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Generalized Operator Valued Integral

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Abstract

Background/Objectives: Hilbert space has been studied in previous researches more than a century. In[1] introduced fusion integral as an operator valued integral over a Hilbert space. **Methods/Statistical Analysis:** A generalized operator valued integral over a Hilbert space is presented. Also we obtain results concerning composition a generalized operator valued integral with a bounded objective operator on Hilbert space. **Findings:** Theory and definitions are developed based on Generalized Operator Valued Integra and its findings include: 2000 Mathematics Subject Classification. Primary: 42C15; Secondary 46L30. The approaches and opinion of the operator valued integral case has validated and assisted to the operator approach. **Applications/Improvements:** One of the problems in the development of frame theory has been the using of an impressive structure for the subject, which is used in this paper.

Keywords: Bounded Operator, Frame of Subspaces, Operator, Operator Valued Integral

1. Introduction

The parameter H is considered as a Hilbert space and $\hat{H}$ is the total of all bounded subspace of H, respectively. In addition, (X, μ) is a measure space, &, $v: X \to [0,\infty)$ a measurable mapping where that $v \neq 0, a.e.$ Also, We the unit closed ball of H by H_1 is defined[2–5].

The study of operators on Hilbert space was conducted for a century, operator valued integral as an operator on a Hilbert space with members of measurable functions is of more recent origin. Studies of operator valued integral that corroborated the analogues of frame theory, frame and subspaces, must be attended for the most share till the recently years. Then, however, this area has been tracked rather strenuously with some noticeable achievements, both for its own proof and connections with other areas of frame theory. Furthermore, the approaches and opinions of the operator amount integral case has validated and assisted to the operator approach. One of the problems in the development of frame theory has been the using of an impressive structure for the subject, which is used in this paper[6–8].

The structure of this article is managed in three main parts. In second section, theorems and presentations which we need from operator theory and definition of generalized operator valued integral will appear. In Section 3, with name main result, many useful properties of generalized operator valued integral will be discussed.

2. Preliminaries

Lemma 2.1

K is considered as a Hilbert space. $u : K \to H$ is considered a limited operator with constant values R_u. In the next, there is a ranged operator $u^\dagger : H \to K$ such that as:

$$uu^\dagger f = f, \qquad f \in R_u.$$

Also $u^\dagger : H \to K$ has bounded range and $(u^*)^\dagger = (u^\dagger)^*$.

The operator $u^\dagger$ is called the pseudo-inverse of u.

Lemma 2.2

$u^\dagger : H \to K$ is considered as a ranged subjective operator. By considering the $y \in H$, the relation $ux = y$ illustrates a just only one solution of minimum norm, $x = u^\dagger y$.

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Lemma 2.3

$u^{\dagger}: H \to K$ is considered as a ranged operator. So:

- $\| u \| = \| u^{*} \| \quad and \quad \| uu^{*} \| = \| u \|^{2}$.
- R_u is bounded, if and only if, R_{u^*} is bounded.
- u is subjective if and only if there is a c > 0 whereas for each h ∈ H $c \| h \| \leq \| u^{*}(h) \|$.

Lemma 2.4

$u^{\dagger}: H \to K$ is considered as a closed operator with closed range R_u. Then this assertions are equivalent:

- Orthogonal projection of H to R_u is specified by $uu^{\dagger}$,
- Orthogonal projection of K to R_{u^*} is specified by $u^{\dagger}u$,
- R_{u^*} is closed and $(u^{*})^{\dagger} = (u^{\dagger})^{*}$,
- $u^{\dagger}$ Operator on R_u is characterized by $u^{\dagger} = u^{*}(uu^{*})^{-1}$.

Lemma 2.5

Let u be a self-adjoint bounded operator on H. Let:

$$m_u = \inf_{h \in H_1} < uh, h > \quad and \quad M_u = \sup_{h \in H_1} < uh, h >.$$

Then, $m_u, M_u \in \sigma(u)$.

Definition 2.6

$\{K_x\}_{x \in X}$ is considered as a total of Hilbert spaces. L^2 (X, K) is considered as the class of total measurable mapping φ : X → K where as for each x ∈ X, φ(x) ∈ K_x and $\int_X \| \varphi(x) \|^2 \, d\mu < \infty$.

It is obvious, it proved that L^2 (X, K) is a Hilbert space with inner product expressed as:

$$< \varphi, \gamma > = \int < \varphi(x), \gamma(x) > d\mu,$$

$$.for \ \varphi, \gamma \in L \ (X, K).$$

Definition 2.7

L(X, K, Λ, υ) is considered as the class of total mapping $F: X \to \hat{H}$ whereas for each h ∈ H, measuring the mapping $x \to \pi_{F(x)}(h)$ is possible and let $\{K_x\}_{x \in X}$ be a set of Hilbert spaces. For each x ∈ X, suppose that $\Lambda_x \in B(F(x), K_x)$ and put

$$\Lambda = \{ \Lambda_x \in B(F(x), K_x) : x \in X \}.$$

and

$$\sup_{h \in H_1} \int_X v^2(x) \| \Lambda_x(\pi_{F(x)}(h)) \|^2 d\mu < \infty,$$

for all h ∈ H, where $\pi_{F(x)}(h)$ be called the orthogonal projection to the subspace F(x).

Remark 2.8 In order to conciseness, L(X, K, Λ, υ) is denoted by L(X, Λ, υ). Let F ∈ L(X, Λ, υ), φ ∈ L^2(X, Λ, υ) and h ∈ H.

So:

$$| \int_X v(x) < \Lambda_x^{*}(\varphi(x)), h > d\mu |$$

$$= | \int_X v(x) < \Lambda_x^{*}(\varphi(x)), \pi_{F(x)}(h) > d\mu |$$

$$= | \int_X v(x) < \varphi(x), \Lambda_x(\pi_{F(x)}(h)) > d\mu | \leq \int_X v(x) \| \varphi(x) \| \cdot \| \Lambda_x(\pi_{F(x)}(h)) \| d\mu$$

$$\leq (\int_X v^2(x) \| \Lambda_x(\pi_{F(x)}(h)) \|^2 d\mu)^{1/2} \times (\int_X \| \varphi(x) \|^2 d\mu)^{1/2} \leq (\int_X \| \varphi(x) \|^2 d\mu)^{1/2}$$

$$\times \sup_{h \in H_1} (\int_X v^2(x) \| \Lambda_x(\pi_{F(x)}(h)) \|^2 d\mu)^{1/2}.$$

For this purpose the following definition is used:

Definition 2.9

F ∈ L(X, Λ, υ). The generalized operator valued integral (for brevity, g-operator valued integral) of F by $\oint_X \Lambda F d\mu$ is denoted as the linear operator of L^2 (X, K) to H defined by[9,10]

$$\left\langle \oint_X \Lambda F d\mu(\varphi), h \right\rangle = \int_X v \ (x) \ \langle \Lambda_x^{*}(\varphi(x)), h \rangle d\mu,$$

Where $\varphi \in L^2(X, K)$ and h ∈ H. The adjoins of $\oint_X \Lambda F d\mu(\varphi)$ are denoted by $\oint_X^{*} \Lambda F d\mu(\varphi)$ and we have,

Lemma 2.10

$F \in L(X, \Lambda, \upsilon)$. So, $\oint_X \Lambda F d\mu$ is bounded, $\oint_X^{*} \Lambda F d\mu = v \Lambda \pi_F$.

Proof

For each $\varphi \in L^2(X, K)$,

$$\| \oint_X \Lambda F d\mu(\varphi) \|$$

$$= \sup_{k \in H_1} | \int_X v(x) < \Lambda_x^{*}(\varphi(x)), k > d\mu |$$

$$= \sup_{k \in H_1} | < \oint_X \Lambda F d\mu(\varphi), k > = \sup_{k \in H_1} | \int_X v(x) < \Lambda_x^{*}(\varphi(x)), k > d\mu |$$

$$= \sup_{k \in H_1} | \int_X v(x) < \varphi(x), \Lambda_x(\pi_{F(x)}(k)) > d\mu |$$

$$\leq \sup_{k \in H_1} \int_X v(x) \| \varphi(x) \| \cdot \| \Lambda_x(\pi_{F(x)}(k)) \| d\mu$$

$$\leq (\int_X \| \varphi(x) \|^2 d\mu)^{1/2} \times \sup_{k \in H_1} (\int_X v^2(x) \| \Lambda_x(\pi_{F(x)}(k)) \|^2 d\mu)^{1/2}.$$

Thus,

$$\| \oint_X \Lambda F d\mu(\varphi) \|^2 = \sup_{k \in H_1} \int_X v^2(x) \| \Lambda_x(\pi_{F(x)}(k)) \|^2 d\mu < \infty.$$

We conclude that $\oint_X \Lambda F d\mu(\varphi)$ is bounded. For each $\varphi \in L^2(X,K)$ and h ∈ H, So, $\left\langle \oint_X^{*} \Lambda F d\mu(h), \varphi \right\rangle$

$$= \left\langle \mathrm{h}, \oint_X \Lambda F d\mu(\varphi) \right\rangle$$
$$= \int_X \nu\ (x)\ \langle h, \Lambda_x^{*}(\varphi(x)) \rangle d\mu$$
$$= \int_X \nu\ (x)\ \langle \pi_{F(x)}(h), \Lambda_x^{*}(\varphi(\mathrm{x})) \rangle d\mu$$
$$= \int_X \nu\ (x)\ \langle \Lambda_x(\pi_{F(x)}(h)), \varphi(\mathrm{x}) \rangle d\mu$$
$$= \langle \nu \Lambda_{(.)} \pi_{F(.)}(h), \varphi \rangle.$$

Hence for each $h \in H$,

$$\oint_X^{*} \Lambda F d\mu = \nu \Lambda \pi_F.$$

3. Main Result

Definition 3.1

For each F ∈ L(X, Λ, υ) the following statement is defined:

$$A_{\Lambda F,\nu} = \inf_{h \in H_1} \| \oint_X^{*} \Lambda F d\mu(h) \|^2, B_{\Lambda F,\nu} = \sup_{h \in H_1} \| \oint_X^{*} \Lambda F d\mu(h) \|^2 .$$

Remark 3.2

F ∈ L(X, Λ, υ). So, for each h ∈ H

$$< \oint_X \Lambda F d\mu \oint_X^{*} \Lambda F d\mu(h), h > = \| \nu \Lambda \pi_F(h) \|^2,$$

$A_{\Lambda F,\nu}$ and $B_{\Lambda F,\nu}$ are optimal scalars which satisfy:

$$A_{\Lambda F,\nu} \le \oint_X \Lambda F d\mu \oint_X^{*} \Lambda F d\mu \le B_{\Lambda F,\nu}.$$

Lemma 3.3

F ∈ L(X, Λ, υ). So, $A_{\Lambda F,\nu} > 0$ if and only if $\oint_X \Lambda F d\mu$ is subjective.

In order to verify it: Let $A_{\Lambda F,\nu} > 0$. Since for each h ∈ H

$$< \oint_X \Lambda F d\mu \oint_X^{*} \Lambda F d\mu(h), h >$$
$$= \int_X \nu^2(x) \| \Lambda_x(\pi_{F(x)}(h)) \|^2\ d\mu$$
$$= \| \nu \Lambda \pi_F(h) \|^2 \ge A_{\Lambda F,\nu}\ \| h \|^2 .$$

Therefore by Lemma 2.3(iii) $\oint_X \Lambda F d\mu$ is subjective.

Now let $\oint_X \Lambda F d\mu$ be subjective. Let

$$\oint_X^{\dagger} \Lambda F d\mu : H \to L^2(X,K)$$

Is presented as its pseudo-inverse. With using Lemma 2.1, for each h ∈ H

$$\| h \| = \| \oint_X^{\dagger *} \Lambda F d\mu \oint_X^{*} \Lambda F d\mu(h) \|$$
$$\le \| \oint_X^{\dagger *} \Lambda F d\mu \| \| \oint_X^{*} \Lambda F d\mu(h) \|$$
$$= \| \oint_X^{\dagger *} \Lambda F d\mu \| \| \nu \Lambda \pi_F(h) \| .$$

So

$$A_{\Lambda F,\nu} \ge \| \oint_X^{\dagger *} \Lambda F d\mu \|^{-2} > 0.$$

Theorem 3.4

F ∈ L(X, Λ, υ). So, the operator $\oint_X \Lambda F d\mu \oint_X^{*} \Lambda F d\mu$ is irreversible if and only if $A_{\Lambda F,\nu} > 0$.

In order to verify it:Let $\oint_X \Lambda F d\mu \oint_X^{*} \Lambda F d\mu$ be irreversible. So:

$$A_{F,\nu} \le \inf_{h \in H_1} \| \oint_X^{*} \Lambda F d\mu \|^2$$
$$= \inf_{h \in H_1} < \oint_X \Lambda F d\mu \oint_X^{*} \Lambda F d\mu(h), h > \in \sigma(\oint_X \Lambda F d\mu \oint_X^{*} \Lambda F d\mu),$$

So, $A_{\Lambda F,\nu} > 0$ Now $A_{\Lambda F,\nu} > 0$ Therefore, by the mentioned Lemma $\oint_X \Lambda F d\mu$ is subjective. So, there exist A > 0 wheras:

$$A \| h \| \le \| \oint_X^{*} \Lambda F d\mu \|, h \in H..$$

So,

$$\oint_X \Lambda F d\mu \oint_X^{*} \Lambda F d\mu \ge A^2 > 0.$$

Definition 3.5

Let $L(X,K,\Lambda,\nu)$ and $L(X,K,\overline{\Lambda},\nu)$ and be two classes of mappings that we said in definition 2.5. $\tilde{F} \in L(X,K,\tilde{\Lambda},\nu)$ and $F \in L(X,K,\Lambda,\nu)$ and are called dual pair if:

$$\oint_X \Lambda F d\mu \oint_X^{*} \tilde{\Lambda} \tilde{F}\, d\mu = I.$$

Theorem 3.5

$L(X,K,\Lambda,v)$ and $L(X,K,\overline{\Lambda},v)$ are considered as two classes of mappings that we said in definition 2.5. For $\widetilde{F}\in L(X,K,\widetilde{\Lambda},v)$ and $F\in L(X,K,\Lambda,v)$ we consider the operator

$$S_{\Lambda F,\widetilde{\Lambda}\widetilde{F}}:H\to H,$$

$$S_{\Lambda F,\widetilde{\Lambda}\widetilde{F}}=\oint_X \Lambda F d\mu \oint_X^{*} \widetilde{\Lambda}\widetilde{F}\, d\mu.$$

Then $S_{\Lambda F,\widetilde{\Lambda}\widetilde{F}}$ is bounded and $S^{*}_{\Lambda F,\widetilde{\Lambda}\widetilde{F}}=S_{\widetilde{\Lambda}\widetilde{F},\Lambda F}$.

Proof.

For any h, k ∈ H by Lemma 2.10 we have:

$$\left\langle S_{\Lambda F,\widetilde{\Lambda}\widetilde{F}}(h),k\right\rangle=\left\langle \oint_X \Lambda F d\mu \oint_X^{*} \overline{\Lambda}\,\overline{F}\, d\mu(h),k\right\rangle$$

$$=\left\langle \oint_X^{*} \overline{\Lambda}\,\overline{F}\, d\mu(h),\oint_X^{*} \Lambda F d\mu(k)\right\rangle=\left\langle v\overline{\Lambda}\pi_{\widetilde{F}}(h),v\Lambda\pi_F(k)\right\rangle$$

$$=\int_X\left\langle v(x)\overline{\Lambda}_x\pi_{\widetilde{F}(x)}(h),v(x)\Lambda_x\pi_{F(x)}(k)\right\rangle d\mu$$

$$=\int_X v^2(x)\left\langle \overline{\Lambda}_x\pi_{\widetilde{F}(x)}(h),\Lambda_x\pi_{F(x)}(k)\right\rangle d\mu.$$

Thus :

$$\left|\left\langle S_{\Lambda F,\widetilde{\Lambda}\widetilde{F}}(h),k\right\rangle\right|^2\leq=\left(\int_X v^2(x)\left\|\Lambda_x\pi_{F(x)}(k)\right\|^2 d\mu\right)\left(\int_X v^2(x)\left\|\overline{\Lambda}_x\pi_{\widetilde{F}(x)}(h)\right\|^2 d\mu\right)$$

$$\leq B^{1/2}{}_{\Lambda F,v}B^{1/2}{}_{\widetilde{\Lambda}\widetilde{F},v}\left\|h\right\|^2\left\|k\right\|^2,$$

Hence :

$$\left\|S_{\Lambda F,\widetilde{\Lambda}\widetilde{F}}\right\|^2\leq B^{1/2}{}_{\Lambda F,v}B^{1/2}{}_{\widetilde{\Lambda}\widetilde{F},v}.$$

Also, $S^{*}_{\Lambda F,\overline{\Lambda}\overline{F}}$ is bounded and we have:

$$S^{*}_{\Lambda F,\widetilde{\Lambda}\widetilde{F}}=\left(\oint_X \Lambda F d\mu \oint_X^{*} \overline{\Lambda}\,\overline{F}\, d\mu\right)^{*}=\oint_X \overline{\Lambda}\,\overline{F}\, d\mu \oint_X^{*} \Lambda F d\mu=S_{\widetilde{\Lambda}\widetilde{F},\Lambda F}.$$

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