z_1 - 2D_1 + 3E_1$$
$$+ 2F_1 + 2G_1 + 4H_1 - I_1 - J_1 - L_1$$

$$u_2 = e^x \left(\frac{3}{4} e^{-x} - \frac{3}{4} e^{-2x} + \frac{1}{4} \% e^{-3x} - \frac{1}{2} e^{-4x} x\right)$$
$$+ c_3 e^x + c_4 e^{-x}$$
$$-e^{-x} \left(\frac{3}{2} x + \frac{3}{4} e^{-x} - e^{-2x} - \frac{3}{4} e^x\right) \tag{25}$$

Taking $c_3, c_4 = 1$,

$$u_2 = e^x \left(\frac{3}{4} e^{-x} - \frac{3}{4} e^{-2x} + \frac{1}{4} \% e^{-3x} - \frac{1}{2} e^{-4x} x\right) + e^x$$
$$+ e^{-x} - e^{-x} \left(\frac{3}{2} x + \frac{3}{4} e^{-x} - e^{-2x} - \frac{3}{4} e^x\right)$$

$$u_2 = \frac{3}{2} + \frac{1}{4} e^{-x} - \frac{1}{2} e^{-2x} + \frac{1}{2} e^{-3x} - \frac{3}{2} e^{-x} x + e^x \tag{26}$$

Taking partial derivative of Eq. (27) and using Eq. (22) and A_2 of Eq. (11), B_2 of Eq. (12), Z_2 of Eq. (13), E_2 of Eq. (14), F_2 of Eq. (15), G_2 of Eq. (16), H_2 of eq. (17), I_2 of Eq. (18), J_2 of Eq. (19), L_2 of Eq. (21) in Eq. (11) we have:

$$A_2 = 2u_0 u_2 \frac{\partial^2}{\partial x^2} u_0^2 + u_1^2 \frac{\partial^2}{\partial x^2} u_0^2$$
$$= 2e^{-2x} e^x \left(\frac{3}{4} e^{-x} - \frac{3}{4} e^{-2x} + \frac{1}{4} e^{-3x} - \frac{1}{2} e^{-4x}\right)$$
$$-2e^{-x} \left(\frac{3}{2} x + \frac{3}{4} e^{-x} - e^{-2x} - \frac{3}{4} e^x\right)$$
$$+2(e^{-x})^2 \left(e^x + e^{-x}\right) - \left(e^x + e^{-x}\right) e^{-x}$$

$$B_2 = -e^{-3x} + \frac{3}{2} e^{-4x} - 2e^{-5x} + 3e^{-3x} x - e^x - \frac{3}{2} e^{-2x}$$

$$Z_2 = \frac{5}{2} e^{-3x} - 2e^{-4x} + 3e^{-5x} - 3e^{-3x} x - e^x$$

$$D_2 = \frac{5}{2} e^{-3x} - 2e^{-4x} + 3e^{-5x} - 3e^{-3x} x - e^x$$

$$E_2 = e^{-3x} - \frac{3}{2} e^{-4x} + 2e^{-5x} - 3e^{-3x} x + e^x + \frac{3}{2} e^{-2x}$$

$$F_2 = -3e^{-2x} + \frac{1}{2} e^{-3x} + e^{-4x} - e^{-5x} + 3e^{-3x} x + e^x$$

$$G_2 = -\frac{5}{2} e^{-3x} + 2e^{-4x} - 3e^{-5x} + 3e^{-3x} x + e^x$$

Substituting the value of $k=2$ in Eq. (11) and substituting the values of $A_2, B_2, Z_2, D_2,$ $E_2, F_2, G_2, H_2, I_2, J_2, L_2$ in Eq. (11),

$$3u_3 = 3\frac{\partial^2}{\partial x^2} u_3 + A_2 - 5B_2 - 2Z_2 - 2D_2$$
$$+3E_2 + 2F_2 + 2G_2 + 4H_2 - I_2 - J_2 - L_2$$

$$u_3 = \frac{11}{16}e^{-3x} - e^{-2x} + \frac{3}{2}e^{x} - \frac{8}{45}e^{-4x} + \frac{7}{24}e^{-5x} - xe^{x} + e^{-x} \quad (27)$$

We will take all the constants appear in u_3 equal to one by following same procedure. We can calculate next to next values. Now, plugging all values like Equation (9), Eq. (25), Eq. (27) and Eq. (28) and so on, we have a generalized form, i.e.

$$u(x,t) = \sum_{k=0}^{n} U_k(x)t^k$$

$$\begin{aligned} u(x,t) = {} & -e^{-x} + \left(e^{x} + e^{-x}\right)t + \frac{3}{2}t^2 + \frac{1}{4}t^2e^{-x} - \frac{1}{2}t^2e^{-2x} \\ & + \frac{1}{2}t^2e^{-3x} - \frac{3}{2}t^2e^{-x}x + t^2e^{x} + \frac{11}{6}t^3e^{-3x} \\ & - t^3e^{-2x} - \frac{3}{2}t^3e^{x} - \frac{8}{45}t^3e^{-4x} + \frac{7}{24}t^3e^{-5x} \\ & - t^3e^{x}x + t^3e^{-x} + \dots \quad . \end{aligned} \quad (28)$$

The values of u_1, u_2, u_3 and $u(x,t)$ have been calculated with the help of MAPLE and MATLAB.

4. Graph of Generalized form of Novikov Equation by RDTM

By plotting the graph of $u(x,t)$ for $(x,t) \in [0,1] \times [0,1]$, One way to interpret above graph is to treat eq. (1) as mathematical model of the velocity flow $u(x,t)$ of shallow water flowing through a rectangular channel (or a cross section of channel), where x denotes the space coordinate/location along the channel axis and t denotes the time. We may also assume that there is no friction and coriolis forcing factor in flow. The graphs shows that as time passes the velocity of the flow is increasing/accelerating smoothly, physically water might experience smooth splashes.

By plotting the graph of $u(x,t)$ for $(x,t) \in [-100,100] \times [-100,100]$, Keeping in view the notation and interpretation same as Figure 1 and 2 can be interpreted as that the solution blow up in finite time i.e., velocity flow is smoother in channel initially then it became singular in finite period of time.This result/observation is in complete agreement with the conclusion of Proposition 3.1 of[1].

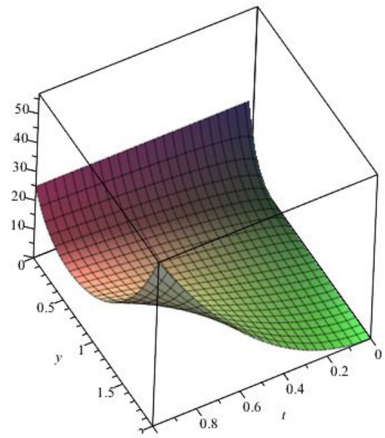

Figure 1. Local behavior of solution $u(x,t)$

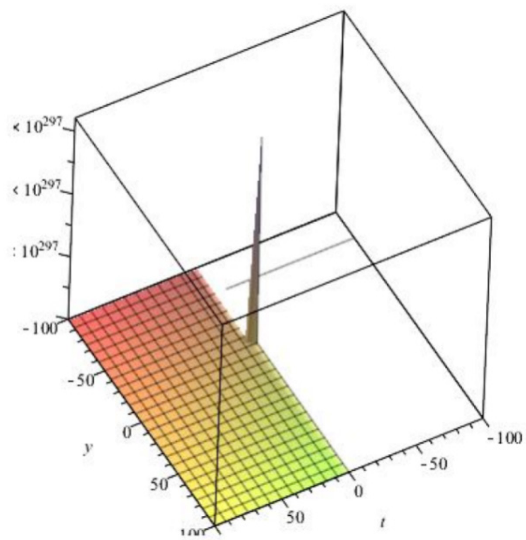

Figure 2. Global behavior of the solution $u(x,t)$

5. Conclusion

Novikovform of models is highly nonlinear in its structure. In this article we have constructed an explicit approximate solution to a highly nonlinear version of the generalized Novikovequation through a highly efficient algorithm knows as Reduced Differential Transform Algorithm. Our results are in agreement with some key abstract conclusions of 1 like blow up of evolution in finite times. Our work shows that Reduced differential transform algorithm is very efficient in constructing the explicit solutions of highly nonlinear problems.

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