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Some New Graceful Lobsters with Pendant Vertices with Central Paths

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Abstract

The objective of this article is to give graceful labeling to some new classes of lobsters in a bid to resolve the three and half decade old Bermond's conjecture that all lobsters are graceful. Here we use the method of component moving transformation such as the transfer of the first and second type and the derived transformations such as BD8TF, 1JTF, and 2JTF for generating graceful trees from a given one. In a bid to resolve Bermond's conjecture here we give graceful labelings to many classes of lobsters possessing at least one of the two distinct features from those found in the literature as detailed below. 1.The central paths of the lobsters contain one or more vertices which do not have any neighbour apart from those on the central path. 2.One or more vertices of the central paths are attached to leaves. 3. The vertices on the central path may be attached to any of the fifteen different combination of odd, even, and pendant branches.

Keywords: AMS classification: 05C78, BD8TF, Graceful Labeling, Lobster, Odd and Even Branches, Transfers of the First and Second Type, 1JTF, 2JTF

1. Introduction

Let G be any graph with q edges. Let $f : V\ (T) \rightarrow \{0,1,2,.....q\ \}$ be an injection. For any edge (x, y) define $g(x,y) = |f(x) - f(y)|$. If g is injection, then we call G a *graceful graph* and f a graceful labeling of G. A *lobster* L is a graph which possesses a unique path such that the distance of each vertex of L from H is at most two. One sees that a lobster L with diameter at least five possesses a unique path $H = y_{0,}\ y_1,.....\ y_n$ such that each vertex y_j, $1 \leq j \leq n - 1$, is adjacent to two vertices in H and the centres of one or more stars $K_{1,r}$. The vertices y_0 and y_m are adjacent to exactly one vertex of H and the centres of at least one star $K_{1,r}$ with $r \geq 1$. The path H is termed as the *central path* of the lobster L. The star $K_{1,r}$ is said to be an *even branch* if r even and, an *odd branch* if r is odd, and a pendant branch if $r = 0$. A vertex y_i on the central path is attached to centres of stars which are represented by a triple (a,b,g), where a,, b and g are the number of odd, even, and pendant branches respectively, adjacent to y_i. Here the symbol o represents an odd number, e represents a non-zero even number. For example, when we say that an y_j is attached to the combination $(e,$ o, $0)$, we mean that y_j is adjacent to an even number of odd branches, odd number of even branch, and no pendant branches.

In 1964 the famous graceful tree conjecture[1] was published which states that "all trees are graceful". Another conjecture due to Bermond got published as a special case of the graceful tree conjecture[2] according to which "all lobsters are graceful". Bermond's conjecture is also unresolved so far and only some specific classes of lobsters are known to be graceful. One may refer to[2–4, 6–17] for the latest updates on the advancement made in resolving Bermond's Conjecture. In this paper we apply the transformation techniques for constructing graceful trees discussed in[5,6] and find some new generalized class of lobsters. The graceful lobsters of this article contains the

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central path $H = x_1, x_2, \ldots, x_m$ in which the vertex x_0 is attached to an even number of odd branches and an odd number of pendant branches and the path $H \backslash \{x_0\}$ partitions into three parts, i.e. $P = P_1 \cup P_2 \cup P_3$ such that a vertex of P_1 is attached to a combination of the form $(x,0,z)$; $x \geq 0$, $z \geq 0$, a vertex of P_2 is attached to a combination of the form $(x,y,0)$; $x \geq 0$, $y \geq 0$ (respectively, $(0,y,z)$, $y \geq 0$, $z \geq 0$), and a vertex of P_3 is attached to a combination of the form $(x,0,0)$, $x \geq 0$ or $(0,y,0)$, $y \geq 0$.

2. Preliminary Results

Here we state some existing terminologies results borrowed from[4,5,9,10] to prove our main result.

2.1 Definition

For an edge $e = \{x; y\}$ of a tree T, we call $x(T)$ as that component of $T - e$ in which x lies.. Here $x(T)$ is said to be a component incident on the vertex y. If a and b are vertices of a tree T, $x(T)$ is a component incident on a, and b does not lie in $x(T)$ then replacing the edge $\{a; x\}$ in T with an edge formed by joining b and x is referred as *the component* $x(T)$ *has been transferred or moved from a to b*. Here by the label of the component "$x(T)$" means the label of the vertex x. Let a and b be two vertices of a tree T. By *$a \rightarrow b$ transfer* we mean that some components from a are transferred to b. We write $a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow a_4$ .....® This as to mean the consecutive transfers $a_1 \rightarrow a_2$, $a_2 \rightarrow a_3$, $a_3 \rightarrow a_4$,...... .

Each vertex a_i, $i = 1,2........ n - 1$ is called a vertex of the transfer $a_1 \rightarrow a_2 \ldots\ldots \rightarrow a_{n-1} \rightarrow a_n$.

2.2 Lemma

[5]Let T be a graceful tree with a graceful labelling f. Let a and b be two vertices of T; let $x(T)$ and $y(T)$ be two components incident on a and b does not belong to $u(T) \cup v(T)$. Then the following hold:

(*i*) If $f(x) + f(y) = f(a) + f(b)$ then the tree $T^{(1)}$ formed from T by transferring the components $x(T)$ and $y(T)$ from a to b also possesses a graceful labeling.

(*ii*) If $2 f(x) = f(a) + f(b)$ then the tree $T^{(2)}$ formed from T by transferring the component $x(T)$ from a to b also possesses a graceful labeling.

2.3 Definition

Let T be a tree with a vertex labeling f. We consider the vertices of T whose labels form the sequence $(s, t, s - 1, t + 1, s - 2, t + 2)$ (respectively, $(s, t, s + 1, t - 1, s + 2, t - 2)$. Let s be adjacent to some vertices having labels distinct from the labels of the above sequence. A $s \rightarrow t$ transfer is called *a* ***transfer of the first type*** if the labels of the vertices (or components) which are moved to t are consecutive integers. A $s \rightarrow t$ transfer is called *a* ***transfer of the second type*** if the labels of the vertices which are moved to t decompose into two disjoint sets of consecutive integers A transfer consisting of eight successive transfers $s \rightarrow t \rightarrow s - 1 \rightarrow t + 1 \rightarrow s \rightarrow t \rightarrow s - 1 \rightarrow t + 1 \rightarrow s - 2$ (respectively, $s \rightarrow t \rightarrow s+1 \rightarrow t-1 \rightarrow s \rightarrow t \rightarrow s+1 \rightarrow t-1 \rightarrow s+2$), with each transfer is a transfer of the first type is termed a ***backward double 8 transfer of the first type*** or ***BD8TF*** s to $s - 2$ (respectively, s to $s+2$). A transfer consisting of four consecutive transfers $s \rightarrow t+1 \rightarrow s-1 \rightarrow t+1 \rightarrow s-2$ (respectively, $s \rightarrow t - 1 \rightarrow s + 1 \rightarrow t - 1 \rightarrow s + 2$), with each transfer is a transfer of the first type is called *a* ***1 - jump transfer of the first type*** or in brief ***1JTF*** s to $s - 2$ (respectively, s to $s + 2$). A transfer consisting of two successive transfers of the first type $s \rightarrow t + 1 \rightarrow s - 2$ (respectively, $s \rightarrow t - 1 \rightarrow s + 2$), is called *a* ***2 - jump transfer of the first type*** or in brief ***2JTF*** s to $s - 2$ (respectively, s to $s + 2$). Figure 1 illustrates these transfers.

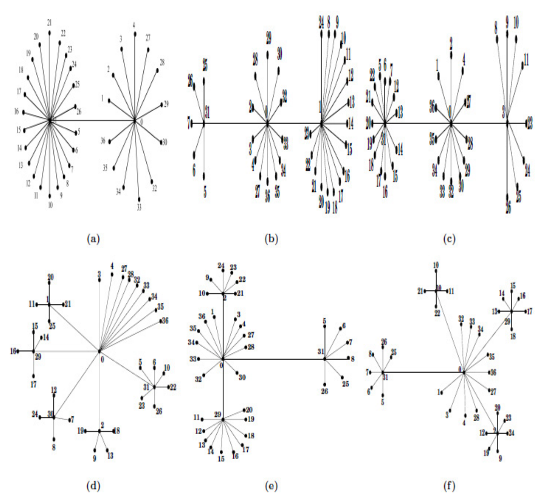

Figure 1. The graceful trees in (b), (c), d), (e), and (f) are obtained from the graceful tree in (a) by applying transfers of the first type 31 → 1, the transfer of second type 31 → 3, BD8TF 31 to 29, 2JTF 31 to 29, and 1JTF 31 to 29, respectively.

2.4 Lemma

[6,10,11]Let f be a graceful labelling of a graceful tree T, let s and t be the labels of two vertices of T. Let s be attached to

a set A of vertices (or components) possessing the labels $r, r + 1, r + 2, ... r + u$ (different from the above vertex labels), which satisfy $(r + 1+i) + (r + u-i) = s + t, i \geq 0$ (respectively, $(r + i)+(r+u-1-i) = s + t,\ i \geq 0$). Then the following hold.

(a) On executing successive transfers of the first type $s \to t \to s-1 \to t+1 \to s-2 \to t+2 \to ....$ (respectively, $s \to t \to s+1 \to t-1 \to s+2 \to t-2 \to ....$), an odd number of elements from A remain at each vertex of the transfer, and we obtain a new graceful tree.

(b) If A contains an even number of elements, then on executing successive transfers of the second type $s \to t \to s-1 \to t+1 \to s-2 \to t+2 \to .....$(respectively, $s \to t \to s+1 \to t-1 \to s+2 \to t-2 \to ....$), an even number of elements from A remain at each vertex of the transfer, and we obtain a new graceful tree.

(c) By a ***BD8TF*** s to $t + 1$ (respectively, $t - 1$), an even number of elements from A remain at $s, t, s - 1$, and $t + 1$ (respectively, $s, t, s+1$, and $t-1$), and the rest get transferred to $s-2$ (respectively, $s+2$). By a ***1JTF*** s to $s+1$ (respectively, $s-1$), we an even number of elements from A remain at $s, s - 1$, and $t + 1$ (respectively, $s, s + 1$, and $t - 1$) and no component at t, and the rest get transferred to $s-2$ (respectively, $s+2$). By a ***2JTF*** s to $t+1$ (respectively, $t-1$), an even number of components remain at s and $t+1$ (respectively, $t-1$) and no component at t and $s-1$ (respectively, $t+1$), and the rest get transferred to $s-2$ (respectively, $s+2$). The resultant tree formed in each of the above cases is graceful.

(d) Consider the transfer $R : s \to t \to s-1 \to t+1 \to .... \to ....$ (respectively, $s \to t \to s+1 \to t-1 \to ..... \to .....$), such that R partitions as $R : T'_1 \to T'_2$, where T'_1 is sequence of transfers in which the transfers are either the transfers of the first type or the derived transfers and T'_2 is a successive transfers of the second type. The tree T^{**} obtained from T on execution of the transfer R possesses a graceful labeling.

3. Results

Here for a given lobster L with q edges and the central path $H = x_0x_1x_2.......x_m$, we first construct a graceful tree (caterpillar) $G(L)$ as shown in Figures 2 and Figure 3. Here we first make the transfer $x_0 \to x_1 \to .... \to x_m \to x_{m+1}$, followed by the transfer $x_{m+1} \to x_m \to\to x_{m-1} \to ...... \to x2 \to x1 \to x0 \to\to q - \frac{m}{2}$ (or $\frac{m-1}{2}$) to move the labels incident on x_0 in $G(L)$. Finally, we carry out the transfer $q- m/2\to \frac{m}{2} \to q\text{-}\frac{m}{2} -1 \to \frac{m}{2}+1 \to ...... \to$ (respectively, $\frac{m-1}{2} \to q-\frac{m-1}{2} -1 \to \frac{m-1}{2} +1 \to q-\frac{m-1}{2} -2 \to ....\to$) and move the labels incident on $q - \frac{m}{2}$ (respectively, $\frac{m-1}{2}$), so as to get back L together with a graceful labeling. We use the ideas involved in Lemmas 2.2 and 2.4 to give graceful labeling to the lobsters of this paper starting from a caterpillar tree of the type in Figures 2 and Figure 3.

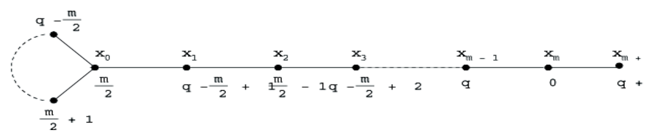

Figure 2. The tree G(L) corresponding to the lobster L for the case m is even.

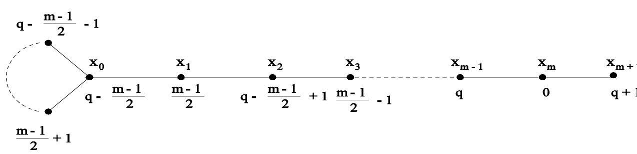

Figure 3. The tree G(L) corresponding to the lobster L for the case m is odd.

3.1 Construction

We construct a class of lobsters with the central path $H = x_0x_1x_2.......x_m$ in which the vertex x_0 is attached to an even number of odd branches and an odd number of pendant branches and the path $H\backslash\{x_0\}$ partitions into finite number of paths $P_i = x_{t_{i-1}} + 1,\ x_{t_{i-1}} +2, ...\ x_{t_{i-1}}$, $1 \leq i \leq n;\ t_0 = 0;\ t_n = m$, such that the combinations of branches incident on the vertices of Pi may have one of the following features. Let n_1 and n_2 be integers, such that $1 \leq n_1 \leq n_2 \leq n$.

1. For $1 \leq i \leq n_1$ we have either (i) or (ii) as given below.

(i) $t_i - t_{i-1} = 4$ and the four vertices of P_i may be attached to any one among the block of combinations -

$B_1 : (e, 0, e), (e, 0, e), (e, 0, e)$, and $(e, 0, e)$; $B_2 : (e, 0, e), (0, 0, 0), (0, 0, 0)$, and $(e, 0, e)$, $B_3 : (e, 0, e), (0, 0, e), (e, 0, 0)$, and $(e, 0, e)$, $B_4 : (e, 0, e), (e, 0, e), (e, 0, 0)$, and $(e, 0, e)$, $B_5 : (e, 0, e), (0, 0, e), (e, 0, e)$, and $(e, 0, e)$, $B_6 : (e, 0, e), (e, 0, 0)$, (e, 0, 0) and (e, 0, e), B_7 : (e, 0, e), (0, 0, e), (0, 0, e), and (e, 0, e); B_8 : (e, 0, e) , (0, 0, 0), (e, 0, 0), and (e, 0, e); B_9: (e, 0, e), (0, 0, e), (0, 0, 0), and (e, 0,e), B_{10} : (o, 0, e), (o, 0, e), (o, 0, e), and (o, 0, e); B_{11} : (e, 0, o), (e, 0, o), (e, 0, o), and (e, 0, o), B_{12} : (o, 0, e), (o, 0, e), (o, 0, 0), and (o, 0, e), B_{13} : (e, 0, o), (0, 0, o), (e, 0, o), and (e, 0, o), B_{14} : (o, 0, e), (o, 0, 0), (o, 0, 0), and (o, 0, e); B_{15} :(e, 0, o), (0, 0, o), (0, 0, o), and (e, 0, o).

(ii) t_i-t_{i-1} is any integer and each vertex of P_i is attached to (o,0,o).

2. For $n_1 + 1 \le i \le m$, the combinations of branches incident on the vertices x_i may be as given in (a) and (b) below.

(a) For $n_1 + 1 \le i \le n_2$ we have either (i) or (ii) as given below.

(i) t_i-t_{i-1} = 4 and the four vertices of P_i may be attached to any one among the block of combinations -

B'_1 : (e, e, 0), (e, e, 0), (e, e, 0), and (e, e, 0); B'_2 : (e, e, 0), (0, 0, 0), (0, 0, 0), and (e, e, 0); B'_3 : (e, e, 0), (0, e, 0), (e, 0, 0), and (e, e, 0); B'_4 : (e, e, 0), (e, e, 0), (e, 0, 0), and (e, e, 0); B'_5 : (e, e, 0), (0, e, 0), (e, e, 0), and (e, e, 0); B'_6 : (e, e, 0), (e, 0, 0), (e, 0, 0),,and (e, e, 0); B'_7 : (e, e, 0), (0, e, 0), (0, e, 0), and (e, e, 0); B'_8 : (e, e, 0), (0, 0, 0), (e, 0, 0), and (e, e, 0); B'_9 : (e, e, 0), (0, e, 0),,(0, 0, 0), and (e, e, 0); B'_{10} : (o, e, 0), (o, e, 0), (o, e, 0), and (o, e, 0); B'_{11} : (e, o, e), (e, o, 0), (e, o, 0), and (e, o, e); B'_{12} : (o, e, 0),(o, e, 0), (o, 0, 0), and (o, e, 0); B'_{13} : (e, o, 0), (0, o, 0), (e, o, 0), and (e, o, 0); B'_{14} : (o, e, 0), (o, 0, 0), (o, 0, 0), and (o, e, 0); B'_{15} :(e, o, 0), (0, o, 0), (0, o, 0), and (e, o, 0).

(ii) t_i-t_{i-1} is any integer and each vertex of P_i is attached to (o, o, 0).

(b) For $n_2 + 1 \le i \le m$ the branches incident on the vertices x_i may satisfy either (i) or (ii) as given below.

(i) t_i-t_{i-1} = 4 and the four vertices of P_i may be attached to any one among the block of combinations - B''_1 :(e, 0, 0), (0, 0, 0), (0, 0, 0), and (e, 0, 0); B''_2 : (e,0,0), (0, 0, 0), (e, 0, 0), and (e, 0, 0); B''_3 : (e, 0, 0), (e, 0, 0), (0, 0, 0), and (e, 0, 0).B''_4 : (o, 0, 0), (o, 0,0), (o, 0, 0), and (o, 0, 0).

(ii) t_i-t_{i-1} is any integer and each vertex of P_i is attached to (e, 0, 0).

3.2 Example

The lobster L in Figure 4 is a lobster of the type in Construction 3.1. Here $o_0 = 4$; $e_0 = 2$; $p_0 = 3$; $o_1 = 2$; $e_1 = 0$; $p_1 = 4$; $o_2 = 0$; $e_2 = 0$; $p_2 = 0$; $o_3 = 0$; $e_3 = 0$; $p_3 = 0$; $o_4 = 4$; $e_4 = 0$; $p_4 = 2$;$o_5 = 3$; $e_5 = 0$; $p_5 = 3$; o6 = 2; $e_6 = 0$ $p_6 = 3$; $o_7 = 0$; $e_7 = 0$; $p_7 = 3$; $o_8 = 2$; $e_8 = 0$; $p_8 = 3$;$o_9 = 2$; $e_9 = 0$; $p_9 = 5$; $o_{10} = 1$; $e_{10} = 0$; $p_{10} = 1$; $o_{11} = 1$; $e_{11} = 2$; $p_{11} = 0$; $o_{12} = 3$; $e_{12} = 2$; $p_{12} = 0$;$o_{13} = 3$; $e_{13} = 0$; $p_{13} = 0$; $o_{14} = 1$; $e_{14} = 2$; $p_{14} = 0$; $o_{15} = 3$; $e_{15} = 1$; $p_{15} = 0$; $o_{16} = 2$; $e_{16} = 2$; $p_{16} = 0$;$o_{17} = 0$; $e_{17} = 4$; $p_{17} = 0$; $o_{18} = 4$; $e_{19} = 0$; $p_{19} = 0$; $o_{19} = 2$; $e_{19} = 2$; $p_{19} = 0$; $o_{20} = 4$; $e_{20} = 0$; $p_{20} = 0$; $o_{21} = 3$; $e_{21} = 0$; $p_{21} = 0$; $o_{22} = 3$; $e_{22} = 0$; $p_{22} = 0$; $o_{23} = 5$; $e_{23} = 0$; $p_{24} = 0$; $o_{24} = 3$;$e_{24} = 0$;$p_{24} = 0$. Thus, x_0 is attached to (e, 0, o), x_1 is attached to (e, 0, e), x_2 is attached to (0, 0, 0), x_3 is attached to (0, 0, 0), x_4 is attached to (e,0,e), x_5 is attached to (o, 0, o), x_6 is attached to (e, 0, o), each of x_7 is attached to (0, 0, o), each of x_8 and x_9 is attached to (e, 0, o), x_{10} is attached to (o, 0, o), each of x_{11} and x_{12} is attached to (o, e, 0), x_{13} is attached to (o, 0, 0), x_{14} is attached to (o, e, o), x_{15} is attached to (o, o, 0), x_{16} is attached to (e, e, 0), x_{17} is attached to (0, e, 0), x_{18} is attached to (e, 0, 0), x_{19} is attached to(e, e, 0), x_{20} is attached to (e, 0, 0), and each of x_{21}, x_{22}, x_{23}, and x_{24} is attached to (o, 0, 0).

3.3 Theorem

The lobster L in Construction 3.1 is graceful.

Proof: Suppose that for $i = 0,1,2,......m$, $o_i + e_i + p_i = \lambda_i$, and

$$|E(L)| = q; \; N_o = \sum_{i=0}^{m} o_i \; , \; N_e = \sum_{i=0}^{m} e_i \; ,$$

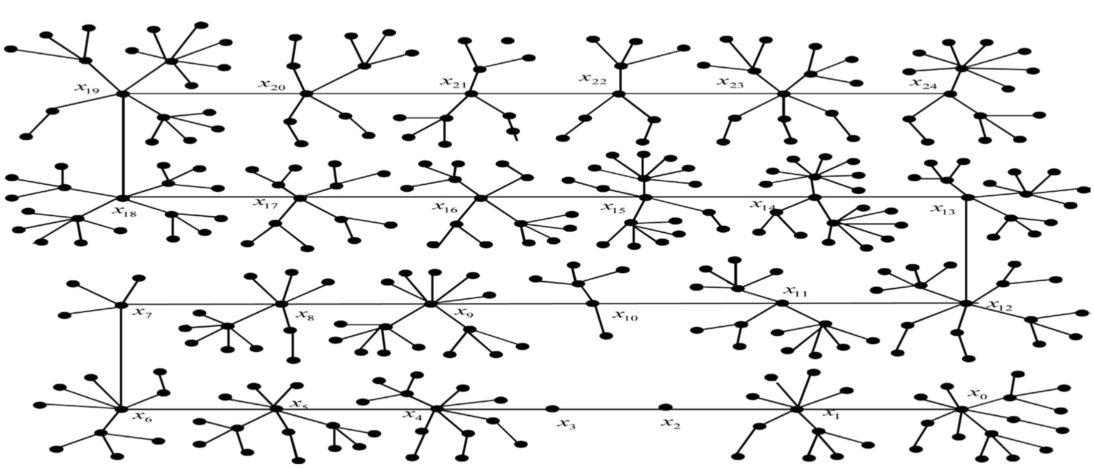

Figure 4. A lobster of the type in construction 3.1.

$N_p = \sum_{i=0}^{m} p_i$. Let $\sum_{i=0}^{m} \lambda_i = k$

Next we proceed as per the following steps.

Step 1: We first form the graceful tree $G(L)$ as shown in Figures 2 and Figure 3 with $|E(G(L))| = q + 1$, i.e. we attach a new pendant vertex x_{m+1} to the vertex x_m, the degree of each vertex x_i; $1 \le i \le m$, is two, and $x0$ is attached to $q - m$ pendant vertices. We consider the following graceful labeling f of $G(L)$.

If *m* is even:

$$f(v) = \begin{cases} \frac{m}{2} - i, v = x_{2i}, i = 0,1,2,.....\frac{m}{2} \\ q - \frac{m}{2} + 1 + i, v = x_{2i+1}, i = 0,1,2,....\frac{m}{2} \\ r, r = \frac{m}{2} + 1, \frac{m}{2} + 2, ..., q - \frac{m}{2} \\ \text{for the } q - m \text{ pendant vertices adjacent to } x_0 \end{cases}$$

If *m* is odd:

$$f(v) = \begin{cases} \frac{m-1}{2} - i, v = x_{2i+1}, i = 0,1,2,.....\frac{m-1}{2} \\ q - \frac{m-1}{2} + i, v = x_{2i}, i = 0,1,2,....\frac{m+1}{2} \\ r, r = \frac{m-1}{2} + 1, \frac{m-1}{2} + 2, ..., q - \frac{m-1}{2} - 1 \\ \text{for the } q - m \text{ pendant vertices adjacent to } x_0 \end{cases}$$

Let A_0 be the set of all pendant vertices adjacent to x_0 in $G(L)$. The set A_0 can be written as $A_0 = \{a_1, a_2 ....\ a_{q-m}\}$, where, for $1 \le i \le q - m$,

$$a_i = \begin{cases} q - \frac{m+i-1}{2} & \text{if } m+1 \text{ is odd} \\ \frac{m+1}{2} & \text{if } m+1 \text{ is even} \end{cases}$$

Further, the sums of consecutive elements of A_0 are alternately, q and $q + 1$ with $a_1 + a_2 = f(x_0) + f(x_1)$. By Lemma 2.2, we form a new graceful tree by moving all the vertices of A_0 from x_0 to x_1.

Step 2: Observe that the set A_0 and the labels of the vertices x_1 and x_2 of the transfer $T_1 : x_1 \to x_2 \to .... \to x_{m-1} \to x_m \to x_{m+1}$ correspond to the set A and the vertex labels a and b of Lemma 2.4. We carry out the transfer T_1 as a sequence of transfers $T_1^{(1)} \to T_1^{(2)} \to ...... \to T_1^{(n)}$ with $T_1^{(i)}$ $1 \le i \le n$, is the transfer

$x_{t_{i-1}+1} \to x_{t_{i-1}+2} \to .... \to x_{t_i}$, $T_0 = 0$ for $1 \le i \le n$, define the set Ai as the set of labels obtained by deleting the elements of A_{i-1} kept at P_{i-1}, $P_0 = \{a_0\}$. By the transfer $T_1^{(i)}$ we keep the required r_j vertices on the vertices x_j of P_i, where $r_j = o_j$ for $1 \le i \le n_2$ and for $n_2 + 1 \le i \le n, r_j$ is an integer with $0 < r_j < \lambda_j$ such that r_j is odd if P_i is attached to the combination B''_4 or Case2(b)(ii) and r_j is even if P_i is attached to one among the combinations B''_1, B''_2 and B''_3. The transfer $T_1^{(i)}$ $,1 \le i \le n$ may be one of the following.

Case I: If the combinations of branches incident on P_i is any one among B_{10}, B_{12}, B_{14},B'_{10}, B'_{12}, B'_{14}, B''_{14} Case 1(ii); Case 2(a)(ii) and Case 2b(ii) then $T_1^{(i)}$ consists of t_i-t_{i-1} successive transfers of the first type.

Case II: If the combinations of branches incident on P_i is any one among B_1, B_4, B_6,B_{11}, B'_1, B'_4, B'_6 and B''_{11} then $T_1^{(i)}$ consists of one BD8TF.

Case III: If the combinations of branches incident on P_i is any one among B_3, B_5, B_8,B_{13}, B'_3, B'_5, B'_8, B'_{13} and B''_2 then $T_1^{(i)}$ consists of one 1JTF.

Case IV: If the combinations of branches incident on P_i is any one among B_2, B_7, B_9,B_{15}, B'_2, B'_7, B'_9, B'_{15} , B''_1 and B''_3 then $T_1^{(i)}$ consists of one 2JTF.

Since a transfer $T_1^{(i)}$ may be a sequence of the transfers of the first type, a BD8TF, a 1JTF, or a 2JTF so each transfer in T_1 is one among the transfer of first type, BD8TF, 1JTF, and 2JTF. By Lemma 2.4, the resultant tree thus formed is graceful.

Let A_{n+1} be the set of vertex labels which are transferred to the vertex x_{m+1} after the transfer T_1 described above. Next make a transfer $x_{m+1} \to x_m$, i.e. $q + 1 \to 0$ of the first type bringing back each element of A_{n+1} to x_m and remove the vertex x_{m+1} and obtain a new graceful tree, say G_1.

Step 3: Next consider the transfer, T_2: $x_m \to x_{m-1} \to x_{m-2} \to ...... \to x_1 \to x_0$. The labels of the vertices x_m and x_{m-1} and the set A_{n+1} satisfy the hypothesis of Lemma 2.4. We partition the transfer T_2 : $T_2^{(1)} \to T_2^{(2)} \to T_2^{(3)} \to .... \to T_2^{(n)}$ and carry out the transfer T_2 by successively carrying out the transfers $T_2^{(1)}$, $T_2^{(2)}$,...., $T_2^{(n)}$ in order, where for $i = 1, 2,......., n$, $T_2^{(i)}$ is the transfer $x_{t_{n-1}+1} \to x_{t_{n-1}+1} \to .... \to x_{t_{n-1+1}} \to x_{t_{n-i}}$. For $1 \le i \le n$, define the set A_{n+i+1} as the set of labels obtained by deleting the elements of A_{n+i} kept at P_{n-i+1}. By the transfer $T_2^{(i)}$ we keep the required $\lambda_j - r_j$ vertices on the vertices x_j of P_j. The transfer $T_2^{(i)}$; $1 \le i \le n$ may be one of the following.

Case I: If the combinations of branches incident on P_{n-i+1} is any one among B_{11}, B_{13}, B_{15} Case1(ii); Case 2(a)(ii), and Case 2b(ii) then $T_2^{(i)}$ consists of $t_i - t_{i-1}$ successive transfers of the first type.

Case II: If the combinations of branches incident on P_{n-i+1} is any one among B_1, B_5, B_7, B_{10}, B'_{14}, B'_1, B'_5, B'_7, B''_{10} B'_{10}, B'_{14} *and* B''_4 then $T_2^{(i)}$ consists of one BD8TF.

Case III: If the combinations of branches incident on P_{n-i+1} is any one among B_3, B_4, B_9, B'_3, B'_4, B'_9, B'_{10}, B'_{12}, *and* B'_3 then $T_2^{(i)}$ consists of one 1JTF.

Case IV: If the combinations of branches incident on P_{n-i+1} is any one among B_2, B_6, B_8, B'_2, B'_6, B'_8, B''_1 *and* B''_2 then $T_2^{(i)}$ consists of one 2JTF.

Since a transfer $T_2^{(i)}$ may be a sequence of the transfers of the first type, a BD8TF, a 1JTF, or a 2JTF so each transfer in T_2 is one among the transfer of first type, BD8TF, 1JTF, and 2JTF. By Lemma 2.4, the resultant tree thus formed, say G_2, thus formed is graceful.

Step 4: A_{2n+1} is the set of vertices transferred to the vertex x_0 in step 3. The set is A_{2n+1} of the form $A_{2n+1} = \{a_{2(k-\lambda_0)+1}, a_{2(k-\lambda_0)+2}, \ldots\ldots\ldots, a_{q-m}\}$ Now carry out the transfer $x_0 \to a_1$ of the first type keeping p_0 vertices from A_{2n+1} and transferring the set say C_1 of the remaining vertices to a_1. The resultant tree thus formed; say G_3 is graceful by Lemma 2.4.

Step 5: Next we carry out the transfer $T_3 : a_1 \to a_2 \to a_3 \to a_4 \to \ldots\ldots \to a_s$, where $s = \sum_{j=1}^{m} \lambda_j$.

Observe that the way we have carried out the transfers T_1 and T_2 in Steps 2 and 3, the first N_0 labels of T_3 are the centers of the odd branches incident on H, the next N_e labels are the centers of the even branches incident on H.

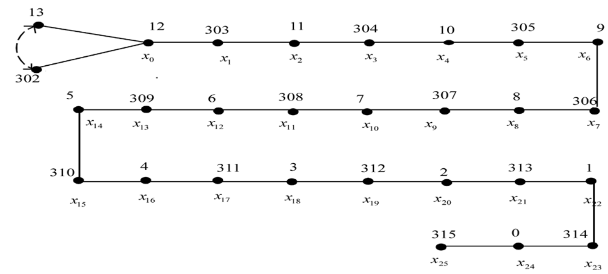

Figure 5. The tree G(L) with graceful labelling.

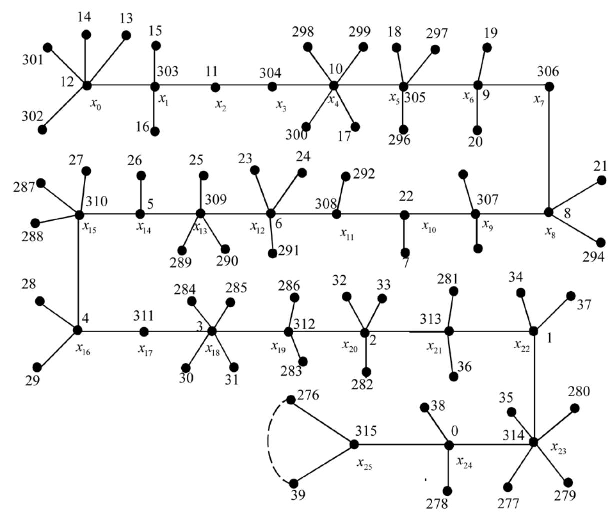

Figure 6. The graceful tree obtained after step 3.

The set C_1 and the transfer T_3 satisfy the hypothesis of Lemma 2.4. The transfer T_3 consists of N_0 successive

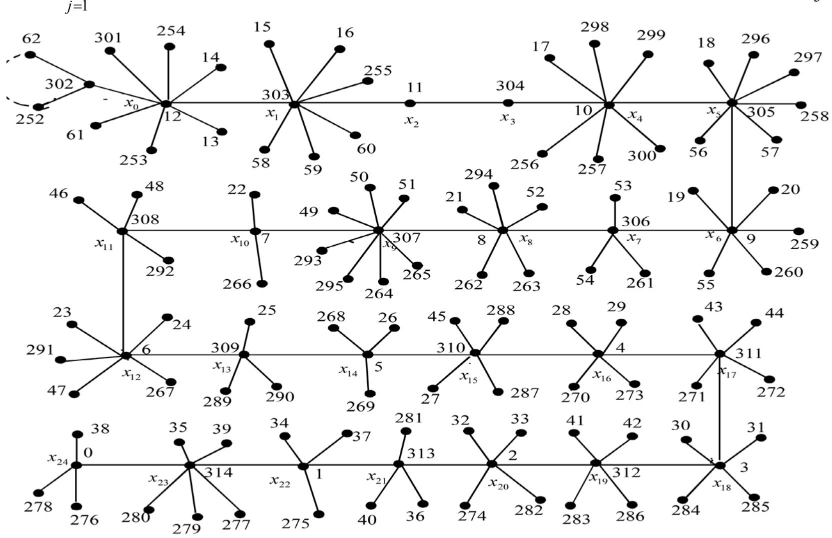

Figure 7. The graceful tree obtained after step 4.

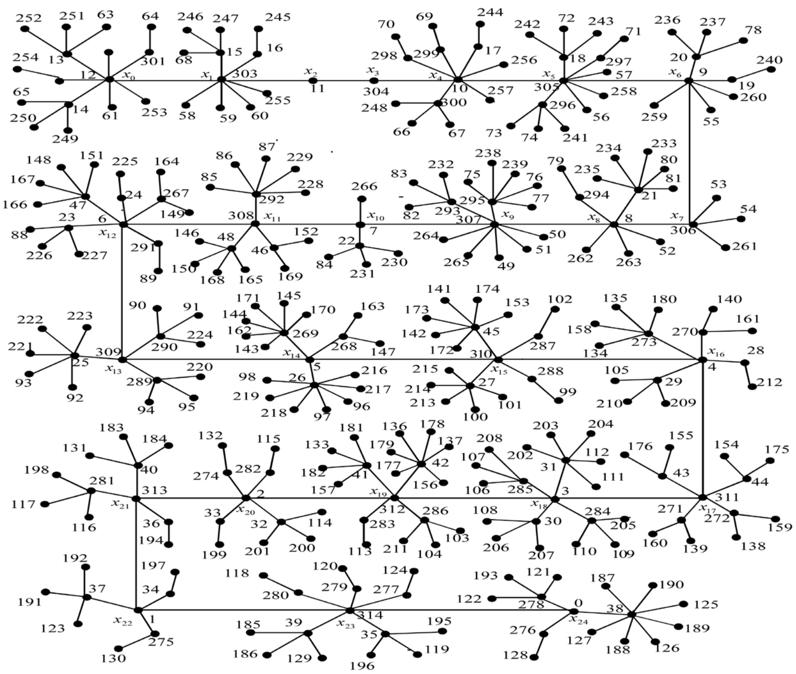

Figure 8. The Lobster L with a graceful labelling.

transfers of the first type, followed by $N_e - 1$ successive transfers of the second type so as to get the desired lobster L. By Lemma 2.4, the lobster L thus obtained is graceful.

Example: The lobster L in Example 3.2 (Figure 4) is a lobster of the type in Theorem 3.3. Here $q = 314$. We first form the graceful tree $G(L)$ as in Figure 5. Figure 6 represents the tree obtained after step 3. Figure 7 represents the tree obtained after step 4. Figure 8 represents the lobster L with a graceful labeling after step 6.

3.4 Construction

We construct a class of lobsters with the central path $H = x_0 x_1 x_2 ....x_m$ *in* which the vertex x_0 is attached to an even number of odd branches and an odd number of pendant branches and the path $H \setminus \{x_0\}$ partitions into finite number of paths $P_i = x_{t_{i-1}+1}, x_{t_{i-1}+2}, ..., x_{t_i}$ $1 \le i \le n$; $t_0 = 0$; $t_n = m$, such that the combinations of branches incident on the vertices of P_i may have one of the following features. Let n_1 and n_2 be integers, such that $1 \le n_1 \le n_2 \le n$.

1. For $1 \le i \le n_1$, the combinations or block of combinations of branches incident on P_i are same as in (1) in Construction 3.1.

2. For $n_1 + 1 \le i \le m$, the combinations of branches incident on the vertices x_i may be as given in (a) and (b) below.

(a) For $n_1 + 1 \le i \le n_2$, the combinations or block of combinations of branches incident on P_i are same as in (2 (a)) in Construction 3.1.

(b) For $n_2 + 1 \le i \le m$ the branches incident on the vertices x_i may be as either in (i) or (ii) given below.

(i) $t_i - t_{i-1} = 4$ and the four vertices of P_i may be attached to any one among the bock of combinations combination - $B_1^{"}$:(0, e, 0), (0, 0, 0), (0, 0, 0), and (0, e, 0); $B_2^{"}$: (0, e, 0), (0, 0, 0), (0, e, 0), and (0, e, 0); $B_3^{"}$: (0, e, 0), (0, e, 0), (0, 0, 0), and (0, e, 0). $B_4^{"}$: (0, o, 0), (0, o, 0), (0, o, 0), and (0, o, 0);

(ii) $t_i - t_{i-1}$ is any integer and each vertex of P_i is attached to (0, e, 0).

3.5 Theorem

The lobster L in Construction 3.4 is graceful.

Proof: Suppose that for i =1,2,3,.....m, $o_i + e_i + p_i = \lambda_i$, and $|E(L)| = q$;

$$N_0 = \sum_{i=0}^{m}(o_i), N_e = \sum_{i=0}^{m}(e_i), N_p = \sum_{i=0}^{m}(p_i). \text{Let} \sum_{i=0}^{m}(\lambda_i) = k.$$

Next we proceed as per the following steps.

Step 1: Repeat Steps 1 to 5 in the proof of Theorem 3.3. *2*

3.6 Construction

We construct a class of lobsters with the central path $H = x_0 x_1 x_2 ....x_m$ in which the vertex x_0 is attached to an even number of odd branches and an odd number of pendant branches and the path $H \backslash \{x_0\}$ partitions into finite number of paths $P_i = x_{t_{i-1}+1}, x_{t_{i-1}+2}, ..., x_{t_i}$, $1 \le i \le n$ such that the combinations of branches incident on the vertices of P_i may have one of the following features. Let n_1 and n_2 be integers, such that $1 \le n_1 \le n_2 \le n$.

1. For $1 \le i \le n_1$, the combinations or block of combinations of branches incident on P_i are same as in (1) in Construction 3.1.

2. For $n_1 + 1 \le i \le m$, the combinations of branches incident on the vertices x_i may be as given in (a) and (b) below.

B'_1 : (0, *e*, *e*), (0, *e*, *e*), (0, *e*, *e*), and (0, *e*, *e*); B'_2 : (0, *e*, *e*), (0, 0, 0), (0, 0, 0), and (0, *e*, *e*); B'_3 : (0, *e*, *e*), (0, 0, *e*), (0, *e*, 0), and (0, *e*, *e*); B'_4 : (0, *e*, *e*), (0, *e*, *e*), (0, *e*, 0), and (0, *e*, *e*); B'_5 : (0, *e*, *e*), (0, 0, *e*), (0, *e*, *e*), and (0, *e*, *e*); B'_6 : (0, *e*, *e*), (0, *e*, 0), (0, *e*, 0),,and (0, *e*, *e*); B'_7 : (0, *e*, *e*), (0, 0, *e*), (0, 0, *e*), and (0, *e*, *e*); B'_8 : (0, *e*, *e*), (0, 0, 0), (0, *e*, 0), and (0, *e*, *e*); B'_9 : (0, *e*, *e*), (0, 0, *e*),(0, 0, 0), and (0, *e*, *e*); B'_{10} : (0, *o*, *e*), (0, *o*, *e*), (0, *o*, *e*), and (0, *o*, *e*); B'_{11} : (0, *e*, *o*), (0, *e*, *o*), (0, *e*, *o*), and (0, *e*, *o*); B'_{12} : (0, *o*, *e*),(0, *o*, *e*), (0, *o*, 0), and (0, *o*, *e*); B'_{13} : (0, *e*, *o*), (0, 0, *o*), (0, *e*, *o*), and (0, *e*, *o*); B'_{14} : (0, *o*, *e*), (0, *o*, 0), (0, *o*, 0), and (0, *o*, *e*); B'_{15} :(0, *e*, *o*), (0, 0, *o*), (0, 0, *o*), and (0, *e*, *o*);

(ii) $t_i - t_{i-1}$ is any integer and each vertex of P_i is attached to (0, *o*, *o*).

(b) For $n_2 + 1 \le i \le n$, the combinations or block of combinations of branches incident on *Pi* are same as in (2 (b)) in Construction 3.4.

3.7 Theorem

The lobster *L* in Construction 3.6 is graceful.

Proof: Suppose that for $i = 0,1,2,......m$, $o_i + e_i + p_i = \lambda_i$ and $|E(L)| = q$;

$$N_0 = \sum_{i=0}^{m} (o_i), N_e = \sum_{i=0}^{m} (e_i), N_p = \sum_{i=0}^{m} (p_i). \text{ Let } \sum_{i=0}^{m} (\lambda_i) = k.$$

Next we proceed as per the following steps.

Step 1: Repeat Step 1 in the proof of Theorem 3.3.

Step 2: Observe that the set A_0 and the labels of the vertices x_1 and x_2 of the transfer $T_1 : x_1 \to x_2 \to .... \to x_{m-1} \to x_m \to x_{m+1}$ correspond to the set *A* and the vertex labels *a* and *b* of Lemma 2.4. We carry out the transfer T_1 as a sequence of transfers $T_1^{(1)} \to T_1^{(2)} \to ...... \to T_1^{(n)}$ with $T_1^{(i)}$ $1 \le i \le n$ is the transfer $x_{t_{i-1}+1} \to x_{t_{i-1}+2} \to .... \to x_{t_i}$, $T_0 = 0$ for $1 \le i \le n$, define the set *Ai* as the set of labels obtained by deleting the elements of A_{i-1} kept at P_{i-1}, $P_0 = \{a_0\}$.

By the transfer $T_1^{(i)}$ we keep the required r_j vertices on the vertices x_j of P_i, where $r_j = o_j$, for $1 \le i \le n_1$, $r_j = e_j$,for $n_1 + 1 \le i \le n_2$, and for $n_2 + 1 \le i \le n$, r_j is an integer with $0 < r_j < \lambda_j$ such that r_j is odd if P_j is attached to the combination B''_4 or Case2(b)(ii) and r_j is even if P_i is attached to one among the combinations B''_1, B''_2 *and* B''_3 .The transfer

$T_1^{(i)}$ $1 \le i \le n$ is same as in Step 2 in the proof involving Theorem 3.3.

Since a transfer $T_1^{(i)}$ may be a sequence of the transfers of the first type, a BD8TF, a 1JTF, or a 2JTF so each transfer in T_1 is one among the transfer of first type, BD8TF, 1JTF, and 2JTF. By Lemma 2.4, the resultant tree thus formed is graceful. Let *An*+1 be the set of vertex labels which are transferred to the vertex x_{m+1} after the transfer T_1 described above. Next make a transfer $x_{m+1} \to x_m$, i.e. $q + 1 \to 0$ of the first type bringing back each element of A_{n+1} to x_m and remove the vertex x_{m+1} and obtain a new graceful tree, say G_1.

Repeat Steps 3 to 5 in the proof involving Theorem 3.3 so as to get a graceful labeling of *L. 2*

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