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Reduced Quadratic Irrational Numbers and Types of G-circuits with Length Four by Modular Group

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Abstract

Objectives: To classify the types of G-circuits with length four in G-orbits α^G where α is a reduced quadratic irrational number and *G* is the modular group. **Methods/Statistical Analysis:** G-orbits of real quadratic fields are evaluated using coset diagrams of modular group. **Findings:** There are five distinct types of the G-circuits in all. The number of disjoint G-orbits containing G-circuits of two types out of these five is four and for the remaining three types of G-circuits corresponding number of disjoint G-orbits is two. **Application/Improvements:** With the help of classification of G-circuits of length four we can find the structure of G-orbits of real quadratic fields.

Keywords: Coset Diagrams, Modular Group, Reduced Quadratic Irrational Numbers, Types of G-circuits of Length Four

1. Introduction and Preliminaries

In this section we give a brief outline of the definitions and known results.

The set of linear fractional transformations of the form $z \to \frac{a_1 z + b_1}{c_1 z + d_1}$, where a_1, b_1, c_1, d_1 are integers and $a_1 d_1 - b_1 c_1 = 1$, is called modular group and it has presentation $G = \langle x, y : x^2 = y^3 = 1 \rangle$. A set X with an action of the group G on it is known as a G-set.
The set

$$Q^*(\sqrt{n}) = \left\{ \frac{a+\sqrt{n}}{c} : a, c, b = \frac{a^2 - n}{c} \in Z \,\&\, (a,b,c) = 1 \right\}$$

is a proper G-subset of $Q(\sqrt{m}) \setminus Q = \bigcup_{k \in N} Q^*(\sqrt{k^2 m})$ where $n = k^2 m$. Throughout this paper $\alpha, \overline{\alpha}, -\alpha$ and $-\overline{\alpha}$ denote $\frac{a+\sqrt{n}}{c}$, $\frac{-a+\sqrt{n}}{-c}$, $\frac{a+\sqrt{n}}{-c}$ and $\frac{-a+\sqrt{n}}{c}$ respectively. If α and $\overline{\alpha}$ have different signs, then α is called ambiguous number. Whereas if α satisfies the conditions $\alpha > 1$ and $-1 < \overline{\alpha} < 0$, then it is called reduced quadratic irrational number. Using coset diagrams idea given it has been shown that there are only a finite number of ambiguous numbers in $Q^*(\sqrt{n})$ and that part of the coset diagram containing such numbers forms a single circuit and it is the only circuit in each orbit α^G with the second property[1]. Number of ambiguous numbers in $Q^*(\sqrt{n})$ have been determined in[2-3]. Properties of elements of $Q^*(\sqrt{n})$ have been discussed in[4]. Some G-subsets of $Q^*(\sqrt{n})$ have been explored[5]. The necessary convention to make meaningful our classification by a type of G-circuit of length 2m, is denoted by $(k_1, k_2, \ldots, k_{2m})$

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we mean the circuit containing one vertex, fixed by $(xy^2)^{k_1}(xy)^{k_2}(xy^2)^{k_3}(xy)^{k_4}......(xy^2)^{k_{2m-1}}(xy)^{k_{2m}}$. In other words, it is the circuit of G-orbit of reduced number α in which k_1 triangles have one vertex outside the circuit, k_2 triangles have one vertex inside the circuit, k_3 triangles have one vertex outside the circuit and so on k_{2m} triangles have one vertex inside the circuit[6]. Length of G-circuits disagrees with the notion of length in graph.

Two types $(t_1,t_2,t_3,t_4,.....,t_{2m})$ and $(u_1,u_2,u_3,u_4.....,u_{2m})$ are equivalent if and only if they represent the same G-circuit and we write it as $(t_1,t_2,t_3,t_4,.....,t_{2m}) \sim (u_1,u_2,u_3,u_4.....,u_{2m})$.

Thus $(k_1,k_2,k_3,k_4,.....,k_{2m-1},k_{2m})$, $(k_3,k_4,.....,k_{2m-1},k_{2m},k_1,k_2)$, $(k_5,k_6,.....,k_{2m-1},k_{2m},k_1,k_2,k_3,k_4)$ and so on $(k_{2m-1},k_{2m},k_1,k_2,k_3,k_4.....,k_{2m-3},k_{2m-2})$ are equivalent as they satisfy the condition to start from the number of triangles having one vertex outside the circuit and end at the number of triangles having one vertex inside the circuit. Throughout this paper G-circuit will be represented in counter-clockwise direction. For example the path from α_1 two α_2 in Figure1 corresponds to the word $(xy^2)^p(xy)^q$. We have discussed properties of reduced quadratic irrational numbers and classified G-circuits of length two in[7]. Throughout this paper p, q, r, s denote pairwise distinct positive integers and the expression

$$\frac{d+\sqrt{d^2+4ef}}{2f} \text{ is replaced by } \frac{\frac{d}{h}+\sqrt{\frac{d^2+4ef}{h^2}}}{\frac{2f}{h}} \text{ when}$$

$(d,(2e,2f))=h>1$. In this paper we classify types of G-circuits with length four in G-orbits of $Q(\sqrt{m})\setminus Q$ and we prove that there are exactly five types $(p,\ q,\ r,\ s)$, $(p,\ p,\ r,\ s)$, $(p,\ q,\ p,\ s)$, $(p,\ p,\ p,\ s)$ and $(p,\ p,\ r,\ r)$ of G-circuits of length four, in G-orbits α^G where $\alpha \in Q(\sqrt{m})\setminus Q$. For a given sequence of positive integers $n_1,n_2,n_3,.....,n_{2k}$ the circuit of the type $(n_1,n_2,n_3,.....,n_{2l},n_1,n_2,n_3,.....,n_{2l},.....n_1,n_2,n_3,.....,n_{2l})$ where l divides k, is said to have a period of length $2l$. It can be deduced from the result in[6] that for a given sequence of positive integers, there exist a circuit which has period of length $2l$, where l divides k. Hence the circuits of the types (p,p,p,p) and (p,q,p,q) cannot exist. We also determine reduced quadratic irrational number α in terms of $p,q,r\ s$. The notations used in this paper are standard and we follow[7-8]. The following theorem is very crucial.

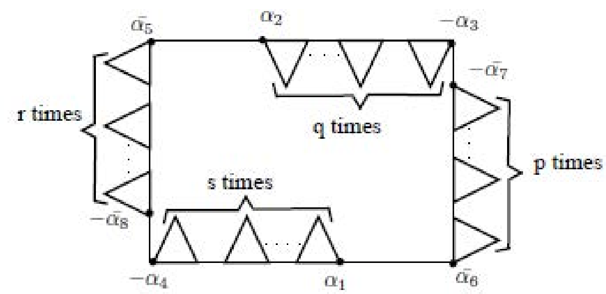

Fig-1: G-circuit of $(\alpha_1)^G=(\alpha_2)^G$

Figure 1. G-circuit of $(\alpha_1)^G=(\alpha_2)^G$.

Theorem 1.1 (Aslam and Sajjad): An ambiguous number $\frac{a+\sqrt{n}}{c}$ where $c>0$ is reduced number if and only if, $|b+c|<2a$.

2. Types of G-Circuits of Length 4

We commence this section with an important theorem of classification of G-circuits with length four that motivated us for the main purpose of this paper.

In it has been proved that there does not exist G-orbits with G-circuits of length two with the property that G-orbits α^G, $(-\alpha)^G$, $(\bar{\alpha})^G$ and $(-\bar{\alpha})^G$ are distinct whereas in this section we prove that all the four G-orbits α^G, $(-\alpha)^G$, $(\bar{\alpha})^G$ and $(-\bar{\alpha})^G$ involving G-circuits of the types $(p,\ q,\ r,\ s)$ and $(p,\ p,\ r,\ s)$ are mutually distinct.

In the following theorem we classify G-circuits of length four.

Theorem 2.1: Let p,q,r,s be mutually distinct positive integers. Then there are exactly five types (p,q,r,s), (p,p,r,s), (p,q,p,s), (p,p,p,s) and (p,p,r,r)

of G-circuits of length four, in G-orbits α^G where $\alpha \in Q(\sqrt{m}) \setminus Q$.

Proof: It is easy to see that quadruple (p,q,r,s) of pair wise distinct positive integers p,q,r,s represent one of the five types of G-circuits of length four. Whereas if first two coordinates of quadruple coincide then (p,p,r,s) is another type. However if first and third coordinates are same while 2nd and fourth are different we get third type namely (p,q,p,s). Similarly if first three coordinates of the quadruple coincide then we get fourth type namely (p,p,p,s). Lastly if first two are same and last two are same we get another type namely (p,p,r,r). As given in, all the four coordinates cannot coincide. Furthermore by the same reference, the type (p,q,p,q) does not exist. This shows that there are exactly five types of G-circuits of length four depending upon pair wise distinct four positive integers p,q,r,s.

Remark 2.2: Obviously $(p,q,r,s) \sim (r,s,p,q)$, $(q,r,s,p) \sim (s,p,r,q)$, $(s,r,q,p) \sim (q,p,s,r)$ and $(p,s,r,q) \sim (r,q,p,s)$.

In the theorem given below we derive a formula to find out the reduced numbers of a G-circuit of length four.

Theorem 2.3: If the G-circuit of an ambiguous number α in G-orbit α^G is of the type (p,q,r,s) then

$$\alpha = \frac{(pq+ps-qr+rs+pqrs) \pm \sqrt{(pq+ps-qr+rs+pqrs)^2 + 4(p+r+pqr)(q+s+qrs)}}{2(p+r+pqr)}.$$

Proof: Let $(\alpha)(xy^2)^p(xy)^q(xy^2)^r(xy)^s = \alpha.$

This implies that $\left(\frac{(1+pq)\alpha+q}{p\alpha+1}\right)(xy^2)^r(xy)^s = \alpha.$

That is
$$\frac{\frac{(1+pq)\alpha+q}{p\alpha+1}}{r\left(\frac{(1+pq)\alpha+q}{p\alpha+1}\right)+1} + s = \alpha.$$

This gives $(p+r+pqr)\alpha^2 - (pq+ps-qr+rs+pqrs)\alpha - (q+s+qrs) = 0.$

Hence $\alpha = \frac{(pq+ps-qr+rs+pqrs) \pm \sqrt{(pq+ps-qr+rs+pqrs)^2 + 4(p+r+pqr)(q+s+qrs)}}{2(p+r+pqr)}$

In the lemma given below we prove that one of these numbers is reduced while the other is not reduced.

Lemma 2.4: $\frac{(pq+ps-qr+rs+pqrs) + \sqrt{(pq+ps-qr+rs+pqrs)^2 + 4(p+r+pqr)(q+s+qrs)}}{2(p+r+pqr)}$ $= \frac{a_1+\sqrt{n}}{c_1} = \alpha_1$ is reduced number while its conjugate $\overline{\alpha_1}$ is not reduced.

Proof: Let $\alpha_1 = \frac{a_1+\sqrt{n}}{c_1}$ be as given in the statement.

Then $a_1 = pq+ps-qr+rs+pqrs$, $b_1 = -2(q+s+qrs)$ and $c_1 = 2(p+r+pqr) > 0$.

Now $(b_1+c_1) = 2(p+r+pqr-q-s-qrs)$ and $2a_1 = 2(pq+ps-qr+rs+pqrs)$. We prove that $|b_1+c_1| < 2a_1$ that is $(b_1+c_1) < 2a_1$ and $-(b_1+c_1) < 2a_1$.

To prove that $(b_1+c_1) < 2a_1$ that is

$2(-q-s-qrs+p+r+pqr) < 2(pq+ps-qr+rs+pqrs)$

it is sufficient to prove that

$(pq+ps-qr+rs+pqrs-p-r-pqr+q+s+qrs) > 0$.

But

$$\begin{aligned} &pq+ps-qr+rs+pqrs-p-r-pqr+q+s+qrs \\ &= p(s-1)+r(s-1)+qr(s-1)+pqr(s-1)+q+s+pq \\ &= (s-1)(p+r+qr+pqr)+q+s+pq > 0 \end{aligned}$$

Because p,q,r,s are positive integers and $s-1 \ge 0$. Similarly it can be proved $-(b_1+c_1) < 2a_1$ and hence $|b_1+c_1| < 2a_1$. By Theorem 1.1 it follows that α_1 is reduced number. Obviously

$$\overline{\alpha_1} = \frac{-(pq+ps-qr+rs+pqrs) + \sqrt{(pq+ps-qr+rs+pqrs)^2 + 4(p+r+pqr)(q+s+qrs)}}{-2(p+r+pqr)}$$

does not satisfy the condition $c_1 > 0$. So $\overline{\alpha_1}$ is not reduced as $-2(p+r+pqr) < 0$.

Theorem 2.5: If β is an ambiguous number such that $(\beta)(yx)^p(y^2x)^q(yx)^r(y^2x)^s = \beta$ then

$$\beta=\frac{-(pq+ps-qr+rs+pqrs)\pm\sqrt{(pq+ps-qr+rs+pqrs)^2+4(p+r+pqr)(q+s+qrs)}}{2(p+r+pqr)}.$$

Proof: Let $(\beta)(yx)^p(y^2x)^q(yx)^r(y^2x)^s=\beta.$

That is $\left(\dfrac{(1+pq)\beta-q}{1-p\beta}\right)(yx)^r(y^2x)^s=\beta.$

This implies $\left(\dfrac{\dfrac{(1+pq)\beta-q}{1-p\beta}}{1-r\left(\dfrac{(1+pq)\beta-q}{1-p\beta}\right)}-s\right)=\beta.$

This gives $(p+r+pqr)\beta^2+(pq+ps-qr+rs+pqrs)\beta-(q+s+qrs)=0.$

Thus β is obtained as required. Obviously

$$\frac{-(pq+ps-qr+rs+pqrs)+\sqrt{(pq+ps-qr+rs+pqrs)^2+4(p+r+pqr)(q+s+qrs)}}{2(p+r+pqr)}=-\overline{\alpha_1}$$

And hence its conjugate is equal to

In the following theorem we prove that α^G, $(-\alpha)^G$ $(\overline{\alpha})^G$ and $(-\overline{\alpha})^G$ are all mutually disjoint.

Theorem 2.6: Let (p,q,r,s) is type of G-circuit contained in α^G. Then (q,r,s,p), (s,r,q,p) and (p,s,r,q) are the types of G-circuits contained in $(-\alpha)^G$, $(\overline{\alpha})^G$ and $(-\overline{\alpha})^G$ respectively.

Proof: Let p,q,r,s be mutually distinct positive integers. Then by Theorems 2.3 and 2.5, G-circuit shown in Figure1 is of the type (p,q,r,s) having reduced numbers α_1 and

$$\alpha_2=\frac{(pq-ps+qr+rs+pqrs)+\sqrt{(pq-ps+qr+rs+pqrs)^2+4(q+s+pqs)(p+r+prs)}}{2(q+s+pqs)}.$$

Also G-circuit shown in Figure 2 is of the type (q,r,s,p) and contains two reduced numbers

$$\alpha_3=\frac{(pq+ps+qr-rs+pqrs)+\sqrt{(pq+ps+qr-rs+pqrs)^2+4(q+s+qrs)(p+r+prs)}}{2(q+s+qrs)}.$$

And $\alpha_4=\dfrac{(-pq+ps+qr+rs+pqrs)+\sqrt{(-pq+ps+qr+rs+pqrs)^2+4(q+s+pqs)(p+r+pqr)}}{2(q+s+pqs)}.$

Also G-circuit shown in Figure 3 is of the type (s,r,q,p) and contains two reduced numbers

$$\alpha_5=\frac{(pq-ps+qr+rs+pqrs)+\sqrt{(pq-ps+qr+rs+pqrs)^2+4(q+s+pqs)(p+r+prs)}}{2(q+s+pqs)}.$$

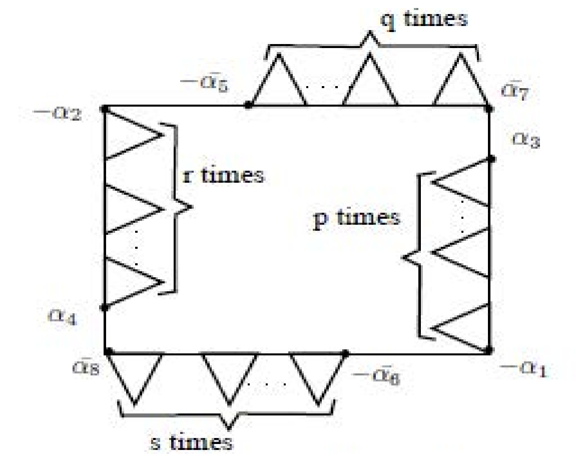

Fig-2: G-circuit of $(-\alpha_1)^G=(\alpha_3)^G=(\alpha_4)^G$

Figure 2. G-circuit of $(-\alpha_1)^G=(\alpha_3)^G=(\alpha_4)^G$.

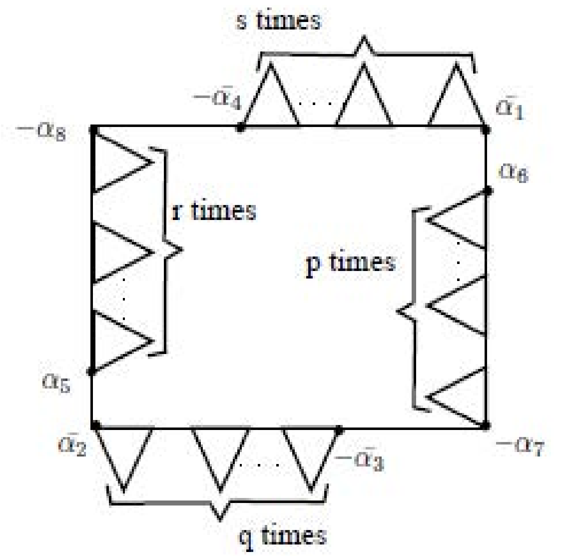

Fig-3: G-circuit of $(\bar{\alpha}_1)^G=(\alpha_5)^G=(\alpha_6)^G$

Figure 3. G-circuit of $(\overline{\alpha}_1)^G=(\alpha_5)^G=(\alpha_6)^G$.

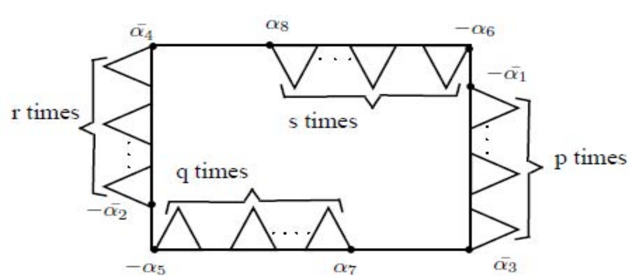

Fig-4: G-circuit of $(-\bar{\alpha}_1)^G=(\alpha_7)^G=(\alpha_8)^G$

Figure 4. G-circuit of $(-\overline{\alpha}_1)^G=(\alpha_7)^G=(\alpha_8)^G$.

And $\alpha_6=\dfrac{(pq+ps-qr+rs+pqrs)+\sqrt{(pq+ps-qr+rs+pqrs)^2+4(q+s+qrs)(p+r+pqr)}}{2(q+s+qrs)}.$

Lastly G-circuit shown in Figure 4 is of the type (s,r,q,p) and contains two reduced numbers

$$\alpha_7=\frac{(pq+ps+qr-rs+pqrs)+\sqrt{(pq+ps+qr-rs+pqrs)^2+4(p+r+prs)(q+s+qrs)}}{2(p+r+prs)}$$

and

$$\alpha_8=\frac{(-pq+ps+qr+rs+pqrs)+\sqrt{(-pq+ps+qr+rs+pqrs)^2+4(p+r+pqr)(q+s+pqs)}}{2(p+r+pqr)}.$$

This completes the proof.

Theorem 2.7: In each α_i, $i=1,2,.....,8$, the expressions under radical signs are equal.

Proof: Expression under radical sign of α_1 is:

$$\begin{aligned}&(pq+ps-qr+rs+pqrs)^2+4(p+r+pqr)(q+s+qrs)=(pq)^2+(ps)^2+(-qr)^2+(rs)^2+(pqrs)^2\\&+2p^2qs-2pq^2r+2pqrs+2p^2q^2rs-2pqrs+2prs^2+2p^2qrs^2-2qr^2s-2pq^2r^2s+2pqr^2s^2+\\&4(pq+ps+qr+rs)+4(pq^2r+qr^2s+pq^2r^2s)\\&=(pq)^2+(ps)^2+(qr)^2+(rs)^2+(pqrs)^2+2p^2qs+2pq^2r+2qr^2s+2pq^2r^2s+4(pq+ps+qr+rs)\\&+8pqrs\end{aligned}$$

And corresponding expression of α_2 is:

$$\begin{aligned}&(pq-ps+qr+rs+pqrs)^2+4(p+r+prs)(q+s+pqs)=(pq)^2+(-ps)^2+(qr)^2+(rs)^2+(pqrs)^2\\&-2p^2qs+2pq^2r+2pqrs+2p^2q^2rs-2pqrs-2prs^2-2p^2qrs^2+2qr^2s+2pq^2r^2s+2pqr^2s^2+\\&4(pq+ps+qr+rs)+4(p^2qs+prs^2+p^2qrs^2)\\&=(pq)^2+(ps)^2+(qr)^2+(rs)^2+(pqrs)^2+2p^2qs+2pq^2r+2qr^2s+2pq^2r^2s+4(pq+ps+qr+rs)\\&+8pqrs\end{aligned}$$

Similarly we can prove that expressions under radical signs of other α_i's are equal to that of α_1.

All the results corresponding to the type (p,p,r,s) can be obtained from the corresponding results related to type (p,q,r,s) just by replacing q by *p*.

Following Corollaries 2.8 and 2.10 are consequences of Theorems 2.3 and 2.5.

Corollary 2.8

(i) If the G-circuit of an ambiguous number α in the orbit α^G

Is of the type (p,q,p,s) then

$$\alpha=\frac{(2ps+p^2qs)+\sqrt{(2ps+p^2qs)^2+4(2p+p^2q)(q+s+pqs)}}{2(2p+p^2q)}$$

(ii) If β is an ambiguous number such that $(\beta)(yx)^p(y^2x)^q(yx)^p(y^2x)^s=\beta$ then

$$\beta=\frac{-(2ps+p^2qs)\pm\sqrt{(2ps+p^2qs)^2+4(2p+p^2q)(q+s+pqs)}}{2(2p+p^2q)}.$$

Proof: Replace r by p in Theorem 2.3 and 2.5 to get the required result.

Obviously if

$$\frac{(2ps+p^2qs)+\sqrt{(2ps+p^2qs)^2+4(2p+p^2q)(q+s+pqs)}}{2(2p+p^2q)}=\alpha_1.$$

Then

$$\frac{(2ps+p^2qs)-\sqrt{(2ps+p^2qs)^2+4(2p+p^2q)(q+s+pqs)}}{2(2p+p^2q)}=\overline{\alpha_1},$$

$$\frac{-(2ps+p^2qs)+\sqrt{(2ps+p^2qs)^2+4(2p+p^2q)(q+s+pqs)}}{2(2p+p^2q)}=-\overline{\alpha_1}$$

And

$$\frac{-(2ps+p^2qs)-\sqrt{(2ps+p^2qs)^2+4(2p+p^2q)(q+s+pqs)}}{2(2p+p^2q)}=-\alpha_1.$$

Theorem 2.9: Let (p,q,p,s) is type of G-circuit contained in $\alpha^G=(-\overline{\alpha})^G$. Then (q,p,s,p) is the type of G-circuits contained in $(-\alpha)^G=(\overline{\alpha})^G$.

Proof: Let p,q,s be distinct positive integers. Then by Theorems 2.3 and 2.5 and Corollary 2.8 G-circuit shown in Figure 5 is of the type (p,q,p,s) having reduced numbers α_1 and

$$\alpha_2=\frac{(2pq+p^2qs)+\sqrt{(2pq+p^2qs)^2+4(2p+p^2s)(q+s+pqs)}}{2(2p+p^2s)}$$

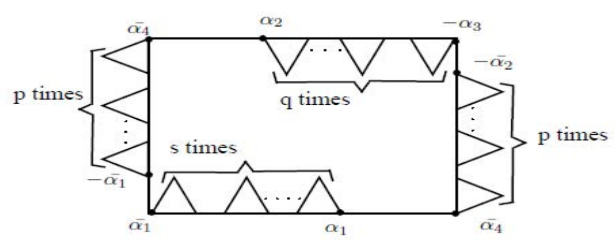

Fig-5: G-circuit of $(\alpha_1)^G=(-\bar{\alpha}_1)^G=(\alpha_2)^G=(-\bar{\alpha}_2)^G$

Figure 5. G-circuit of $(\alpha_1)^G=(-\overline{\alpha}_1)^G=(\alpha_2)^G=(-\overline{\alpha}_2)^G$.

And G-circuit shown in Figure 6 is of the type (q,p,s,p) and contains two reduced numbers

$$\alpha_3=\frac{(2pq+p^2qs)+\sqrt{(2pq+p^2qs)^2+4(q+s+pqs)(2p+p^2s)}}{2(q+s+pqs)}.$$

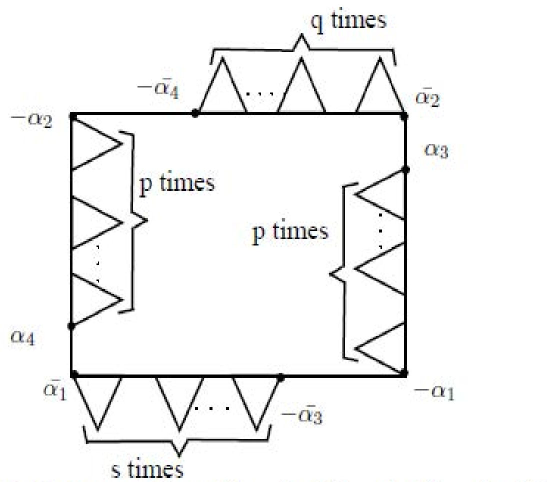

Fig-6: G-circuit of $(-\alpha_1)^G = (\bar{\alpha}_1)^G = (\alpha_3)^G = (\alpha_4)^G$

Figure 6. G-circuit of $(-\alpha_1)^G = (\overline{\alpha}_1)^G = (\alpha_3)^G = (\alpha_4)^G$.

And

$$\alpha_4 = \frac{(2ps+p^2qs)+\sqrt{(2ps+p^2qs)^2+4(q+s+pqs)(2p+p^2q)}}{2(q+s+pqs)}.$$

This completes the proof.

The results corresponding to the type (p,p,p,s) can be obtained from the corresponding results related to type (p,q,p,s) just by replacing q by p in Corollary 2.8.

Corollary 2.10:

(i) if the G-circuit of an ambiguous number α in the orbit α^G is of the type (p,p,r,r) then

$$\alpha = \frac{(p^2+r^2+p^2r^2)+\sqrt{(p^2+r^2+p^2r^2)^2+4(p+r+p^2r)(p+r+pr^2)}}{2(p+r+p^2r)}.$$

(ii) If β is an ambiguous number such that $(\beta)(yx)^p(y^2x)^p(yx)^r(y^2x)^r=\beta$ then

$$\beta = \frac{-(p^2+r^2+p^2r^2)+\sqrt{(p^2+r^2+p^2r^2)^2+4(p+r+p^2r)(p+r+pr^2)}}{2(p+r+p^2r)}.$$

Proof: Replace q by p and s by r in Theorem 2.3 and 2.5 to get the required result.

Obviously if

$$\frac{(p^2+r^2+p^2r^2)+\sqrt{(p^2+r^2+p^2r^2)^2+4(p+r+p^2r)(p+r+pr^2)}}{2(p+r+p^2r)}=\alpha_1$$

then

$$\frac{(p^2+r^2+p^2r^2)-\sqrt{(p^2+r^2+p^2r^2)^2+4(p+r+p^2r)(p+r+pr^2)}}{2(p+r+p^2r)}=\overline{\alpha}_1,$$

$$\frac{-(p^2+r^2+p^2r^2)+\sqrt{(p^2+r^2+p^2r^2)^2+4(p+r+p^2r)(p+r+pr^2)}}{2(p+r+p^2r)}=-\overline{\alpha}_1$$

and

$$\frac{-(p^2+r^2+p^2r^2)-\sqrt{(p^2+r^2+p^2r^2)^2+4(p+r+p^2r)(p+r+pr^2)}}{2(p+r+p^2r)}=-\alpha_1.$$

Theorem 2.11: Let (p,p,r,r) be type of G-circuit contained in $\alpha^G=(\overline{\alpha})^G$. Then (p,r,r,p) is the type of G-circuits contained in $(-\alpha)^G=(-\overline{\alpha})^G$.

Proof: Let p,r be distinct positive integers. Then by Theorems 2.3 and 2.5 and Corollary 2.10 G-circuit shown in Figure 7 is of the type (p,p,r,r) having reduced numbers α_1 and

$$\alpha_2 = \frac{(p^2+r^2+p^2r^2)+\sqrt{(p^2+r^2+p^2r^2)^2+4(p+r+prs)(p+r+p^2r)}}{2(p+r+prs)}.$$

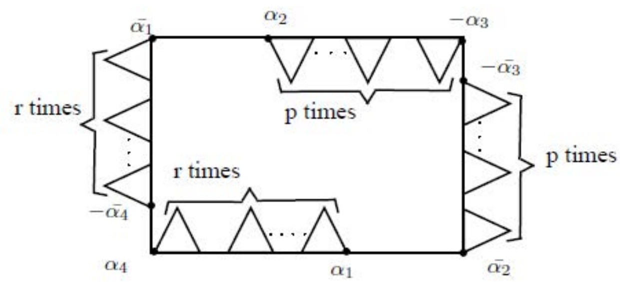

Fig-7: G-circuit of $(\alpha_1)^G = (\bar{\alpha}_1)^G = (\alpha_2)^G = (\bar{\alpha}_2)^G$

Figure 7. G-circuit of $(\alpha_1)^G = (\overline{\alpha}_1)^G = (\alpha_2)^G = (\overline{\alpha}_2)^G$.

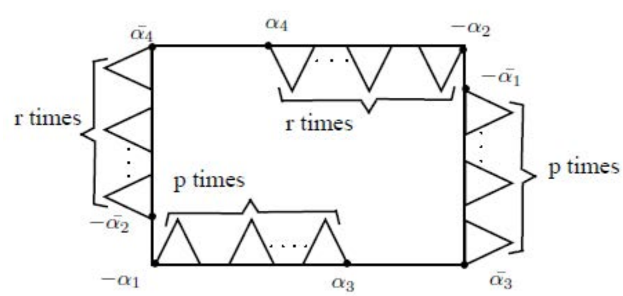

Fig-8: G-circuit of $(-\alpha_1)^G = (-\bar{\alpha}_1)^G = (\alpha_3)^G = (\alpha_4)^G$

Figure 8. G-circuit of $(-\alpha_1)^G = (-\overline{\alpha}_1)^G = (\alpha_3)^G = (\alpha_4)^G$.

Similarly G-circuit shown in Figure 8 is of the type (q,p,s,p) and contains two reduced numbers

$$\alpha_3 = \frac{(p^2+2pr-r^2+p^2r^2)+\sqrt{(p^2+2pr-r^2+p^2r^2)^2+4(p+r+pr^2)^2}}{2(p+r+pr^2)},$$

$$\alpha_4 = \frac{(-p^2+2pr+r^2+p^2r^2)+\sqrt{(-p^2+2pr+r^2+p^2r^2)^2+4(p+r+p^2r)^2}}{2(p+r+p^2r)}.$$

3. Conclusion

The idea of types of G-circuits in G-orbits of real quadratic field by modular group, which is given in this paper, is new and original. We have proved that for mutually distinct positive integers p,q,r,s there are exactly five types (p,q,r,s), (p,p,r,s), (p,q,p,s), (p,p,p,s) and (p,p,r,r) of G-circuits of length four, in G-orbits α^G where $\alpha \in Q(\sqrt{m})\setminus Q$. There are four disjoint G-orbits α^G, $(-\alpha)^G$, $(\overline{\alpha})^G$ and $(-\overline{\alpha})^G$ containing G-circuits of the types (p,q,r,s), (p,p,r,s). Similarly there are two disjoint G-orbits $\alpha^G=(-\overline{\alpha})^G$ and $(-\alpha)^G=(\overline{\alpha})^G$ containing G-circuits of the types (p,q,p,s), (p,p,p,s). At the end we find that there are two disjoint G-orbits $\alpha^G=(\overline{\alpha})^G$ and $(-\alpha)^G=(-\overline{\alpha})^G$ containing G-circuits of the type (p,p,r,r).

4. References

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