Abstract

Objectives/Aim: To find fixed points for integral types of contraction mappings in a complex valued metric spaces. Methods: We used mathematical analysis to prove the results as a generalization of weakly contraction self mappings defined in a complex metric space. Findings: We are finding by weakly integral types of contraction self mappings that there are common fixed points. In this paper we used commutative self mappings for our conclusions. Application/Improvements: In future it can also be used for the existence and uniqueness of the differential equations solutions.

Keywords: Complex Valued Metric Space, Contraction Mapping, Fixed Point, Integral Type, Weakly Commuting

1. Introduction

A fixed point theory contains a great number of generalizations of Banach contraction principle by using different form of contraction condition in various spaces. But majority of such generalizations are obtained by improving underlying contraction conditions which also includes contraction conditions described by rational expressions. The concept of complex valued metric space was introduced by \(^4\) in 2011. The complex valued b-metric space is introduced by \(^5\), which is more general than well-known complex valued metric space. There are many fixed point results in complex valued metric spaces \(^5,11\) also in complex valued b-metric spaces \(^2,12\). In this study we present a common fixed point result for two self-mappings satisfying a contractive condition in complex valued metric spaces. This idea is intended to define rational expressions which are not meaningful in cone metric spaces and thus many such results of analysis cannot be generalized to cone metric spaces but to complex valued metric spaces. A contractive condition of integral type was introduced by \(^6\) in 2002. After the paper of Branciari, a lot of research works have been carried out on generalizing contractive conditions of integral type for different contractive mappings satisfying various known properties. There are various types of weakly contractive self mappings in the last thirty years.

2. Preliminaries

Let \(C\) be the set of complex numbers, and for any two complex numbers \(z_1\) and \(z_2\) in \(C\) is denoted by \(z_1, z_2 \in C\). If Real part and imaginary part of the complex number are denoted by \(\text{Re}(z)\) and \(\text{Im}(z)\) respectively. Define a partial order on \(C\) as follows: \(z_1 \preceq z_2\) (that is
mean subordination if and only if Re(z₁) ≤ Re(z₂) and Im(z₁) ≤ Im(z₂).

Thus \( z_1 \preceq z_2 \) if one of the following holds:

1. Real parts \( z_1 \) and \( z_2 \) are equal and imaginary parts and \( z_1 \) are \( z_2 \) equal.
2. Real part \( z_1 \) is less than Real part \( z_2 \) and imaginary parts \( z_1 \) and \( z_2 \) are equal.
3. Real parts \( z_1 \) and \( z_2 \) are equal and imaginary part \( z_1 \) is less than imaginary part of \( z_2 \).
4. Real part \( z_1 \) is less than real part and imaginary part \( z_1 \) is less than imaginary part \( z_2 \).

We will write \( z_1 \preceq z_2 \) (subordination and isn’t equal \( z_2 \)) if \( z_1 \) and \( z_2 \) are n’t equal and one of (2), (3) and (4) is satisfied. Also, we will write \( z_1 \prec z_2 \) (\( z_1 \) is strictly subordination) if only (4) is satisfied. It follows that:

(i) \( 0 < z_1 \preceq z_2 \) implies \( |z_1| \leq |z_2| \)
(ii) \( z_1 \prec z_2 \) and \( z_2 \prec z_3 \) imply \( z_1 \prec z_3 \)
(iii) \( 0 \preceq z_1 \preceq z_2 \) implies \( |z_1| \leq |z_2| \)
(iv) For any two real numbers \( a \) and \( b \), \( 0 \leq a \leq b \) and \( z_1 \preceq z_2 \), then \( a \cdot z_1 \preceq b \cdot z_2 \).

Definition 2.1. Let \( X \) be a nonempty set and \( C \) be the set of complex numbers. Suppose that the mapping \( d:X \times X \rightarrow C \) satisfies the following conditions:

1. \( d(x,y) \) is nonnegative for all \( x,y \in X \) and \( x=y \) if and only if \( d(x,y) = 0 \)
2. \( d(x,y) \) and \( d(y,x) \) are equal for all \( x \) and \( y \) in \( C \)
3. \( d(x,y) \) is subordination of the sum of \( d(x,z) \) and \( d(x,y) \) for all \( x \), \( y \) and \( z \) in \( X \).

Then \( d \) is called a complex valued metric on \( X \), and \((X;d)\) is called a complex valued metric space.

Definition 2.2. Let \((X;d)\) be a complex valued metric space. Then

1. A point \( x \) in \( X \) is called an interior point of a set \( A \) in \( X \) whenever there exists positive real numberr such that the open set \( B(x,r) \) in \( A \)
2. A point \( x \) in \( X \) is called a limit point of a set \( A \) whenever there exists positive real numberr such that \( B(x,r) \) and \( (A-x) \) are nonempty set.
3. A subset \( A \) in \( X \) is called closed whenever each element of \( A \) belongs to \( A \).
4. A subcbasis for a Hausdorff topology \( \tau \) on \( X \) is a family \( F=\{B(x,r):x \in X \text{and } 0<r\} \).

Definition 2.3. Let \((X;d)\) be a complex valued metric space, \( x_n \) be a sequence in \( X \) and \( x \in X \). Then

1. If for every \( 0 < c \in C \) there is \( n \in N \) such that for all \( n \in N, d(x_n,x) < c \), then \( x_n \) is said to be convergent to \( x \) and \( x \) is called a limit point of \( x_n \). We denote this by \( \lim_{n \to \infty} x_n = x \) or \( \{x_n\} \to x \) as \( n \to \infty \).
(2) If for every \(0 < c \in C\) there is \(n \in N\) such that for all \(n \in N\), \(d(x_n, x_{n+m}) < c\) where \(m \in N\), then \(x_n\) is said to be Cauchy sequence.

(3) If every Cauchy sequence in \(X\) is convergent, then \((X, d)\) is said to be a complete complex valued metric space.

**Lemma 2.4** \(^1\). Let \((X, d)\) be a complex valued metric space and let \(x_n\) be a sequence in \(X\). Then \(x_n\) converges to \(x\) if and only if \(\lim\limits_{n \to \infty} d(x_n, x) = 0\).

**Lemma 2.5** \(^1\). Let \((X, d)\) be a complex valued metric space and let \(x_n\) be a sequence in \(X\). Then \(x_n\) is a Cauchy sequence if and only if \(\lim\limits_{n \to \infty} d(x_n, x_{n+m}) = 0\) where \(m \in N\).

In 2002, a general contractive condition of integral type is given and analyzed by \(^6\) in the following theorem.

**Theorem 2.6** \(^5\). Let \((X; d)\) be a complete metric space, \(c \in (0, 1)\) and let \(T : X \to X\) be a mapping such that for each \(x, y \in X\),

\[
\int_{0}^{c} d(Tx, Ty) \varphi(t) \leq c \int_{0}^{d(x, y)} \varphi(t)
\]

where \(\varphi : [0; +\infty) \to [0; +\infty)\) is a Lesbegue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of \([0; +\infty)\), non-negative, and such that for each \(\phi > 0\), \(\int_{0}^{\phi} \varphi(t) > 0\), then \(T\) has a unique fixed point \(a \in X\) such that for each \(x \in X\), \(\lim\limits_{n \to \infty} T^n x = a\).

Also, the pairwise commuting is defined.

**Definition 2.7** \(^14\). The two families of self mappings \(\{f_i\}_i^n\) and \(\{g_j\}_i^n\) are said to be pairwise commuting if

\[
(1) \quad f_i f_j = f_j f_i, \quad \text{for } i, j \in \{1, 2, ..., m\}
\]

\[
(2) \quad g_k g_l = g_l g_k, \quad \text{for } k, l \in \{1, 2, ..., n\}
\]

\[
(3) \quad f_i g_j = g_j f_i, \quad \text{for } i \in \{1, 2, ..., m\} \text{ and } l \in \{1, 2, ..., n\}
\]

The concepts of weakly commuting, weak * commuting and weak ** commuting are introduced by \(^14-17\) respectively. Next, we introduce the weakly contraction self mappings in the complex valued metric space.

**Definition 2.8.** Let \((X, d)\) be a complex valued metric space. Then

(1) The two self maps \(f\) and \(g\) are weakly commuting if

\[
d(fgx, gfx) \leq d(fx, gx),
\]

(2) The two self maps \(f\) and \(g\) are weak * commuting if

\[
d(fgx, gfx) \leq d(f^2x, g^2x)
\]

(3) The two self maps \(f\) and \(g\) of a complete space \((X, d)\) are weak ** commuting if

\[
f(x) \subseteq g(x) \quad \text{for } x \in X
\]

(4) The two self maps \(f\) and \(g\) of a complete space \((X, d)\) are \(m - \text{weak} \; \text{** commute if}

\[
f(x) \subseteq g(x) \quad \text{for any } x \in X
\]
In this article, we denote $X$ to be the complex metric space $(X,d)$, and to a self mapping $f:(X,d) \to (X,d)$ by a self mapping $f$ on $X$. And we define new concept of weak commuting in the complex valued metric space as follows.

**Definition 2.9.** For a real number $\alpha \geq \frac{1}{2}$, the two self mappings $f_1$ and $f_2$ defined in a complex valued metric space $(X,d)$ are said to be $2\alpha$-weak commuting if $f_1(X) \subset f_2(X)$ and

$$d(f_1^{2\alpha}z,f_2^{2\alpha}z) \leq d(f_1^{2\alpha}z,f_2^{2\alpha}z) \leq d(f_1f_2^{2\alpha}z,f_2^{2\alpha}f_1z) \leq d(f_1f_2z,f_2f_1z) \leq d(f_1^{2\alpha}z,f_2^{2\alpha}z).$$

The $2\alpha$-weak commuting mapping is a generalization of various types of weak commuting maps.

### 3. The Main Results

By a rational expression of contraction mapping defined in a complex valued metric space $X$, we prove the next result of a common fixed point.

**Theorem 3.1** Let $f_1$ and $f_2$ be two self sequentially continuous mappings on a complete complex valued metric space $X$ such that $f_1(X) \subset f_2(X)$ and the order pair $\{f_1,g\}$ is $2\alpha-$ weakly commuting, also the condition is satisfied

$$d(f_1^{2\alpha}z,f_2^{2\alpha}z,f_2^{2\alpha}w) \leq \lambda d(f_1^{2\alpha}z,f_2^{2\alpha}z)$$

+ $\delta \left( \frac{d(f_1^{2\alpha}z,f_2^{2\alpha}z,f_2^{2\alpha}w)[1+d(f_2^{2\alpha}z,f_1^{2\alpha}z)]}{1+d(f_2^{2\alpha}z,f_1^{2\alpha}z)} \right)
+ \gamma \left( \frac{d(f_1^{2\alpha}z,f_2^{2\alpha}z,f_2^{2\alpha}w)[1+d(f_2^{2\alpha}z,f_1^{2\alpha}z)]}{1+d(f_2^{2\alpha}z,f_1^{2\alpha}z)} \right)$

where, $\alpha,\beta,\gamma \geq 0$ and $\alpha + \beta + \gamma < 1$. Then $f_1$ and $f_2$ have a unique common fixed point.

**Proof.** Let $z_0$ be an arbitrary point in $X$, and construct a sequence $\{z_n\}$ such that

$f_1^{2\alpha}z_2=_{z_{2n+1}}$ and $f_2^{2\alpha}z_2=_{z_{2n+2}}$, for $n=0,1,2,...$

To show that $\{z_n\}$ is a Cauchy sequence in $X$. Let

$$d(z_{2n+1},z_{2n+2}) = d(f_2^{2\alpha}z_{2n+1},f_2^{2\alpha}z_{2n}) \leq \lambda d(f_1^{2\alpha}z_{2n-1},f_2^{2\alpha}z_{2n})$$

+ $\delta \left( \frac{d(f_1^{2\alpha}z_{2n-1},f_1^{2\alpha}z_{2n+1})[1+d(f_2^{2\alpha}z_{2n},f_1^{2\alpha}z_{2n})]}{1+d(f_2^{2\alpha}z_{2n},f_1^{2\alpha}z_{2n})} \right)
+ \gamma \left( \frac{d(f_1^{2\alpha}z_{2n-1},f_1^{2\alpha}z_{2n})[1+d(f_2^{2\alpha}z_{2n},f_1^{2\alpha}z_{2n})]}{1+d(f_2^{2\alpha}z_{2n},f_1^{2\alpha}z_{2n})} \right)$

But $d(f_1^{2\alpha}z_{2n},f_1^{2\alpha}z_{2n-1}) = 0$ since $f_1^{2\alpha}z_{2n-1}=z_{2n}$.

So that

$$d(z_{2n+1},z_{2n+2}) \leq \lambda d(f_1^{2\alpha}z_{2n-1},f_2^{2\alpha}z_{2n})$$

+ $\delta \left( \frac{d(f_1^{2\alpha}z_{2n-1},f_1^{2\alpha}z_{2n})[1+d(f_2^{2\alpha}z_{2n+1},f_1^{2\alpha}z_{2n})]}{1+d(f_2^{2\alpha}z_{2n+1},f_1^{2\alpha}z_{2n})} \right).$
Therefore,
\[|d(z_{2n+1}, z_{2n+2})| \leq \lambda |d(f_1^{2a}z_{2n-1}, f_2^{2a}z_{2n})| + \delta (d(f_1^{2a}z_{2n+1}, f_2^{2a}z_{2n+2})[1 + d(f_1^{2a}z_{2n-1}, f_2^{2a}z_{2n+2})]) \]

\[+ \delta (d(f_1^{2a}z_{2n+1}, f_2^{2a}z_{2n+2})[1 + d(f_1^{2a}z_{2n-1}, f_2^{2a}z_{2n+2})]) \]

Since
\[|1 + d(f_2^{2a}z_{2n-1}, f_1^{2a}z_{2n})| = |1 + d(f_2^{2a}z_{2n-1}, f_1^{2a}z_{2n})| > |d(f_2^{2a}z_{2n-1}, f_1^{2a}z_{2n})| \]

which implies that
\[|d(z_{2n+1}, z_{2n+2})| \leq \lambda |d(f_1^{2a}z_{2n-1}, f_2^{2a}z_{2n})| + 2\delta |d(f_1^{2a}z_{2n-1}, f_2^{2a}z_{2n})| \]

and
\[|d(z_{2n+1}, z_{2n+2})| \leq \lambda |d(f_1^{2a}z_{2n-1}, f_2^{2a}z_{2n})| + 2\delta |d(f_1^{2a}z_{2n-1}, f_2^{2a}z_{2n})| \]

Moreover,
\[|d(z_{2n+1}, z_{2n+2})| \leq \lambda |d(f_1^{2a}z_{2n-1}, f_2^{2a}z_{2n})| + 2\delta |d(f_1^{2a}z_{2n-1}, f_2^{2a}z_{2n})| \]

Let \( \hat{\lambda} = \frac{\lambda}{1 - 2\delta} \). Then
\[|d(z_{n+1}, z_{n+2})| \leq \hat{\lambda} |d(z_{n-1}, z_{n})| \leq \hat{\lambda}^2 |d(z_{n-2}, z_{n-1})| \leq \hat{\lambda}^3 |d(z_{n-3}, z_{n-2})| \leq \cdots \leq \hat{\lambda}^m |d(z_0, z_1)|. \]

So that for \( m > n \),
\[|d(z_n, z_m)| \leq |d(z_n, z_{n+1})| + |d(z_{n+1}, z_{n+2})| + \cdots + |d(z_{m-1}, z_m)| \]
\[\leq (\dot{\alpha}^{-n} + \dot{\alpha}^{-n+1} + \dot{\alpha}^{-n+2} + \ldots + \dot{\alpha}^{-m-1}) |d(z_0, z_1)|\]

\[\leq \left(\frac{\dot{\alpha}^{-n}}{1 - \dot{\alpha}}\right) |d(z_0, z_1)| \to 0 \text{ as } n \to \infty.\]

Hence, the sequence \(\{z_n\}\) is Cauchy. By the completeness of the complex valued metric space \(X\), there exists a number \(t \in X\) such that \(x_n \to 0\) as \(n \to \infty\). Assume by the contradiction that \(t \neq f_1^2 t\) and \(0 \leq z = d(t, f_1^{2\alpha} t)\) so that

\[
z = d(t, f_1^{2\alpha} t) \leq d(t, f_1^{2\alpha} z_{2n+1}) + d(f_2^{2\alpha} z_{2n+1}, f_1^{2\alpha} t)\]

\[
l(t, z_{2n+2}) + \lambda d(t, z_{2n+1}) + \delta\left(\frac{d(t, f_1^{2\alpha} t)[1 + d(z_{2n+1}, f_2^{2\alpha} z_{2n+1})]}{1 + d(t, f_1^{2\alpha} z_{2n})}\right)\]

\[
+ \gamma\left(\frac{d(z_{2n+1}, f_1^{2\alpha} t)[1 + d(t, z_{2n+1})]}{1 + d(t, f_1^{2\alpha} z_{2n})}\right)\]

Hence,

\[
|d(t, f_1^{2\alpha} t)| \leq |d(t, z_{2n+2})| + \lambda |d(t, z_{2n+1})|\]

\[
+ \delta\left(\frac{|z[1 + d(z_{2n+1}, z_{2n+2})]|}{1 + d(t, z_{2n+1})}\right)\]

\[
+ \gamma\left(\frac{|d(z_{2n+1}, f_1^{2\alpha} t)[1 + |d(t, z_{2n+2})]|}{1 + d(t, z_{2n+1})}\right)\]

Therefore, \(|d(t, f_1^{2\alpha} t)| = 0\) as \(n \to \infty\) which is a contradiction. Hence \(t = f_1^{2\alpha} t\). Similarly we get \(f_2^2 t = t\). Thus \(t\) is a common fixed point of \(f\) and \(g\). Let \(s \in X\) be another common fixed point of \(f_1\) and \(f_2\) (i.e. \(s = f_1 s = f_2 s\)). Then

\[
|d(t, f_1^{2\alpha} t)| \leq \lambda |d(f_1^{2\alpha} s, f_2^{2\alpha} t) - \lambda d(f_1^{2\alpha} s, f_2^{2\alpha} t)\]

So that \(f_1^{2\alpha} f_2^2 t = f_1^{2\alpha} t\).

Since \(f_1^{2\alpha} f_2^2 f_1 (t) = f_1^{2\alpha} f_1 (t)\) and \(f_2^{2\alpha} f_1 (t) = f_1^{2\alpha} f_1 (t)\)

then

\[
|d(t, f_1^{2\alpha} t)| \leq \lambda |d(f_1^{2\alpha} t, f_2^{2\alpha} f_1 (t)) + \lambda |d(f_1^{2\alpha} t, f_2^{2\alpha} f_1 (t))|\]

\[
+ \delta(|d(f_1^{2\alpha} t, f_2^{2\alpha} f_1 (t))| + |d(f_1^{2\alpha} t, f_2^{2\alpha} f_1 (t))|)\]

\[
+ \gamma(|d(f_1^{2\alpha} t, f_2^{2\alpha} f_1 (t))| + |d(f_1^{2\alpha} t, f_2^{2\alpha} f_1 (t))|)\]

Therefore,

\[
|d(t, f_1^{2\alpha} t)| \leq \lambda |d(f_1^{2\alpha} t, f_2^{2\alpha} f_1 (t))|\]

\[
+ \delta(|d(f_1^{2\alpha} t, f_2^{2\alpha} f_1 (t))| + |d(f_1^{2\alpha} t, f_2^{2\alpha} f_1 (t))|)\]

\[
+ \gamma(|d(f_1^{2\alpha} t, f_2^{2\alpha} f_1 (t))| + |d(f_1^{2\alpha} t, f_2^{2\alpha} f_1 (t))|)\]

\[
\leq [\delta + \lambda] |d(t, f_1^{2\alpha} t)|.\]

So that \(|d(t, f_1^{2\alpha} t)| = 0\) and \(f_1^{2\alpha} t = t\) since \(\alpha + \beta < 1\).

Similarly we get \(f_2^2 t = t\). Thus \(t\) is a common fixed point of \(f\) and \(g\). Let \(s \in X\) be another common fixed point of \(f_1\) and \(f_2\) (i.e. \(s = f_1 s = f_2 s\)). Then

\[
d(s, t) = d(f_1^{2\alpha} s, f_2^{2\alpha} f_1^{2\alpha} t) \leq \lambda d(f_1^{2\alpha} s, f_2^{2\alpha} t)\)
\[
\alpha^2 + \alpha^2 + \alpha^2 + \alpha^2 + \alpha^2 + \alpha^2 + \alpha^2
\]

\[
\delta(1 + d(f(s, f(t)), f(s)) + \gamma(1 + d(f(s, f(t))), f(s))
\]

\[
d(f(w, f(w)), f(w)) + \delta(1 + d(f(w, f(w))), f(w)) + \gamma(1 + d(f(w, f(w))), f(w))
\]

\[
\lambda \geq \alpha, \beta, \gamma \geq 0 \quad \text{and} \quad \alpha + 2\beta + \gamma < 1
\]

\[
d(F(z, G(z)), F(z)) + \delta(1 + d(F(z, G(z)), G(z)) + \gamma(1 + d(F(z, G(z)), G(z)))
\]

\[
d(F(z, G(z)), F(z)) + \delta(1 + d(F(z, G(z)), G(z)) + \gamma(1 + d(F(z, G(z)), G(z)))
\]

\[
\lambda d(F(G(z)), G(F(z))) + \delta(1 + d(F(G(z)), G(F(z))) + \gamma(1 + d(F(G(z)), G(F(z))))
\]

\[
\lambda \geq \alpha, \beta, \gamma \geq 0 \quad \text{and} \quad \alpha + 2\beta + \gamma < 1
\]

\[
\text{where, } \alpha, \beta, \gamma \geq 0 \quad \text{and} \quad \alpha + 2\beta + \gamma < 1
\]

\[
\text{This complete the proof of the theorem.}
\]

\[
\text{Corollary 3.2. Let } \{f_i \}_{i=1}^m \text{ and } \{g_i \}_{i=1}^n \text{ be two families of pairwise commuting self mappings on a complete complex valued metric space } X. \text{ Define } F \text{ and } G \text{ to be a finite products of self mappings on } X \text{ such that } F(f_1 f_2 \cdots f_n) \text{ and } G = g_1 g_2 \cdots g_n. \text{ Then we have the next result.}
\]

\[
\text{Corollary 3.3. Let } f_1 \text{ and } f_2 \text{ be two self sequentially continuous mappings on a complete complex valued metric space } X \text{ such that } f_1(X) \subseteq f_2(X) \text{ and the order pair } \{f, g\} \text{ is weakly commuting, also the condition is satisfied}
\]

\[
d(F(z, G(z)), F(z)) + \delta(1 + d(F(z, G(z)), G(z)) + \gamma(1 + d(F(z, G(z)), G(z)))
\]

\[
\lambda \geq \alpha, \beta, \gamma \geq 0 \quad \text{and} \quad \alpha + 2\beta + \gamma < 1
\]

\[
\text{where, } \alpha, \beta, \gamma \geq 0 \quad \text{and} \quad \alpha + 2\beta + \gamma < 1
\]
\( \varnothing : [0; +\infty) \rightarrow [0; +\infty) \) is a Lesbesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of \([0; +\infty)\), non-negative, and such that for each \( \varnothing > 0 \), \( \int_0^\varnothing \varnothing(t) > 0 \). Then \( f_1 \) and \( f_2 \) have a unique common fixed point.

**Corollary 3.4.** Let \( \{f_i\}^m_1 \) and \( \{g_j\}^n_1 \) be two families of pairwise sequentially continuous commuting self mappings on the complex valued metric space \( X \). Define \( F \) and \( G \) to be a finite products of self mappings on \( X \) such that \( F = f_1 f_2 \cdots f_n \) and \( G = g_1 g_2 \cdots g_n \). If \( F(X) \subset G(X) \), the order pair \( \{F, G\} \) is \( 2\alpha - \) weakly commuting and the condition is satisfied

\[
\sum_{j=1}^n d(f_{j+1} f_{j+2} \cdots f_n w, f_{j+1} f_{j+2} \cdots f_n w) \leq \lambda \sum_{j=1}^n \varnothing(t) + \delta \int_0^\infty \frac{d(f_{j+1} f_{j+2} \cdots f_n w, f_{j+1} f_{j+2} \cdots f_n w)}{d(G^2 a z, f_{j+1} f_{j+2} \cdots f_n w)} \varnothing(t) + \gamma \int_0^\infty \frac{d(f_{j+1} f_{j+2} \cdots f_n w, f_{j+1} f_{j+2} \cdots f_n w)}{d(G^2 a z, f_{j+1} f_{j+2} \cdots f_n w)} \varnothing(t)
\]

where, \( \lambda, \delta, \gamma \geq 0 \), \( \lambda + 2\delta + \gamma < 1 \) and

\( \varnothing : [0; +\infty) \rightarrow [0; +\infty) \) is a Lesbesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of \([0; +\infty)\), non-negative, and such that for each \( \varnothing > 0 \), \( \int_0^\varnothing \varnothing(t) > 0 \). Then \( F \) and \( G \) have a unique common fixed point.

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### 5. References
