

Wavelet-Collocation Method of Solving Singular Integral Equation

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Abstract

Objectives: This article describes one of the approaches to the approximate solution of the singular integral equation of the first kind with the Cauchy kernel on material axis interval, based on the approximation of the desired function by Chebyshev's wavelets of the II-nd kind. **Methods:** The theory of such equations states that the solution in a closed form can be obtained only in rare cases. Therefore, various approximation methods are used, followed by their theoretical basis. The Uniform error estimates of obtained approximate solutions are very relevant ones for practice. **Results:** However, the incorrect problem of this equation solution caused primarily by universal values of first and reversible singular operators, on the pair of continuous function spaces, leads to particular difficulties at a numerical solution. The work as the area of desired elements and right sides considers weighted spaces, some of which are the restrictions of continuous functions on which a correct task is set. A computational scheme of wavelet collocation method is developed. The theorem on the unique solvability of obtained linear algebraic equation system is proved. Uniform estimates for the relative solution errors are set depending on the structural properties of original data. **Conclusion:** The performed numerical experiment in Wolfram Mathematical package showed a real convergence of the approximate solution, obtained by the method of wavelet collocation, with the exact one.

Keywords: Chebyshev's Wavelets, Singular Integral Equation, Wavelet Collocation Method

1. Introduction

Let us consider the singular integral equation (i.e.) of the first kind with Cauchy kernel:

$$Kx \equiv \frac{1}{\pi} \int_{-1}^1 \frac{\sqrt{1-\tau^2} x(\tau)}{\tau-t} d\tau + \frac{1}{\pi} \int_{-1}^1 \sqrt{1-\tau^2} h(t,\tau) x(\tau) d\tau = f(t), \quad |t| < 1, \quad (1)$$

Where $h(t, \tau)$, $f(t)$ are known continuous functions in their ranges of definition, $x(\tau)$ is the required function and the singular integral

$$I\varphi = I(\varphi; t) = \frac{1}{\pi} \int_{-1}^1 \frac{\varphi(\tau)}{\tau-t} d\tau$$

is understood in the sense of Cauchy principal value.

The equation (1) is widely used in many fields of science and technology. The theory of such equations is well developed¹. According to this theory the obtaining of i.e. exact solution (1) in a closed form is possible only in certain cases. Therefore, various approximate methods of its solution are developed, which are based on the idea of the desired solution in the form of classical orthogonal polynomials².

It is known that the problem of the equation (1) solution is an incorrectly posed in many pairs of functional spaces, including the pair of continuous function spaces. However, there is the possibility of finding a cor-

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rect formulation of a problem for equation solution (1). The work³ sets the correctness of the specified equation problem solution (1) on a pair of weighted spaces (X, Y), where X is the space of continuous functions $x(t)$ on [-1; 1], for which the singular integral $I(rx; t)$ is also a continuous function, $\rho \equiv \rho(t) = \sqrt{1-t^2}$, and Y is the space of continuous functions $f(t)$ on [-1; 1] such, that $\frac{1}{q(t)}I(qf; t)$ is the continuous function on (-1; 1), which allows a continuous extension on the ends of the segment, $q \equiv q(t) = \frac{1}{\sqrt{1-t^2}}$ and the following term is performed

$$\int_{-1}^1 q(t)y(t)dt = 0 \tag{2}$$

The norms in these areas are determined respectively in the following way:

$$\|x\|_X = \|\rho x\|_C + \|I(\rho x)\|_C, \quad x \in X;$$

$$\|f\|_Y = \|f\|_C + \left\| \frac{1}{q} I(qf) \right\|_C, \quad f \in Y,$$

Where $\|x\|_C = \max_{-1 \leq t \leq 1} |x(t)|$.

Then i.e. (1) is equivalent to operator equation

$$Kx \equiv Sx + Vx = f \quad (x \in X, f \in Y), \tag{3}$$

$$Sx = \frac{1}{\pi} \int_{-1}^1 \frac{\sqrt{1-\tau^2} x(\tau)}{\tau-t} d\tau, \quad Vx = \frac{1}{\pi} \int_{-1}^1 \sqrt{1-\tau^2} h(t, \tau) x(\tau) d\tau$$

This pair of spaces carried out³ the substantiation of various approximation methods based on the idea of the approximate solution in the form of polynomials. At the end of the last century, a new class of basic functions named wavelets appeared. Nowadays, the wavelet theory is being actively developed, finds its application in various fields of science^{4,5}, while leaving a vast field for research. In recent years, the theory of wavelets obtained an intensive development in the works of many authors based on trigonometric and algebraic polynomials, as well as the methods of function expansion in series according to polynomial wavelets⁶⁻¹¹. Wavelet has been widely used to solve image processing problems including classification by extracting better features from each image¹²⁻¹⁴. In adopted the concept of integral form to

solve partial differential equation in mechanical problem¹⁵⁻²⁰. The collocation method was applied in this paper to solve the singular integral equation (1). The method was based on the approximation of the desired function by Chebyshev's wavelets of the II-nd kind.

2. Wavelet Collocation Method

Let's consider the "regularized" equation

$$\overline{Kx} \equiv Sx + Vx + \gamma = f \quad (\overline{x} \in \overline{X}, f \in Y, \gamma \in R), \tag{4}$$

Where $\overline{X} = (X, \gamma)$ with the norm

$$\|\overline{x}\|_{\overline{X}} = \|x\|_X + |\gamma|,$$

$\overline{x} = (x, \gamma)$ is a vector function with the following components $x(t) \in X$ and $\gamma \in R$.

The equation (2) shows that the regularization parameter is the following one:

$$\gamma = \int_{-1}^1 \frac{f(t) - V(x; t)}{\sqrt{1-t^2}} dt$$

An approximate solution of the equation (1) will be sought in the form of the vector function

$$\overline{x}_m = (x_m, \gamma_m), \quad x_m(t) = a_0 \Phi_{0,0}(t) + a_1 \Phi_{0,1}(t) + \sum_{j=0}^{m-1} \sum_{k=0}^{2^j-1} b_{j,k} \Psi_{j,k}(t), \tag{5}$$

Where

$$\Phi_{m,k}(t) = \sum_{j=0}^{2^m} U_j(t) U_j(\xi_k^{(2^{m+1})}) \frac{2 \left| \sin \frac{\pi(k+1)}{2^m+2} \right|}{\sqrt{\pi(2^m+2)}}, \quad m=0,1,\dots, \quad k = \overline{0, 2^m - 1}$$

$$\Psi_{m,k}(t) = \sum_{j=2^{m+1}}^{2^{m+1}} U_j(t) U_j(\xi_k^{(2^{m+1})}) \frac{2 \left| \sin \frac{\pi(k+1)}{2^m+1} \right|}{\sqrt{\pi(2^m+1)}}, \quad m=1,2,\dots, \quad k = \overline{0, 2^m - 1}$$

The so-called scaling function and Chebyshev's wavelet function of the II-nd kind, respectively²¹,

$$U_j(t) = \frac{\sin((j+1)\arccos t)}{\sqrt{1-t^2}}, \quad j = 0,1,2,\dots$$

are Chebyshev's polynomial of the II-nd kind, $\xi_k^{(n)} = \cos \frac{\pi(k+1)}{n+1}, \quad k = 0, \dots, n-1$ are polynomial

zeros $U_n(t), \gamma_m \in R$.

The unknown coefficients $a_0, a_1, b_{j,k}$ ($j = \overline{0, m-1}, k = \overline{0, 2^j-1}$), γ_m will be sought from the condition of discrepancy zero equality in the nodes of collocation

$$t_k^{(2^m+2)} = \cos \frac{\pi \left(k + \frac{1}{2}\right)}{2^m + 2}, \quad m = 1, 2, \dots, k = 0, 2^m + 1. \quad (6)$$

Taking into account the known ratios $SU_n = -T_{n+1}$, $T_n(t) = \cos n \arccos t$ -Chebyshev's polynomials of the I-st kind, we obtain the system of linear algebraic equations

$$\begin{aligned} \gamma_m + \sum_{j=0}^1 T_{j+1}(t_k^{(2^m+2)}) \left(-a_0 \frac{U_j(\xi_0^{(2)})}{\sqrt{\pi}} - a_1 \frac{U_j(\xi_1^{(2)})}{\sqrt{\pi}} \right) - \\ - \sum_{j=0}^{m-1} \sum_{k=0}^{2^j-1} b_{j,k} T_{j+1}(t_k^{(2^{m+2})}) U_j(\xi_k^{(2)}) \frac{2 \left| \sin \frac{\pi(k+1)}{2^j+1} \right|}{\sqrt{\pi(2^j+1)}} + \\ + \frac{1}{\pi} \int_{-1}^1 \sqrt{1-\tau^2} h(t_k^{(2^m+2)}, \tau) \left(a_0 \varphi_{0,0}(\tau) + a_1 \varphi_{0,1}(\tau) + \sum_{j=0}^{m-1} \sum_{k=0}^{2^j-1} b_{j,k} \psi_{j,k}(\tau) \right) d\tau = f(t_k^{(2^m+2)}), \quad k = 0, 2^m + 1. \end{aligned} \quad (7)$$

Let's denote the set of functions with a continuous derivative of r-th order, satisfying Holder's condition with the value $\alpha, 0 < \alpha \leq 1, r \geq 0$ via $W^r H_\alpha = W^r H[-1, 1]$.

Theorem: Let's the following terms are satisfied:

- a) the equation (4) has a single solution $x_m^* \in \bar{X}$ at any right part $f \in Y$;
- b) the function $f(t) \in W^r H_\alpha$, the kernel $h(t, \tau) \in W^r H_\alpha$ according to variable t is uniform relative to τ .

Then starting from some $m \in N$, the system of collocation method (7) has a single value $\hat{a}_0^*, \hat{a}_1^*, b_{j,k}^* (j = 0, m-1, k = 0, 2^j - 1), \gamma_m^*$ and the approximate solutions $\bar{x}_m^* = (x_m^*, \gamma_m^*)$ are converged to exact solution x^* in the area X with the speed

$$\|\bar{x} - \bar{x}_m^*\|_0 = \left(\frac{\ln(2^m + 1)}{(2^m + 1)^{r+\alpha}} \right), \quad 0 < \alpha \leq 1, r \geq 0.$$

Proving: Let's H_{2^m+1} is the set of all algebraic polynomials of the degree not exceeding 2^m+1 . Let's introduce the subspaces of vector functions $\bar{X}_m = \{\bar{x}_m\}, \bar{x}_m = (x_m, \gamma_m), x_m \in H_{2^m+1}, \gamma_m \in R$ with the following norm $\|\bar{x}_m\|_{\bar{X}_m} = \|x_m\|_{H_{2^m+1}} + |\gamma_m|; Y_m = H_{2^m+2}$. Then the system of the collocation method may be written in an operator form as follows:

$$\bar{K}_m \bar{x}_m \equiv \gamma_m + Sx_m + L_m Vx_m = L_m f, \quad (8)$$

$$(\bar{x}_m \in \bar{X}_m, L_m f \in H_{2^m+1})'$$

Where $L_m f: Y \rightarrow H_{2^m+1}$ is the operator, which assigns the correspondence of continuous function f with

Lagrange interpolation poly nomial according to node system (6).

We obtain the following one from (4) and (8) for any $\bar{x}_m \in \bar{X}_m$

$$\|(\bar{K} - \bar{K}_m) \bar{x}_m\|_Y = \|Vx_m - L_m Vx_m\|_Y \leq \|h - L'_m h\|_{Y \otimes C} \|x_m\|_X,$$

where the operator L'_m is applied to the function $h(t; \tau)$ according to variable t.

From results³ and theorem conditions we obtain the following one:

$$\bar{\varepsilon}_n \equiv \|(\bar{K} - \bar{K}_m) \bar{x}_m\|_{\bar{X}_m \rightarrow Y} = O\left(\frac{\ln(2^m + 1)}{(2^m + 1)^{r+\alpha}}\right) \rightarrow 0, \quad m \rightarrow \infty. \quad (9)$$

Under the conditions of the theorem the operator $\bar{K}: \bar{X} \rightarrow Y$ is continuously reversible. Then due to (9) and the following ratios

$$\bar{\delta}_n \equiv \|f - L_m f\|_Y = O\left(\frac{\ln(2^m + 1)}{(2^m + 1)^{r+\alpha}}\right) \rightarrow 0, \quad m \rightarrow \infty$$

The required statement follows from the theorem 7 of the chapter I²².

3. Numerical Experiment

Let's consider the following singular integral equation

$$\frac{1}{\pi} \int_{-1}^1 \frac{\sqrt{1-\tau^2} x(\tau)}{\tau-t} d\tau + \frac{1}{\pi} \int_{-1}^1 \sqrt{1-\tau^2} (t+\tau^3) x(\tau) d\tau = -t^5 - \frac{3}{2}t^3 - \frac{33}{16}t, \quad |t| < 1. \quad (10)$$

The exact equation (10) at that

$$x^*(\tau) = \tau^4 + 2\tau^2 + 7.$$

The search of an approximate solution according to the above-stated calculation scheme implemented in Wolfram Mathematic.

Let's m = 1. The solution of equation (10) under the scheme of the wavelet collocation (7) will be the following one is shown in Figures 1 and 2.

$$\bar{x}_1^* = (x_1^*, \gamma_1^*),$$

Where

$$x_1^*(t) = \frac{0,625 \sin 3 \arccos t}{\sqrt{1-t^2}} + 7,66, \quad \gamma_1^* = 0.$$

Absolute error standard:

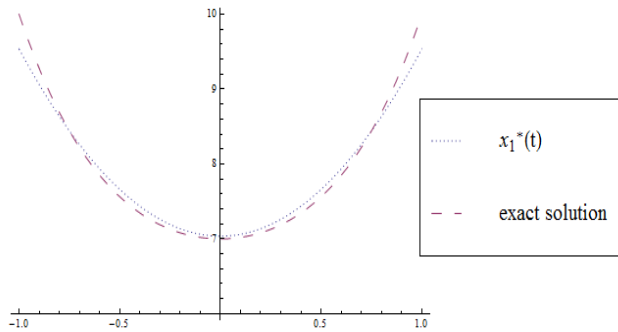


Figure 1. $x_1^*(t)$ graph and the exact solution of the equation (10), $m = 1$.

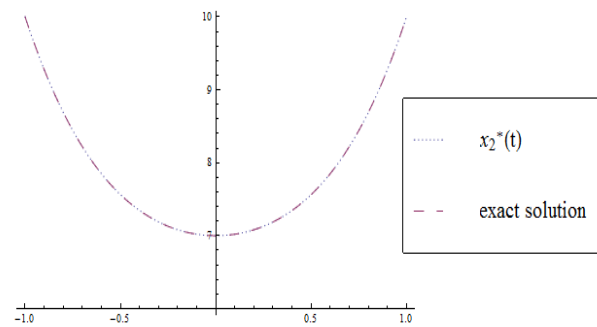


Figure 2. $x_2^*(t)$ graph and exact equation solution (10), $m = 2$.

$$\|x^*(t) - x_1^*(t)\|_{\mathcal{O}} = 0,192.$$

Let $m = 2$. An approximate solution will be the following one:

$$\overline{x_2^*} = (x_2^*, \gamma_2^*),$$

Where

$$x_2^*(t) = \frac{0,6875 \sin 3 \arccos t + 0,0625 \sin 5 \arccos t}{\sqrt{1-t^2}} + 7,625,$$

$$\gamma_2^* = 0.$$

Absolute error norm:

$$\|x^*(t) - x_2^*(t)\|_X = 0,000.$$

4. Summary

The peculiarity of wavelet analysis is that it may use a large number of basic wavelet functions. Therefore, there is a possibility of choice between the families of wavelet functions and a flexible application of those which solve a particular task most efficiently. Because of the singular integral equation nature, Chebyshev’s wavelets of the II-nd type were chosen for the study. A computational scheme of collocation methods was developed by the selected wavelets. The approximation of wavelet functions is checked nearby using computing algebra system Wolfram Mathematic. The operation of the system showed that a good rate of convergence of an approximate equation solution to the exact one was obtained in the second layer.

4.1 Conflict of Interest

The authors acknowledge that the presented data do not contain any conflict of interest.

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